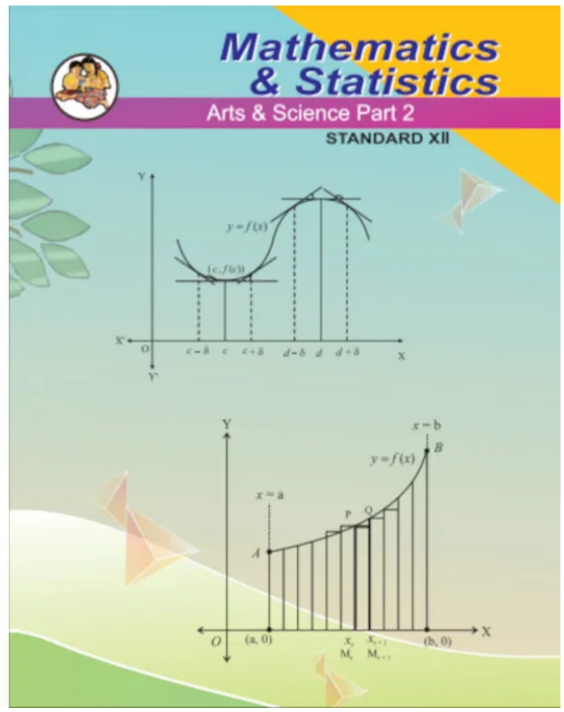
Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 6- Differential Equations

Class 12 - Chapter 6 Differential Equations



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Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 6- Differential Equations

Class 12: Maths Chapter 6 solutions. Complete Class 12 Maths Chapter 6 Notes.

Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 6- Differential Equations

Maharashtra Board 12th Maths Chapter 6, Class 12 Maths Chapter 6 solutions

Ex 6.1

1. Determine the order and degree of each of the following differential equations:

Question (i).
$$\frac{dy}{dx^2}+X\left(\frac{dy}{dx}\right)+y=2\sin x$$
Solution:
The given D.E. is $\frac{dy}{dx^2}+X\left(\frac{dy}{dx}\right)+y=2\sin x$
This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.
∴ the given D.E. is of order 2 and degree 1.

Question (ii).
$$\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}=\frac{d^2y}{dx^2}$$
Solution:
The given D.E. is $\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}=\frac{d^2y}{dx^2}$
On cubing both sides, we get
$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{d^2y}{dx^2}\right)^3$$
This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 3.
∴ the given D.E. is of order 2 and degree 3.

Question (iii).
$$\frac{dy}{dx}=\frac{2\sin x+3}{\frac{dy}{dx}}$$
Solution:

The given D.E. is $\frac{dy}{dx} = \frac{2\sin x+3}{\frac{dy}{dx}}$

$\therefore \left(\frac{dy}{dx}\right)^2$ = 2 sin x + 3

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 2.

∴ the given D.E. is of order 1 and degree 2.

Question (iv).

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$

Solution:

The given D.E. is $\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$

On squaring both sides, we get

$\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\right)^2 = 1 + \frac{d^3y}{dx^3}$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 1.

∴ the given D.E. has order 3 and degree 1.

Question (v).

$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + 7x + 5 = 0$

Solution:

The given D.E. is $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + 7x + 5 = 0$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.
∴ the given D.E. has order 2 and degree 1.

Question (vi).
(y"')2 + 3y" + 3xy' + 5y = 0
Solution:
The given D.E. is (y"')2 + 3y" + 3xy' + 5y = 0
This can be written as:

$$\left(\frac{d^3y}{dx^3}\right)^2 + 3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = 0$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 2.
∴ The given D.E. has order 3 and degree 2.

Question (vii).

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

Solution:
The given D.E. is $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$
This D.E. has highest order derivative $\frac{d^2y}{dx^2}$
∴ order = 2
Since this D.E. cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

Question (viii).

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=8\frac{d^2y}{dx^2}$$

Solution:

The given D.E. is $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=8\frac{d^2y}{dx^2}$

On squaring both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=8^2\cdot\left(\frac{d^2y}{dx^2}\right)^2$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 2.

∴ the given D.E. has order 2 and degree 2.

Question (ix).

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}}\cdot\left(\frac{dy}{dx}\right)^{\frac{1}{3}}=20$$

Solution:

The given D.E. is $\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}}\cdot\left(\frac{dy}{dx}\right)^{\frac{1}{3}}=20$

$$\therefore\left(\frac{d^3y}{dx^3}\right)^3\cdot\left(\frac{dy}{dx}\right)^2=20^6$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 3.

∴ the given D.E. has order 3 and degree 3.

Question (x).

$$x+\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{d^2y}{dx^2}\right)^2}$$

Solution:

The given D.E. is $x+\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{d^2y}{dx^2}\right)^2}$

On squaring both sides, we get

$$\left(x+\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore\ x^2+2x\frac{d^2y}{dx^2}+\left(\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore\ x^2+2x\frac{d^2y}{dx^2}-1=0$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

∴ the given D.E. has order 2 and degree 1.

Ex 6.2

Question 1.
Obtain the differential equation by eliminating the arbitrary constants from the following equations:
(i) $x^3 + y^3 = 4ax$
Solution:
$x^3 + y^3 = 4ax$ ……..(1)
Differentiating both sides w.r.t. x, we get
$3x^2 + 3y^2 \frac{dy}{dx} = 4a \times 1$
$\therefore 3x^2 + 3y^2 \frac{dy}{dx} = 4a$
Substituting the value of 4a in (1), we get
$x^3 + y^3 = (3x^2 + 3y^2 \frac{dy}{dx})\, x$
$\therefore x^3 + y^3 = 3x^3 + 3xy^2 \frac{dy}{dx}$
$\therefore 2x^3 + 3xy^2 \frac{dy}{dx} - y^3 = 0$
This is the required D.E.

(ii) $Ax^2 + By^2 = 1$

Solution:

$Ax^2 + By^2 = 1$

Differentiating both sides w.r.t. x, we get

$A \times 2x + B \times 2y \frac{dy}{dx} = 0$

$\therefore Ax + By \frac{dy}{dx} = 0$ ....(1)

Differentiating again w.r.t. x, we get

$$A \times 1 + B\left[y\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{dy}{dx}\right] = 0$$

$$\therefore \; A + B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$$

$$\therefore \; A = -B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

Substituting the value of A in (1), we get

$$-Bx\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + By\frac{dy}{dx} = 0$$

$$\therefore \; -x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + y\frac{dy}{dx} = 0$$

$$\therefore \; -xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$$

$$\therefore \; xy\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y\frac{dy}{dx} = 0$$

This is the required D.E.

Alternative Method:

$Ax^2 + By^2 = 1$ ....(1)

Differentiating both sides w.r.t. x, we get

$A \times 2x + B \times 2y \frac{dy}{dx} = 0$

$\therefore Ax + By \frac{dy}{dx} = 0$ ....(2)

Differentiating again w.r.t. x, we get,

$$A \times 1 + B\left[y \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{dy}{dx}\right] = 0$$

$$\therefore \; A + B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \quad \text{......(3)}$$

The equations (1), (2) and (3) are consistent in A and B.

$\therefore$ determinant of their consistency is zero.

$$\therefore \begin{vmatrix} x^2 & y^2 & 1 \\ x & y\frac{dy}{dx} & 0 \\ 1 & y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix} = 0$$

$$\therefore \; x^2(0-0) - y^2(0-0) + 1\left[xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}\right] = 0$$

$$\therefore \; xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

This is the required D.E.

(iii) y = A cos(log x) + B sin(log x)

Solution:

y = A cos(log x) + B sin (log x) ……. (1)
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin(\log x)\cdot\frac{d}{dx}(\log x) + B\cos(\log x)\cdot\frac{d}{dx}(\log x)$$

$$= \frac{-A\sin(\log x)}{x} + \frac{B\cos(\log x)}{x}$$

$$\therefore\ x\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-A\cos(\log x)}{x} - \frac{B\sin(\log x)}{x}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[A\cos(\log x) + B\sin(\log x)]$$

$$= -y \qquad \text{... [By (1)]}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ is the required D.E.}$$

(iv) $y^2 = (x + c)^3$
Solution:
$y^2 = (x + c)^3$
Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx}=3(x+c)^2\cdot(1)=3(x+c)^2$$

$$\therefore\ (x+c)^2=\frac{2y}{3}\cdot\frac{dy}{dx}$$

$$\therefore\ (x+c)^6=\left(\frac{2y}{3}\cdot\frac{dy}{dx}\right)^3$$

$$\therefore\ (y^2)^2=\frac{8y^3}{27}\cdot\left(\frac{dy}{dx}\right)^3 \quad \text{....... [By (1)]}$$

$$\therefore\ 27y^4=8y^3\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ 27y=8\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ 8\left(\frac{dy}{dx}\right)^3-27y=0$$

This is the required D.E.

(v) $y = Ae^{5x} + Be^{-5x}$

Solution:

$y = Ae^{5x} + Be^{-5x}$ ....(1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx}=Ae^{5x}\times 5+Be^{-5x}\times(-5)$$

$$\therefore\ \frac{dy}{dx}=5Ae^{5x}-5Be^{-5x}$$

and $\frac{d^2y}{dx^2} = 5Ae^{5x} \times 5 - 5Be^{-5x} \times (-5)$

$= 25Ae^{5x} + 25Be^{-5x}$

$= 25(Ae^{5x} + Be^{-5x}) = 25y$ .... [By (1)]

$\therefore \frac{d^2y}{dx^2} - 25y = 0$

This is the required D.E.

(vi) $(y - a)^2 = 4(x - b)$
Solution:
$(y - a)^2 = 4(x - b)$
Differentiating both sides w.r.t. x, we get
$2(y - a) . \frac{d}{dx}(y - a) = 4 \frac{d}{dx}(x - b)$
$\therefore 2(y - a) . (\frac{dy}{dx} - 0) = 4(1 - 0)$
$\therefore 2(y - a) \frac{dy}{dx} = 4$
$\therefore (y - a) \frac{dy}{dx} = 2$ ....(1)
Differentiating w.r.t. x, we get

$(y-a) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(y-a) = 0$

$\therefore (y-a)\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0\right) = 0$

$\therefore (y-a)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

$$\therefore \frac{2}{\left(\frac{dy}{dx}\right)} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \text{....... [By (1)]}$$

$$\therefore 2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

This is the required D.E.

(vii) y = a + $\frac{a}{x}$

Solution:

y = a + $\frac{a}{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(a + \frac{a}{x}\right) = 0 + a\left(-\frac{1}{x^2}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{a}{x^2}$$

$$\therefore a = -x^2\frac{dy}{dx}$$

Substituting the value of a in (1), we get

$$y = -x^2\frac{dy}{dx} + \frac{1}{x}\left(-x^2\frac{dy}{dx}\right)$$

$$\therefore y = -x^2\frac{dy}{dx} - x\frac{dy}{dx}$$

$$\therefore (x^2 + x)\frac{dy}{dx} + y = 0$$

$$\therefore\ x(x+1)\frac{dy}{dx}+y=0$$

This is the required D.E.

(viii) $y = c_1e^{2x} + c_2e^{5x}$

Solution:

$y = c_1e^{2x} + c_2e^{5x}$ ....(1)

Differentiating twice w.r.t. x, we get

$\frac{dy}{dx} = c_1e^{2x} \times 2 + c_2e^{5x} \times 5$

$$\therefore\ \frac{dy}{dx}=2c_1e^{2x}+5c_2e^{5x} \quad ....(2)$$

$$\text{and } \frac{d^2y}{dx^2}=2c_1e^{2x}\times 2+5c_2e^{5x}\times 5$$

$$\therefore\ \frac{d^2y}{dx^2}=4c_1e^{2x}+25c_2e^{5x} \quad ....(3)$$

The equations (1), (2) and (3) are consistent in c_1e^{2x} and c_2e^{5x}

∴ determinant of their consistency is zero.

$$\therefore\ \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 2 & 5 \\ \frac{d^2y}{dx^2} & 4 & 25 \end{vmatrix}=0$$

$$\therefore\ y(50-20)-1\left(25\frac{dy}{dx}-5\frac{d^2y}{dx^2}\right)+1\left(4\frac{dy}{dx}-2\frac{d^2y}{dx^2}\right)=0$$

$$\therefore\ 30y - 25\frac{dy}{dx} + 5\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2\frac{d^2y}{dx^2} = 0$$

$$\therefore\ 3\frac{d^2y}{dx^2} - 21\frac{dy}{dx} + 30y = 0$$

$$\therefore\ \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$$

This is the required D.E.

Alternative Method:
$y = c_1e^{2x} + c_2e^{5x}$
Dividing both sides by e^{5x}, we get

$$e^{-5x} \cdot y = c_1e^{-3x} + c_2$$

Differentiating w.r.t. x, we get

$$e^{-5x} \cdot \frac{dy}{dx} + y \times e^{-5x} \times (-5) = c_1e^{-3x} \times (-3) + 0$$

$$\therefore\ e^{-5x}\left(\frac{dy}{dx} - 5y\right) = -3c_1e^{-3x}$$

Dividing both sides by e^{-3x}, we get

$$e^{-2x}\left(\frac{dy}{dx} - 5y\right) = -3c_1$$

Differentiating w.r.t. x, we get

$$e^{-2x}\left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx}\right) + \left(\frac{dy}{dx} - 5y\right) \cdot e^{-2x}(-2) = 0$$

$$\therefore e^{-2x}\left(\frac{d^2y}{dx^2}-5\frac{dy}{dx}-2\frac{dy}{dx}+10y\right)=0$$

$$\therefore \frac{d^2y}{dx^2}-7\frac{dy}{dx}+10y=0$$

This is the required D.E.

(ix) $c_1x^3 + c_2y^2 = 5$.

Solution:

$c_1x^3 + c_2y^2 = 5$ ....(1)

Differentiating w.r.t. x, we get

$$c_1 \times 3x^2 + c_2 \times 2y\frac{dy}{dx}=0$$

$$\therefore 3c_1x^2 + 2c_2y\frac{dy}{dx}=0 \quad ......(2)$$

Differentiating again w.r.t. x, we get

$$3c_1 \times 2x + 2c_2\left[y\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{dy}{dx}\right]=0$$

$$\therefore 6c_1x + 2c_2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0 \quad ....(3)$$

The equations (1), (2) and (3) in c_1, c_2 are consistent.

∴ determinant of their consistency is zero.

$$\therefore \begin{vmatrix} x^3 & y^2 & 5 \\ 3x^2 & 2y\frac{dy}{dx} & 0 \\ 6x & 2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix} = 0$$

$$\therefore x^3(0-0) - y^2(0-0) + 5\left[6x^2y\frac{d^2y}{dx^2}+6x^2\left(\frac{dy}{dx}\right)^2-12xy\frac{dy}{dx}\right]=0$$

$$\therefore 6x^2y\frac{d^2y}{dx^2}+6x^2\left(\frac{dy}{dx}\right)^2-12xy\frac{dy}{dx}=0$$

$$\therefore xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}=0$$

This is the required D.E.

(x) $y = e^{-2x}(A \cos x + B \sin x)$

Solution:

$y = e^{-2x}(A \cos x + B \sin x)$

$\therefore e^{2x} . y = A \cos x + B \sin x$ ....(1)

Differentiating w.r.t. x, we get

$$e^{2x}\cdot\frac{dy}{dx}+y\cdot e^{2x}\times 2 = A(-\sin x)+B\cos x$$

$$\therefore e^{2x}\left(\frac{dy}{dx}+2y\right) = -A\sin x + B\cos x$$

Differentiating again w.r.t. x, we get

$$\therefore \begin{vmatrix} x^3 & y^2 & 5 \\ 3x^2 & 2y\frac{dy}{dx} & 0 \\ 6x & 2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix} = 0$$

$$\therefore x^3(0-0) - y^2(0-0) +$$

$$5\left[6x^2y\frac{d^2y}{dx^2} + 6x^2\left(\frac{dy}{dx}\right)^2 - 12xy\frac{dy}{dx}\right] = 0$$

$$\therefore 6x^2y\frac{d^2y}{dx^2} + 6x^2\left(\frac{dy}{dx}\right)^2 - 12xy\frac{dy}{dx} = 0$$

$$\therefore xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} = 0$$

This is the required D.E.

(x) $y = e^{-2x}(A \cos x + B \sin x)$

Solution:

$y = e^{-2x}(A \cos x + B \sin x)$

$\therefore e^{2x} . y = A \cos x + B \sin x$ ....(1)

Differentiating w.r.t. x, we get

$$e^{2x}\cdot\frac{dy}{dx} + y\cdot e^{2x} \times 2 = A(-\sin x) + B \cos x$$

$$\therefore e^{2x}\left(\frac{dy}{dx} + 2y\right) = -A \sin x + B \cos x$$

Differentiating again w.r.t. x, we get

Differentiating again w.r.t. x, we get

$$e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}+2y\right)\cdot e^{2x}\times 2$$

$$= -A\cos x+B(-\sin x)$$

$$\therefore\ e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2\frac{dy}{dx}+4y\right)=-(A\cos x+B\sin x)$$

$$\therefore\ e^{2x}\left(\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y\right)=-e^{2x}\cdot y \qquad \text{... [By (1)]}$$

$$\therefore\ \frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=-y$$

$$\therefore\ \frac{d^2y}{dx^2}+4\frac{ay}{dx}+5y=0$$

This is the required D.E.

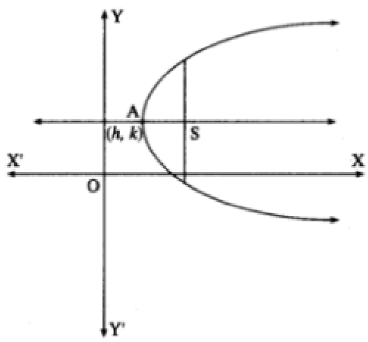
Question 2.
Form the differential equation of family of lines having intercepts a and b on the coordinate axes respectively.
Solution:
The equation of the line having intercepts a and b on the coordinate axes respectively, is
$\frac{x}{a}+\frac{y}{b}=1$ ....(1)
where a and b are arbitrary constants.
[For different values of a and b, we get, different lines. Hence (1) is the equation of family of lines.]
Differentiating (1) w.r.t. x, we get

$$\frac{1}{a}(1)+\left(\frac{1}{b}\right)\cdot\frac{dy}{dx}=0$$

$$\therefore \left(\frac{1}{b}\right)\frac{dy}{dx}=-\frac{1}{a} \qquad \therefore \frac{dy}{dx}=-\frac{b}{a}$$

Differentiating again w.r.t. x, we get $\frac{d^2y}{dx^2}=0$
This is the required D.E.

Question 3.
Find the differential equation all parabolas having length of latus rectum 4a and axis is parallel to the X-axis.
Solution:



Let A(h, k) be the vertex of the parabola whose length of latus rectum is 4a.
Then the equation of the parabola is $(y-k)^2 = 4a(x-h)$, where h and k are arbitrary constants.
Differentiating w.r.t. x, we get

$$2(y-k)\cdot\frac{d}{dx}(y-k)=4a\frac{d}{dx}(x-h)$$

$$\therefore\ 2(y-k)\left(\frac{dy}{dx}-0\right)=4a(1-0)$$

$$\therefore\ 2(y-k)\frac{dy}{dx}=4a$$

$$\therefore\ (y-k)\frac{dy}{dx}=2a \quad \text{........(1)}$$

Differentiating again w.r.t. x, we get

$$(y-k)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(y-k)=0$$

$$\therefore\ (y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\left(\frac{dy}{dx}-0\right)=0$$

$$\therefore\ (y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\therefore\ \frac{2a}{\left(\frac{dy}{dx}\right)}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \quad \text{........ [By (1)]}$$

$$\therefore\ 2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$$

This is the required D.E.

Question 4.
Find the differential equation of the ellipse whose major axis is twice its minor axis.
Solution:
Let 2a and 2b be lengths of major axis and minor axis of the ellipse.
Then 2a = 2(2b)

∴ a = 2b

∴ equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i.e., $\frac{x^2}{(2b)^2} + \frac{y^2}{b^2} = 1$

∴ $\frac{x^2}{4b^2} + \frac{y^2}{b^2} = 1$

∴ $x^2 + 4y^2 = 4b^2$

Differentiating w.r.t. x, we get

$2x + 4 \times 2y \frac{dy}{dx} = 0$

∴ $x + 4y \frac{dy}{dx} = 0$

This is the required D.E.

Question 5.

Form the differential equation of family of lines parallel to the line 2x + 3y + 4 = 0.

Solution:

The equation of the line parallel to the line 2x + 3y + 4 = 0 is 2x + 3y + c = 0, where c is an arbitrary constant.

Differentiating w.r.t. x, we get

$2 \times 1 + 3 \frac{dy}{dx} + 0 = 0$

∴ $3 \frac{dy}{dx} + 2 = 0$

This is the required D.E.

Question 6.
Find the differential equation of all circles having radius 9 and centre at point (h, k).
Solution:
Equation of the circle having radius 9 and centre at point (h, k) is
$(x - h)^2 + (y - k)^2 = 81$ …… (1)
where h and k are arbitrary constant.
Differentiating (1) w.r.t. x, we get

$$2(x-h)\cdot\frac{d}{dx}(x-h)+2(y-k)\cdot\frac{d}{dx}(y-k)=0$$

$$\therefore\ (x-h)(1-0)+(y-k)\left(\frac{dy}{dx}-0\right)=0$$

$$\therefore\ (x-h)+(y-k)\frac{dy}{dx}=0 \quad \text{...... (2)}$$

Differentiating again w.r.t. x, we get

$$\frac{d}{dx}(x-h)+(y-k)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(y-k)=0$$

$$\therefore\ (1-0)+(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\left(\frac{dy}{dx}-0\right)=0$$

$$\therefore\ (y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0$$

$$\therefore\ (y-k)\frac{d^2y}{dx^2}=-\left[\left(\frac{dy}{dx}\right)^2+1\right]$$

$$\therefore\ y-k=\frac{-\left[\left(\frac{dy}{dx}\right)^2+1\right]}{\left(\frac{d^2y}{dx^2}\right)} \quad \text{....... (3)}$$

From (2), x – h = -(y – k) $\frac{dy}{dx}$
Substituting the value of (x – h) in (1), we get

$$(y-k)^2\left(\frac{dy}{dx}\right)^2+(y-k)^2=81$$

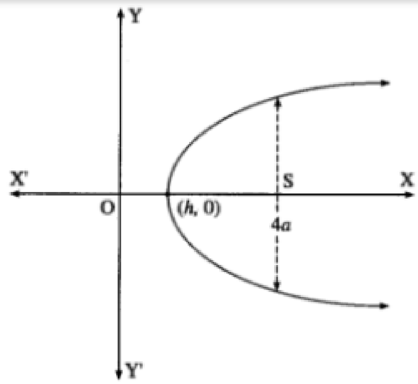
$$\therefore\ \left(\frac{dy}{dx}\right)^2+1=\frac{81}{(y-k)^2}$$

$$\therefore\ \left(\frac{dy}{dx}\right)^2+1=\frac{81\cdot\left(\frac{d^2y}{dx^2}\right)^2}{\left[\left(\frac{dy}{dx}\right)^2+1\right]^2} \quad \text{....... [By (3)]}$$

$$\therefore\ 81\left(\frac{d^2y}{dx^2}\right)^2=\left[\left(\frac{dy}{dx}\right)^2+1\right]^3$$

This is the required D.E.

Question 7.
Form the differential equation of all parabolas whose axis is the X-axis.
Solution:



The equation of the parbola whose axis is the X-axis is

$y^2 = 4a(x - h)$ …… (1)

where a and h are arbitrary constants.

Differentiating (1) w.r.t. x, we get

$2y \frac{dy}{dx} = 4a(1 - 0)$

$\therefore y \frac{dy}{dx} = 2a$

Differentiating again w.r.t. x, we get

$$y \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{dy}{dx} = 0$$

$$\therefore\ y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$

This is the required D.E.

Ex 6.3

Question 1.
In each of the following examples verify that the given expression is a solution of the corresponding differential equation.

(i) xy = log y + c; $\frac{dy}{dx} = \frac{y^2}{1-xy}$

Solution:
xy = log y + c
Differentiating w.r.t. x, we get

$$x \cdot \frac{dy}{dx} + y \times 1 = \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore \ x\frac{dy}{dx} + y = \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\left(x - \frac{1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore \ \left(\frac{xy-1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore \ \frac{dy}{dx} = \frac{-y^2}{xy-1} = \frac{y^2}{1-xy}, \text{ if } xy \neq 1$$

Hence, xy = log y + c is a solution of the D.E.
$\frac{dy}{dx} = \frac{y^2}{1-xy'}, xy \neq 1$

(ii) y = $(\sin^{-1}x)^2 + c$; $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

Solution:

y = $(\sin^{-1} x)^2 + c$ ....(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^2 + 0$$

$$\therefore \frac{dy}{dx} = 2(\sin^{-1}x)\cdot\frac{d}{dx}(\sin^{-1}x)$$

$$= 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(y-c) \quad \text{......... [By (1)]}$$

Differentiating again w.r.t. x, we get

$$(1-x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(1-x^2) = 4\frac{d}{dx}(y-c)$$

$$\therefore (1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{dy}{dx} - 0\right)$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

Hence, $y=(\sin^{-1}x)^2+c$ is a solution of the D.E.

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2.$$

(iii) y = e^{-x} + Ax + B; $e^x\frac{d^2y}{dx^2}=1$

Solution:

y = e^{-x} + Ax + B

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{-x}\times(-1)+A\times 1+0$$

$$\therefore\ \frac{dy}{dx}=-e^{-x}+A$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-e^{-x}\times(-1)+0$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{1}{e^x}$$

$$\therefore e^x\frac{d^2y}{dx^2}=1$$

Hence, y = e^{-x} + Ax + B is a solution of the D.E.

$$e^x\frac{d^2y}{dx^2}=1$$

(iv) y = x^m; $x^2\frac{d^2y}{dx^2}-mx\frac{dy}{dx}+my=0$

Solution:
$y = x^m$
Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^m) = mx^{m-1}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(mx^{m-1}) = m\frac{d}{dx}(x^{m-1}) = m(m-1)x^{m-2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my$$

$$= x^2 \cdot m(m-1)x^{m-2} - mx \cdot mx^{m-1} + m \cdot x^m$$

$$= m(m-1)x^m - m^2x^m + mx^m$$

$$= (m^2 - m - m^2 + m)x^m = 0$$

This shows that $y = x^m$ is a solution of the D.E.
$x^2\frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my = 0$

(v) $y = a + \frac{b}{x}$; $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$
Solution:
$y = a + \frac{b}{x}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 + b\left(-\frac{1}{x^2}\right) = -\frac{b}{x^2}$$

$$\therefore x^2\frac{dy}{dx} = -b$$

Differentiating again w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 2x = 0$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Hence, y = a + $\frac{b}{x}$ is a solution of the D.E.

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

(vi) $y = e^{ax}$; x $\frac{dy}{dx}$ = y log y

Solution:

$y = e^{ax}$

log y = log e^{ax} = ax log e

log y = ax …….(1) ……..[∵ log e = 1]

Differentiating w.r.t. x, we get

$\frac{1}{y} \cdot \frac{dy}{dx}$ = a × 1

∴ $\frac{dy}{dx}$ = ay

∴ x $\frac{dy}{dx}$ = (ax)y

∴ x $\frac{dy}{dx}$ = y log y ………[By (1)]

Hence, $y = e^{ax}$ is a solution of the D.E.
$x \frac{dy}{dx} = y \log y$.

Question 2.
Solve the following differential equations.
(i) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$
Solution:

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\therefore \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx$$

$$\therefore \tan^{-1} y = \tan^{-1} x + c$$

This is the general solution.

(ii) $\log(\frac{dy}{dx}) = 2x + 3y$
Solution:

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

$$\therefore \frac{dy}{dx} = e^{2x+3y} = e^{2x} \cdot e^{3y}$$

$$\therefore \frac{1}{e^{3y}} dy = e^{2x} dx$$

Integrating both sides, we get

$\int e^{-3y}\,dy = \int e^{2x}dx$

$\therefore \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c_1$

$\therefore 2e^{-3y} = -3e^{2x} + 6c_1$

$\therefore 2e^{-3y} + 3e^{2x} = c$, where $c = 6c_1$

This is the general solution.

(iii) $y - x\frac{dy}{dx} = 0$

Solution:

$y - x\frac{dy}{dx} = 0$

$\therefore x\frac{dy}{dx} = y$

$\therefore \frac{1}{x}dx = \frac{1}{y}dy$

Integrating both sides, we get

$\int \frac{1}{x}dx = \int \frac{1}{y}dy$

$\therefore \log|x| = \log|y| + \log c$

$\therefore \log|x| = \log|cy|$

$\therefore x = cy$

This is the general solution.

(iv) $\sec^2 x \cdot \tan y\,dx + \sec^2 y \cdot \tan x\,dy = 0$

$\sec^2 x \cdot \tan y\, dx + \sec^2 y \cdot \tan x\, dy = 0$

$\therefore \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

Integrating both sides, we get

$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = c_1$

Each of these integrals is of the type

$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

∴ the general solution is

∴ log|tan x| + log|tan y | = log c, where c_1 = log c

∴ log |tan x . tan y| = log c

∴ tan x . tan y = c

This is the general solution.

(v) cos x . cos y dy – sin x . sin y dx = 0

Solution:

cos x . cos y dy – sin x . sin y dx = 0

$\frac{\cos y}{\sin y} dy - \frac{\sin x}{\cos x} dx = 0$

Integrating both sides, we get

∫cot y dy – ∫tan x dx = c_1

∴ log|sin y| – [-log|cos x|] = log c, where c_1 = log c

∴ log |sin y| + log|cos x| = log c

∴ log|sin y . cos x| = log c

∴ sin y . cos x = c

This is the general solution.

(vi) $\frac{dy}{dx}$ = -k, where k is a constant.
Solution:
$\frac{dy}{dx}$ = -k
∴ dy = -k dx
Integrating both sides, we get
∫dy = -k∫dx
∴ y = -kx + c
This is the general solution.

(vii) $\frac{\cos^2 y}{x} dy + \frac{\cos^2 x}{y} dx = 0$
Solution:
$\frac{\cos^2 y}{x} dy + \frac{\cos^2 x}{y} dx = 0$
∴ y $\cos^2$y dy + x $\cos^2$x dx = 0
$\therefore x\left(\frac{1+\cos 2x}{2}\right) dx + y\left(1+\frac{\cos 2y}{2}\right) dy = 0$
∴ x(1 + cos 2x) dx + y(1 + cos 2y) dy = 0
∴ x dx + x cos 2x dx + y dy+ y cos 2y dy = 0
Integrating both sides, we get
∫x dx + ∫y dy + ∫x cos 2x dx + ∫y cos 2y dy = c_1 ……..(1)
Using integration by parts

$$\int x\cos 2x dx = x\int \cos 2x\, dx - \int\left[\frac{d}{dx}(x)\int \cos 2x\ dx\right]dx$$

$$= x\left(\frac{\sin 2x}{2}\right) - \int 1 \cdot \frac{\sin 2x}{2}\,dx$$

$$= \frac{x \sin 2x}{2} + \frac{1}{2} \cdot \frac{\cos 2x}{2} = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

Similarly,

$$\int y \cos 2y\,dy = \frac{y \sin 2y}{2} + \frac{\cos 2y}{4}$$

$\therefore$ from (1), we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + \frac{y \sin 2y}{2} + \frac{\cos 2y}{4} = c_1$$

Multiplying throughout by 4, this becomes

$2x^2 + 2y^2 + 2x \sin 2x + \cos 2x + 2y \sin 2y + \cos 2y = 4c_1$

$\therefore 2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + \cos 2y + \cos 2x + c = 0$, where $c = -4c_1$

This is the general solution.

(viii) $y^3 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$

Solution:

$$y^3 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$$

$$\therefore\ y^3 = \frac{dy}{dx} + x^2 \frac{dy}{dx}$$

$$\therefore\ y^3 = (1 + x^2)\frac{dy}{dx} \qquad \therefore\ \frac{1}{1 + x^2}\,dx = \frac{1}{y^3}\,dy$$

Integrating both sides, we get

$$\int \frac{1}{1+x^2}dx = \int y^{-3}dy$$

$$\therefore \tan^{-1}x = \frac{y^{-2}}{-2} + c_1$$

$$\therefore \tan^{-1}x = -\frac{1}{2y^2} + c_1$$

$$\therefore 2y^2\tan^{-1}x = -1 + 2c_1y^2$$

$$\therefore 2y^2\tan^{-1}x + 1 = cy^2, \text{ where } c = 2c_1$$

This is the general solution.

(ix) $2e^{x+2y}\,dx - 3\,dy = 0$

Solution:

$$2e^{x+2y}dx - 3dy = 0$$

$$\therefore 2e^x \cdot e^{2y}dx - 3dy = 0$$

$$\therefore 2e^x dx - \frac{3}{e^{2y}}dy = 0$$

Integrating both sides, we get

$$2\int e^x dx - 3\int e^{-2y}dy = c_1$$

$$\therefore 2e^x - 3\cdot\frac{e^{-2y}}{(-2)} = c_1$$

$$\therefore 4e^x + 3e^{-2y} = 2c_1$$

$$\therefore 4e^x + 3e^{-2y} = c, \text{ where } c = 2c_1$$

This is the general solution.

(x) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

Solution:

$$\frac{dy}{dx} = e^{x+y} + x^2 e^y$$

$$\therefore \frac{dy}{dx} = e^x \cdot e^y + x^2 e^y = e^y (e^x + x^2)$$

$$\therefore \frac{1}{e^y} dy = (e^x + x^2)\, dx$$

Integrating both sides, we get

$$\int e^{-y} dy = \int (e^x + x^2)\, dx$$

$$\therefore \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c_1$$

$$\therefore e^x + e^{-y} + \frac{x^3}{3} = -c_1$$

$\therefore 3e^x + 3e^{-y} + x^3 = -3c_1$

$\therefore 3e^x + 3e^{-y} + x^3 = c$, where $c = -3c_1$

This is the general solution.

Question 3.

For each of the following differential equations, find the particular solution satisfying the given condition:

(i) $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$, when $x = 0$, $y = \pi$

Solution:

$3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$

$\therefore \frac{3e^x}{1+e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$

Integrating both sides, we get

$3\int \frac{e^x}{1+e^x}dx + \int \frac{\sec^2 y}{\tan y}dy = c_1$

Each of these integrals is of the type

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

$\therefore$ the general solution is

$3\log|1+e^x| + \log|\tan y| = \log c$, where $c_1 = \log c$

$\therefore \log|(1+e^x)^3 \cdot \tan y| = \log c$

$\therefore (1+e^x)^3 \tan y = c$

When $x = 0$, $y = \pi$, we have

$(1+e^0)^3 \tan \pi = c$

$\therefore c = 0$

$\therefore$ the particular solution is $(1+e^x)^3 \tan y = 0$.

(ii) $(x - y^2x)\, dx - (y + x^2y)\, dy = 0$, when $x = 2$, $y = 0$

Solution:

$(x - y^2x)\, dx - (y + x^2y)\, dy = 0$

$\therefore x(1 - y^2)\, dx - y(1 + x^2)\, dy = 0$

$\therefore \frac{x}{1+x^2}dx - \frac{y}{1-y^2}dy = 0$

$\therefore \frac{2x}{1+x^2} - \frac{2y}{1-y^2}dy = 0$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

∴ the general solution is

$\log|1+x^2| + \log|1-y^2| = \log c$, where $c_1 = \log c$

∴ $\log|(1+x^2)(1-y^2)| = \log c$

∴ $(1+x^2)(1-y^2) = c$

When x = 2, y = 0, we have

(1 + 4)(1 – 0) = c

∴ c = 5

∴ the particular solution is $(1 + x^2)(1 - y^2) = 5$.

(iii) $y(1 + \log x) \frac{dx}{dy} - x \log x = 0$, $y = e^2$, when x = e

Solution:

$y(1 + \log x) \frac{dx}{dy} - x \log x = 0$

$$\therefore \frac{1+\log x}{x \log x}dx - \frac{dy}{y} = 0$$

Integrating both sides, we get

$$\therefore \int \frac{1+\log x}{x \log x}dx - \int \frac{dy}{y} = c_1 \qquad \text{... (1)}$$

Put $x \log x = t$.

Then $\left[x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)\right] dx = dt$

$\therefore \left[\frac{x}{x} + (\log x)(1)\right] dx = dt \quad \therefore (1 + \log x)\, dx = dt$

$\therefore \int \frac{1 + \log x}{x \log x} dx = \int \frac{dt}{t} = \log |t| = \log |x \log x|$

$\therefore$ from (1), the general solution is

$\log |x \log x| - \log |y| = \log c$, where $c_1 = \log c$

$\therefore \log \left|\frac{x \log x}{y}\right| = \log c \quad \therefore \frac{x \log x}{y} = c$

$\therefore x \log x = cy$

This is the general solution.

Now, $y = e^2$, when $x = e$

$\therefore e \log e = c \cdot e^2 \qquad 1 = c \cdot e \qquad \ldots [\because \log e = 1]$

$\therefore c = \frac{1}{e}$

$\therefore$ the particular solution is $x \log x = \left(\frac{1}{e}\right) y$

$\therefore y = ex \log x$.

(iv) $(e^y + 1) \cos x + e^y \sin x \frac{dy}{dx} = 0$, when $x = \frac{\pi}{6}$, $y = 0$

Solution:

$(e^y + 1)\cos x + e^y \sin x \frac{dy}{dx} = 0$

$$\therefore \frac{\cos x}{\sin x}dx + \frac{e^y}{e^y+1}dy = 0$$

Integrating both sides, we get

$$\int \frac{\cos x}{\sin x}dx + \int \frac{e^y}{e^y+1}dy = c_1 \quad \ldots (1)$$

Now, $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(e^y + 1) = e^y$ and

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

$\therefore$ from (1), the general solution is

$\log|\sin x| + \log|e^y + 1| = \log c$, where $c_1 = \log c$

$\therefore \log|\sin x . (e^y + 1)| = \log c$

$\therefore \sin x . (e^y + 1) = c$

When $x = \frac{\pi}{4}$, $y = 0$, we get

$\left(\sin \frac{\pi}{4}\right)\left(e^0 + 1\right) = c$

$\therefore c = \frac{1}{\sqrt{2}}(1 + 1) = \sqrt{2}$

$\therefore$ the particular solution is $\sin x . (e^y + 1) = \sqrt{2}$

(v) $(x + 1)\frac{dy}{dx} - 1 = 2e^{-y}$, $y = 0$, when $x = 1$

Solution:

$$(x+1)\frac{dy}{dx} - 1 = 2e^{-y}$$

$$\therefore (x+1)\frac{dy}{dx} = \frac{2}{e^y} + 1 = \frac{2+e^y}{e^y}$$

$\therefore \frac{e^y}{2+e^y}dy = \frac{1}{x+1}dx$

Integrating both sides, we get

$\int \frac{e^y}{2+e^y}dy = \int \frac{1}{x+1}dx$

$\therefore \log|2+e^y| = \log|x+1| + \log c$

$\ldots \left[\because \frac{d}{dy}(2+e^y) = e^y \text{ and } \int \frac{f'(y)}{f(y)}dy = \log|f(y)| + c\right]$

$\therefore \log|2+e^y| = \log|c(x+1)|$

$\therefore 2+e^y = c(x+1)$

This is the general solution.

Now, y = 0, when x = 1

$\therefore 2 + e^0 = c(1 + 1)$

$\therefore 3 = 2c$

$\therefore c = \frac{3}{2}$

$\therefore$ the particular solution is $2 + e^y = \frac{3}{2}(x + 1)$

$\therefore 2(2 + e^y) = 3(x + 1)$.

(vi) $\cos\left(\frac{dy}{dx}\right) = a$, $a \in R$, $y(0) = 2$

Solution:

$\cos\left(\frac{dy}{dx}\right) = a$

$\therefore \frac{dy}{dx} = \cos^{-1} a$

$\therefore dy = (\cos^{-1} a)\, dx$

dx

∴ dy = $(\cos^{-1} a)$ dx

Integrating both sides, we get

$\int dy = (\cos^{-1} a) \int dx$

∴ $y = (\cos^{-1} a)\, x + c$

∴ $y = x \cos^{-1} a + c$

This is the general solution.

Now, y(0) = 2, i.e. y = 2,

when x = 0, 2 = 0 + c

∴ c = 2

∴ the particular solution is

∴ $y = x \cos^{-1} a + 2$

∴ $y - 2 = x \cos^{-1} a$

∴ $\frac{y-2}{x} = \cos^{-1} a$

∴ $\cos\left(\frac{y-2}{x}\right) = a$

Question 4.

Reduce each of the following differential equations to the variable separable form and hence solve:

(i) $\frac{dy}{dx} = \cos(x + y)$

Solution:

$$\frac{dy}{dx} = \cos(x+y) \qquad \ldots (1)$$

Put $x + y = u$. Then $1 + \frac{dy}{dx} = \frac{du}{dx}$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} - 1$$

$\therefore$ (1) becomes, $\frac{du}{dx} - 1 = \cos u$

$\therefore \frac{du}{dx} = 1 + \cos u$

$\therefore \frac{1}{1+\cos u} du = dx$

Integrating both sides, we get

$\int \frac{1}{1+\cos u} du = \int dx$

$\therefore \int \frac{1}{2\cos^2\left(\frac{u}{2}\right)} du = \int dx$

$\therefore \frac{1}{2}\int \sec^2\left(\frac{u}{2}\right) du = \int dx$

$\therefore \frac{1}{2} \cdot \frac{\tan\left(\frac{u}{2}\right)}{\left(\frac{1}{2}\right)} = x + c$

$\therefore \tan\left(\frac{x+y}{2}\right) = x + c$

This is the general solution.

(ii) $(x - y)^2 \frac{dy}{dx} = a^2$

Solution:

$$(x-y)^2 \frac{dy}{dx} = a^2 \qquad \dots (1)$$

Put $x - y = u \quad \therefore x - u = y \quad \therefore 1 - \frac{du}{dx} = \frac{dy}{dx}$

$\therefore$ (1) becomes, $u^2\left(1 - \frac{du}{dx}\right) = a^2$

$\therefore u^2 - u^2 \frac{du}{dx} = a^2$

$\therefore u^2 - a^2 = u^2 \frac{du}{dx} \quad \therefore dx = \frac{u^2}{u^2 - a^2} du$

Integrating both sides, we get

$\int dx = \int \frac{(u^2 - a^2) + a^2}{u^2 - a^2} du$

$\therefore x = \int 1\, du + a^2 \int \frac{du}{u^2 - a^2} + c_1$

$= u + a^2 \cdot \frac{1}{2a} \log\left|\frac{u-a}{u+a}\right| + c_1$

$\therefore x = x - y + \frac{a}{2} \log\left|\frac{x-y-a}{x-y+a}\right| + c_1$

$\therefore -c_1 + y = \frac{a}{2} \log\left|\frac{x-y-a}{x-y+a}\right|$

$\therefore \quad -2c_1 + 2y = a \log\left|\frac{x-y-a}{x-y+a}\right|$

$\therefore \quad c + 2y = a \log\left|\frac{x-y-a}{x-y+a}\right|$, where $c = -2c_1$

This is the general solution.

(iii) $x + y \frac{dy}{dx} = \sec(x^2 + y^2)$

Solution:

$$x + y\frac{dy}{dx} = \sec(x^2 + y^2) \quad \ldots (1)$$

Put $x^2 + y^2 = u \qquad \therefore \quad 2x + 2y\frac{dy}{dx} = \frac{du}{dx}$

$\therefore \quad x + y\frac{dy}{dx} = \frac{1}{2}\cdot\frac{du}{dx}$

$\therefore$ (1) becomes, $\frac{1}{2}\cdot\frac{du}{dx} = \sec u \qquad \therefore \quad \frac{1}{\sec u} du = 2\cdot dx$

Integrating both sides, we get

$\int \cos u \, du = 2 \int dx$

$\therefore \sin u = 2x + c$

$\therefore \sin(x^2 + y^2) = 2x + c$

This is the general solution.

(iv) $\cos^2(x - 2y) = 1 - 2 \frac{dy}{dx}$

Solution:

$$\cos^2(x-2y) = 1 - 2\frac{dy}{dx} \quad \text{.....(1)}$$

Put $x - 2y = u$. Then $1 - 2\frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ (1) becomes, $\cos^2 u = \frac{du}{dx}$

$\therefore dx = \frac{1}{\cos^2 u} du$

Integrating both sides, we get

$\int dx = \int \sec^2 u \, du$

$\therefore x = \tan u + c$

$\therefore x = \tan(x - 2y) + c$

This is the general solution.

(v) (2x – 2y + 3) dx – (x – y + 1) dy = 0, when x = 0, y = 1

Solution:

(2x – 2y + 3) dx – (x – y + 1) dy = 0

$\therefore$ (x – y + 1) dy = (2x – 2y + 3) dx

$\therefore \frac{dy}{dx} = \frac{2(x-y)+3}{(x-y)+1}$ ....(1)

Put x – y = u, Then $1 - \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{dy}{dx} = 1 - \frac{du}{dx}$

$\therefore$ (1) becomes, $1 - \frac{du}{dx} = \frac{2u+3}{u+1}$

$$\therefore \frac{du}{dx} = 1 - \frac{2u+3}{u+1} = \frac{u+1-2u-3}{u+1}$$

$$\therefore \frac{du}{dx} = \frac{-u-2}{u+1} = -\left(\frac{u+2}{u+1}\right)$$

$$\therefore \frac{u+1}{u+2}\,du = -dx$$

Integrating both sides, we get

$$\int \frac{u+1}{u+2}\,du = -\int 1dx$$

$$\therefore \int \frac{(u+2)-1}{u+2}\,du = -\int 1dx$$

$$\therefore \int \left(1 - \frac{1}{u+2}\right) du = -\int 1dx$$

∴ u – log|u + 2| = -x + c

∴ x – y – log|x – y + 2| = -x + c

∴ (2x – y) – log|x – y + 2| = c

This is the general solution.

Now, y = 1, when x = 0.

∴ (0 – 1) – log|0 – 1 + 2| = c

∴ -1 – o = c

∴ c = -1

∴ the particular solution is

(2x – y) – log|x – y + 2| = -1

∴ (2x – y) – log|x – y + 2| + 1 = 0

Ex 6.4

I. Solve the following differential equations:

Question 1.
$x \sin\left(\frac{y}{x}\right) dy = \left[y \sin\left(\frac{y}{x}\right) - x\right] dx$
Solution:

$$x \sin\left(\frac{y}{x}\right) dy = \left[y \sin\left(\frac{y}{x}\right) - x\right] dx$$

$$\therefore \frac{dy}{dx} = \frac{y \sin\left(\frac{y}{x}\right) - x}{x \sin\left(\frac{y}{x}\right)} \quad \dots (1)$$

Put $y = vx$ $\quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{y}{x} = v$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{vx \sin v - x}{x \sin v}$

$$\therefore x\frac{dv}{dx} = \frac{v \sin v - 1}{\sin v} - v$$

$$\therefore x\frac{dv}{dx} = \frac{v \sin v - 1 - v \sin v}{\sin v} = \frac{-1}{\sin v}$$

$$\therefore \sin v \, dv = -\frac{1}{x} dx$$

Integrating both sides, we get

$$\int \sin v \, dv = -\int \frac{1}{x} dx$$

$$\therefore -\cos v = -\log x - c$$

$$\therefore \cos\left(\frac{y}{x}\right) = \log x + c$$

This is the general solution.

Question 2.

$(x^2 + y^2)\, dx - 2xy \,.\, dy = 0$

Solution:

$(x^2 + y^2)\, dx - 2xy\, dy = 0$

$\therefore 2xy\, dy = (x^2 + y^2)\, dx$

$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ ....(1)

Put $y = vx$ $\quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$

$$\therefore v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2}\, dv = \frac{1}{x}\, dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2}\, dv = \int \frac{1}{x}\, dx$$

$$-\int \frac{-2v}{1-v^2}\, dv = \int \frac{1}{x}\, dx$$

$$\therefore -\log|1-v^2| = \log x + \log c_1$$

$$\dots \left[\because \frac{d}{dv}(1-v^2) = -2v \text{ and } \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c\right]$$

$$\therefore \log \left|\frac{1}{1-v^2}\right| = \log c_1 x$$

$$\therefore \log \left|\frac{1}{1-\left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$$

$$\therefore \log \left|\frac{x^2}{x^2-y^2}\right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1} x$$

$$\therefore x^2 - y^2 = cx, \text{ where } c = \frac{1}{c_1}$$

This is the general solution.

Question 3.

$$\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 0$$

Solution:

$$(1 + 2e^{\frac{x}{y}}) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 0$$

$$\therefore\ (1+2e^{\frac{x}{y}})+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\cdot\frac{1}{\left(\frac{dx}{dy}\right)}=0$$

$$\therefore\ (1+2e^{\frac{x}{y}})\frac{dx}{dy}+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=0 \qquad \dots (1)$$

Put $\frac{x}{y}=u \qquad \therefore\ x=uy$

$$\therefore\ \frac{dx}{dy}=u+y\frac{du}{dy}$$

$\therefore$ (1) becomes, $(1+2e^u)\left(u+y\frac{du}{dy}\right)+2e^u(1-u)=0$

$$\therefore\ u+2ue^u+y\left(1+2e^u\right)\frac{du}{dy}+2e^u-2ue^u=0$$

$$\therefore\ (u+2e^u)+y(1+2e^u)\frac{du}{dy}=0$$

$$\therefore\ \frac{dy}{y}+\frac{1+2e^u}{u+2e^u}\,du=0$$

Integrating both sides, we get

$$\int\frac{1}{y}\,dy+\int\frac{1+2e^u}{u+2e^u}\,du=c_1$$

$\therefore\ \log|y|+\log|u+2e^u|=\log c$, where $c_1=\log c$

$$\dots\left[\because\ \frac{d}{du}(u+2e^u)=1+2e^u \text{ and } \int\frac{f'(u)}{f(u)}\,du=\log|f(u)|+c\right]$$

$$\therefore\ \log|y(u+2e^u)|=\log c$$

$\therefore \; y(u+2e^{u})=c$

$\therefore \; y\left(\frac{x}{y}+2e^{\frac{x}{y}}\right)=c$

$\therefore \; x+2ye^{\frac{x}{y}}=c$

This is the general solution.

Question 4.

$y^2\,dx + (xy + x^2)\,dy = 0$

Solution:

$y^2\,dx + (xy + x^2)\,dy = 0$

$\therefore (xy + x^2)\,dy = -y^2\,dx$

$\therefore \frac{dy}{dx} = \frac{-y^2}{xy+x^2}$ ....(1)

Put y = vx

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting these values in (1), we get

$$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x\cdot vx+x^2}=\frac{-v^2}{v+1}$$

$$\therefore \; x\frac{dv}{dx}=\frac{-v^2}{v+1}-v=\frac{-v^2-v^2-v}{v+1}$$

$$\therefore \; x\frac{dv}{dx}=\frac{-2v^2-v}{v+1}=-\left(\frac{2v^2+v}{v+1}\right)$$

$$\therefore \; \frac{v+1}{2v^2+v}dv=-\frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{v+1}{2v^2+v}dv = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{v+1}{v(2v+1)}dv = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{(2v+1)-v}{v(2v+1)}dv = -\int \frac{1}{x}dx$$

$$\therefore \int \left(\frac{1}{v} - \frac{1}{2v+1}\right)dv = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{1}{v}dv - \int \frac{1}{2v+1}dv = -\int \frac{1}{x}dx$$

$$\therefore \log|v| - \frac{1}{2}\log|2v+1| = -\log|x| + \log c$$

$$\therefore 2\log|v| - \log|2v+1| = -2\log|x| + 2\log c$$

$$\therefore \log|v^2| - \log|2v+1| = -\log|x^2| + \log c^2$$

$$\therefore \log\left|\frac{v^2}{2v+1}\right| = \log\left|\frac{c^2}{x^2}\right|$$

$$\therefore \frac{v^2}{2v+1} = \frac{c^2}{x^2}$$

$$\therefore \frac{\left(\frac{y^2}{x^2}\right)}{2\left(\frac{y}{x}\right)+1} = \frac{c^2}{x^2} \qquad \therefore \frac{y^2}{x(2y+x)} = \frac{c^2}{x^2}$$

$$\therefore xy^2 = c^2(x+2y)$$

This is the general solution.

Question 5.

$(x^2 - y^2)$ dx + 2xy dy = 0

Solution:

$$(x^2 - y^2)dx + 2xy\,dy = 0$$

$$\therefore -2xy\,dy = (x^2 - y^2)dx$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{-}2xy \quad ....(1)$$

put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{-2x(vx)}$$

$$\therefore v + x\frac{dv}{dx} = \frac{1 - v^2}{-2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 - v^2}{-2v} - v = \frac{1 - v^2 + 2v^2}{-2v}$$

$\therefore x \frac{dv}{dx} = \frac{1+v^2}{-2v}$

$\therefore \frac{-2v}{1+v^2} dv = \frac{1}{x} dx$

Integrating both sides, we get

$\therefore \int \frac{-2v}{1+v^2} dv = \int \frac{1}{x} dx$

$\therefore \log\left|1+v^2\right| = \log x + \log c_1$

$\ldots\left[\because \frac{d}{dx}\left(1+v^2\right) = 2v \text{ and } \int\left[\frac{f\prime(x)}{f(x)} dx = \log|f(x)| + c\right]\right.$

$\therefore \log\left|\frac{1}{1+v^2}\right| = \log c_1 x$

$\therefore \log\left|\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$

$\therefore \log\left|\frac{x^2}{x^2+y^2}\right| = \log c_1 x$

$\therefore \frac{x^2}{x^2+y^2} = c_1 x$

$\therefore x^2 + y^2 = \frac{1}{c_1}x$

$\therefore x^2 + y^2 = cx$ where $c = \frac{1}{c_1}$

This is the general solution.

Question 6.
$\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$
Solution:

$$\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x-2y}{2x-y}\right) \quad \text{.........(1)}$$

Put $y = vx$. $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = -\left(\frac{x-2vx}{2x-vx}\right)$

$$\therefore v + x\frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right)$$

$$\therefore x\frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right) - v$$

$$\therefore x\frac{dv}{dx} = \frac{-1+2v-2v+v^2}{2-v}$$

$\therefore\ x\frac{dv}{dx}=\frac{v^2-1}{2-v}$

$\therefore\ \frac{2-v}{v^2-1}dv=\frac{1}{x}dx$

Integrating both sides, we get

$\int\frac{2-v}{v^2-1}dv=\int\frac{1}{x}dx$

$\therefore\ 2\int\frac{1}{v^2-1}dv-\frac{1}{2}\int\frac{2v}{v^2-1}dv=\int\frac{1}{x}dx$

$\therefore\ 2\times\frac{1}{2}\log\left|\frac{v-1}{v+1}\right|-\frac{1}{2}\log|v^2-1|=\log|x|+\log c_1$

$\ldots\left[\because\ \frac{d}{dv}(v^2-1)=2v \text{ and } \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$

$\therefore\ \log\left|\frac{v-1}{v+1}\right|-\log|(v^2-1)^{\frac{1}{2}}|=\log|c_1x|$

$\therefore\ \log\left|\frac{v-1}{v+1}\cdot\frac{1}{\sqrt{v^2-1}}\right|=\log|c_1x|$

$\therefore\ \frac{v-1}{v+1}\cdot\frac{1}{\sqrt{v^2-1}}=c_1x$

$\therefore\ \frac{\frac{y}{x}-1}{\frac{y}{x}+1}\cdot\frac{1}{\sqrt{\frac{y^2}{x^2}-1}}=c_1x$

$$\therefore \frac{y-x}{y+x} \cdot \frac{x}{\sqrt{y^2-x^2}} = c_1 x$$

$$\therefore \frac{y-x}{y+x} = c_1 \sqrt{y^2-x^2}$$

$$\therefore \frac{y-x}{y+x} = c_1 \sqrt{y-x} \cdot \sqrt{y+x}$$

$$\therefore \sqrt{y-x} = c_1 (y+x)^{\frac{3}{2}}$$

$$\therefore y-x = c_1{}^2 (x+y)^3$$

$$\therefore y-x = c(x+y)^3, \text{ where } c = c_1{}^2$$

$$\therefore y = c(x+y)^3 + x$$

This is the general solution.

Question 7.

$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

Solution:

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0 \qquad (1)$$

Put $y = vx$ $\qquad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{y}{x} = v$

$$\therefore \text{(1) becomes, } x\left(v + x\frac{dv}{dx}\right) - vx + x\sin v = 0$$

$$\therefore vx + x^2\frac{dv}{dx} - vx + x\sin v = 0$$

$$\therefore\ vx + x^2\frac{dv}{dx} - vx + x\sin v = 0$$

$$\therefore\ x^2\frac{dv}{dx} + x\sin v = 0$$

$$\therefore\ \frac{1}{\sin v}dv + \frac{1}{x}dx = 0$$

Integrating, we get

$$\int \operatorname{cosec} v\, dv + \int \frac{1}{x}dx = c_1$$

$$\therefore\ \log|\operatorname{cosec} v - \cot v| + \log|x| = \log c, \text{ where } c_1 = \log c$$

$$\therefore\ \log|x(\operatorname{cosec} v - \cot v)| = \log c$$

$$\therefore\ x\left(\frac{1}{\sin v} - \frac{\cos v}{\sin v}\right) = c$$

$$\therefore\ x(1 - \cos v) = c\sin v$$

$$\therefore\ x\left[1 - \cos\left(\frac{y}{x}\right)\right] = c\sin\left(\frac{y}{x}\right)$$

This is the general solution.

Question 8.

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{X}{y}\right)dy = 0$$

Solution:

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\therefore\ (1 + e^{x/y})\frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0 \qquad \ldots (1)$$

Put $\frac{x}{y} = u$ $\therefore x = uy$ $\therefore \frac{dx}{dy} = u + y\frac{du}{dy}$

$\therefore$ (1) becomes, $(1+e^u)\left(u + y\frac{du}{dy}\right) + e^u(1-u) = 0$

$\therefore u + ue^u + y(1+e^u)\frac{du}{dy} + e^u - ue^u = 0$

$\therefore (u+e^u) + y(1+e^u)\frac{du}{dy} = 0$

$\therefore \frac{dy}{y} + \frac{1+e^u}{u+e^u}du = 0$

$\therefore \int\frac{dy}{y} + \int\frac{1+e^u}{u+e^u}du = c_1$ ... (2)

$\frac{d}{du}(u+e^u) = 1+e^u$ and $\int\frac{f'(u)}{f(u)}du = \log|f(u)| + c$

$\therefore$ from (2), the general solution is

$\log|y| + \log|u+e^u| = \log c$, where $c_1 = \log c$

$\therefore \log|y(u+e^u)| = \log c$ $\therefore y(u+e^u) = c$

$\therefore y\left(\frac{x}{y} + e^{x/y}\right) = c$ $\therefore x + ye^{x/y} = c$

This is the general solution.

Question 9.

$y^2 - x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Solution:

$$y^2 - x^2\frac{dy}{dx} = xy\frac{dy}{dx}$$

$$\therefore x^2\frac{dy}{dx} + xy\frac{dy}{dx} = y^2$$

$$\therefore (x^2 + xy)\frac{dy}{dx} = y^2$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x^2 + xy} \quad \ldots (1)$$

Put $y = vx$ $\quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2 + x \cdot vx} = \frac{v^2}{1+v}$$

$$\therefore x\frac{dv}{dx} = \frac{v^2}{1+v} - v = \frac{v^2 - v - v^2}{1+v}$$

$$\therefore x\frac{dv}{dx} = \frac{-v}{1+v} \quad \therefore \frac{1+v}{v}dv = -\frac{1}{x}dx$$

Integrating, we get

$$\int\left(\frac{1+v}{v}\right)dv = -\int\frac{1}{x}dx$$

$$\int\left(\frac{1}{v}+1\right)dv = -\int\frac{1}{x}dx$$

$$\therefore \int\frac{1}{v}dv + \int 1\,dv = -\int\frac{1}{x}dx$$

$\therefore \log|v| + v = -\log|x| + c$

$\therefore \log\left|\frac{y}{x}\right| + \frac{y}{x} = -\log|x| + c$

$\therefore \log|y| - \log|x| + \frac{y}{x} = -\log|x| + c$

$\therefore \frac{y}{x} + \log|y| = c$

This is the general solution.

Question 10.

xy $\frac{dy}{dx}$ = x^2 + $2y^2$, y(1) = 0

Solution:

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \quad \ldots (1)$$

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 2v^2x^2}{x \cdot vx} = \frac{1 + 2v^2}{v}$

$\therefore x\frac{dv}{dx} = \frac{1 + 2v^2}{v} - v = \frac{1 + 2v^2 - v^2}{v}$

$\therefore x\frac{dv}{dx}=\frac{1+v^2}{v}$

$\therefore \frac{v}{1+v^2}dv=\frac{1}{x}dx$

Integrating, we get

$\int\frac{v}{1+v^2}dv=\int\frac{1}{x}dx$

$\therefore \frac{1}{2}\int\frac{2v}{1+v^2}dv=\int\frac{1}{x}dx+\log c_1$

$\therefore \frac{1}{2}\log|1+v^2|=\log|x|+\log c_1$

$\therefore \log|1+v^2|=2\log|x|+2\log c_1$

$\therefore \log|1+v^2|=\log|x^2|+\log {c_1}^2$

$\therefore \log|1+v^2|=\log|cx^2|$, where $c={c_1}^2$

$\therefore 1+v^2=cx^2$

$\therefore 1+\frac{y^2}{x^2}=cx^2$

$\therefore \frac{x^2+y^2}{x^2}=cx^2$

$\therefore x^2+y^2=cx^4$

This is the general solution.

Now, $y(1)=0$, i.e. when $x=1$, $y=0$, we get

$1+0=c(1) \qquad \therefore c=1$

$\therefore$ the particular solution is $x^2+y^2=x^4$.

Question 11.

x dy + 2y · dx = 0, when x = 2, y = 1

Solution:

∴ x dy + 2y · dx = 0

∴ x dy = -2y dx

∴ $\frac{1}{y}dy = \frac{-2}{x}dx$

Integrating, we get

$$\int \frac{1}{y}dy = -2\int \frac{1}{x}dx$$

$$\therefore \log|y| = -2\log|x| + \log c$$

$$\therefore \log|y| = -\log|x^2| + \log c$$

$$\therefore \log|y| = \log\left|\frac{c}{x^2}\right|$$

$$\therefore y = \frac{c}{x^2} \qquad \therefore x^2y = c$$

This is the general solution.

When x = 2, y = 1, we get

4(1) = c

∴ c = 4

∴ the particular solution is $x^2y = 4$.

Question 12.

$x^2 \frac{dy}{dx} = x^2 + xy + y^2$

Solution:

$x^2 \frac{dy}{dx} = x^2 + xy + y^2$

du x^2+xu+u^2 . .

$\therefore \frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$ ....(1)

Put y = vx

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + x \cdot vx + v^2x^2}{x^2}$

$\therefore v + x\frac{dv}{dx} = 1 + v + v^2$

$\therefore x\frac{dv}{dx} = 1 + v^2$

$\therefore \frac{1}{1+v^2}dv = \frac{1}{x}dx$

Integrating, we get

$\int \frac{1}{1+v^2}dv = \int \frac{1}{x}dx + c$

$\therefore \tan^{-1} v = \log|x| + c$

$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$

This is the general solution.

Question 13.
(9x + 5y) dy + (15x + 11y) dx = 0
Solution:
(9x + 5y) dy + (15x + 11y) dx = 0

(9x + 5y) dy + (15x + 11y) dx = 0

∴ (9x + 5y) dy = -(15x + 11y) dx

∴ $\frac{dy}{dx} = \frac{-(15x+11y)}{9x+5y}$ ....(1)

Put y = vx

∴ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

∴ (1) becomes, $v + x\frac{dv}{dx} = \frac{-(15x+11vx)}{9x+5vx}$

$\therefore\ v + x\frac{dv}{dx} = \frac{-(15+11v)}{9+5v}$

$\therefore\ x\frac{dv}{dx} = \frac{-(15+11v)}{9+5v} - v = \frac{-15-11v-9v-5v^2}{9+5v}$

$\therefore\ x\frac{dv}{dx} = \frac{-5v^2-20v-15}{9+5v} = -\left(\frac{5v^2+20v+15}{5v+9}\right)$

$\therefore\ \frac{5v+9}{5v^2+20v+15}dv = -\frac{1}{x}dx$ ... (2)

Integrating, we get

$\frac{1}{5}\int\frac{5v+9}{v^2+4v+3}dv = -\int\frac{1}{x}dx$

Let $\frac{5v+9}{v^2+4v+3} = \frac{5v+9}{(v+3)(v+1)} = \frac{A}{v+3} + \frac{B}{v+1}$

$\therefore\ 5v+9 = A(v+1) + B(v+3)$

Put $v+3=0$, i.e. $v=-3$, we get

$-15+9 = A(-2) + B(0)$

$\therefore\ -6 = -2A \qquad \therefore\ A = 3$

Put $v + 1 = 0$, i.e. $v = -1$, we get

$-5 + 9 = A(0) + B(2)$

$\therefore\ 4 = 2B \qquad \therefore\ B = 2$

$$\therefore\ \frac{5v+9}{v^2+4v+3} = \frac{3}{v+3} + \frac{2}{v+1}$$

$\therefore$ (2) becomes,

$$\frac{1}{5}\int\left(\frac{3}{v+3} + \frac{2}{v+1}\right)dv = -\int\frac{1}{x}dx$$

$$\therefore\ \frac{3}{5}\int\frac{1}{v+3}dv + \frac{2}{5}\int\frac{1}{v+1}dv = -\int\frac{1}{x}dx$$

$$\therefore\ \frac{3}{5}\log|v+3| + \frac{2}{5}\log|v+1| = -\log|x| + c_1$$

$$\therefore\ 3\log|v+3| + 2\log|v+1| = -5\log x + 5c_1$$

$$\therefore\ \log|(v+3)^3| + \log|(v+1)^2| = -\log|x^5| + \log c,$$

where $5c_1 = \log c$

$$\therefore\ \log|(v+3)^3(v+1)^2| = \log\left|\frac{c}{x^5}\right|$$

$$\therefore\ (v+3)^3(v+1)^2 = \frac{c}{x^5}$$

$$\therefore\ \left(\frac{y}{x}+3\right)^3\left(\frac{y}{x}+1\right)^2 = \frac{c}{x^5}$$

$$\therefore \frac{(y+3x)^3}{x^3} \times \frac{(y+x)^2}{x^2} = \frac{c}{x^5}$$

$$\therefore (x+y)^2(3x+y)^3 = c$$

This is the general solution.

Question 14.
$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$
Solution:
$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$
$\therefore x^2\,dy = (x^2 + 3xy + y^2)\,dx$
$\therefore \frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$ ....(1)

Put $y = vx \qquad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 3x \cdot vx + v^2x^2}{x^2}$

$$\therefore v + x\frac{dv}{dx} = 1 + 3v + v^2$$

$$x\frac{dv}{dx} = v^2 + 2v + 1 = (v+1)^2$$

$$\therefore \frac{1}{(v+1)^2}dv = \frac{1}{x}dx$$

Integrating, we get

$$\int (v+1)^{-2}\,dv = \int \frac{1}{x}dx$$

$$\therefore \frac{(v+1)^{-1}}{-1} = \log|x| + c_1$$

$$\therefore -\frac{1}{v+1} = \log|x| + c_1$$

$$\therefore -\frac{1}{\frac{y}{x}+1} = \log|x| + c_1$$

$$\therefore -\frac{x}{y+x} = \log|x| + c_1$$

$$\therefore \log|x| + \frac{x}{x+y} = -c_1$$

$$\therefore \log|x| + \frac{x}{x+y} = c, \text{ where } c = -c_1$$

This is the general solution.

Question 15.
$(x^2 + y^2)\, dx - 2xy\, dy = 0.$
Solution:

$(x^2 + y^2)dx - 2xy\, dy = 0$

$\therefore 2xy\, dy = (x^2 + y^2)dx$

$$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{....(1)}$$

Put y = vx

$\therefore \frac{dy}{dx} = v + x du/dx$

$\therefore$ (1) becomes, $v + x\frac{du}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$

$\therefore v + x\frac{du}{dx} = \frac{1+v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$

$\therefore \frac{2v}{1-v^2}dv = \frac{1}{x}dx$

Integrating both sides, we get

$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$

$-\int \frac{-2v}{1-v^2}dv = \int \frac{1}{x}dx$

$\therefore - \log|1 - v^2| = \log x + \log c_1$ .... $\left[\because \frac{d}{dv}\left(1-v^2\right) = -2v \text{ and } \int \frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right]$

$$\therefore \log\left|\frac{1}{1-v^2}\right| = \log c_1x$$

$$\therefore \log\left|\frac{1}{1-\left(\frac{y^2}{x^2}\right)}\right| = \log c_1x$$

$$\therefore \log\left|\frac{x^2}{x^2-y^2}\right| = \log c_1x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1x$$

$$\therefore x^2-y^2 = \frac{1}{c_1}x$$

$\therefore x^2-y^2 = cx$, where $c = \dfrac{1}{c_1}$

This is the general solution.

Ex 6.5

Question 1.
Solve the following differential equations:
(i) $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$
Solution:
$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$ ....(1)
This is the linear differential equation of the form
$\frac{dy}{dx}$ + P . y = Q, where P = $\frac{1}{x}$ and Q = $x^3 - 3$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx}$

$= e^{\log x} = x$

$\therefore$ the solution of (1) is given by

$y(\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c_1$

$\therefore y \cdot x = \int (x^3 - 3) x \, dx + c_1$

$\therefore xy = \int (x^4 - 3x) dx + c_1$

$\therefore xy = \frac{x^5}{5} - 3 \cdot \frac{x^2}{2} + c_1$

$\therefore \frac{x^2}{5} - \frac{3x^2}{2} - xy = c$, where $c = -c_1$

This is the general solution.

(ii) $\cos^2 x \cdot \frac{dy}{dx} + y = \tan x$

Solution:

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x$$

$$\therefore \frac{dy}{dx} + \frac{1}{\cos^2 x} \cdot y = \frac{\tan x}{\cos^2 x}$$

$$\therefore \frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x \quad \ldots (1)$$

This is the linear differential equation of the form $\frac{dy}{dx} + P \cdot y = Q$, where $P = \sec^2 x$ and $Q = \tan x \cdot \sec^2 x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \sec^2 x \, dx} = e^{\tan x}$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \, (\text{I.F.}) \, dx + c$

$\therefore y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + c$

Put $\tan x = t \quad \therefore \sec^2 x \, dx = dt$

$\therefore y \cdot e^{\tan x} = \int t \cdot e^t \, dt + c$

$$\therefore y \cdot e^{\tan x} = t \int e^t \, dt - \int \left[\frac{d}{dt}(t) \int e^t \, dt \right] dt + c$$

$$= t \cdot e^t - \int 1 \cdot e^t \, dt + c$$

$$= t \cdot e^t - e^t + c$$

$$= e^t (t - 1) + c$$

$\therefore \; y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + c$

This is the general solution.

(iii) $(x + 2y^3)\frac{dy}{dx} = y$

Solution:

$$(x + 2y^3)\frac{dy}{dx} = y$$

$$\therefore \; \frac{x + 2y^3}{y} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\therefore \; \frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\therefore \; \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2 \qquad \ldots (1)$$

This is the linear differential equation of the form $\frac{dx}{dy} + P \cdot x = Q$, where $P = -\frac{1}{y}$ and $Q = 2y^2$

$$\therefore \; \text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy}$$

$$= e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c$$

$\therefore\ x\left(\frac{1}{y}\right)=\int 2y^2\times\frac{1}{y}\,dy+c$

$\therefore\ \frac{x}{y}=2\int y\ dx+c$

$\therefore\ \frac{x}{y}=2\cdot\frac{y^2}{2}+c \qquad \therefore\ x=y(c+y^2)$

This is the general solution.

(iv) $\frac{dy}{dx}$ + y . sec x = tan x

Solution:

$\frac{dy}{dx}$ + y sec x = tan x

∴ $\frac{dy}{dx}$ + (sec x) . y = tan x ……..(1)

This is the linear differential equation of the form

$\frac{dy}{dx}$ + P . y = Q, where P = sec x and Q = tan x

∴ I.F. = $e^{\int P dx}$

= $e^{\int \sec x dx}$

= $e^{\log(\sec x+\tan x)}$

= sec x + tan x

∴ the solution of (1) is given by

y (I.F.) = ∫Q . (I.F.) dx + c

∴ y(sec x + tan x) = ∫tan x (sec x + tan x) dx + c

∴ (sec x + tan x) . y = ∫(sec x tan x + $\tan^2$x) dx + c

∴ (sec x + tan x) . y = ∫(sec x tan x + $\sec^2$x – 1) dx + c

∴ (sec x + tan x) . y = sec x + tan x – x + c

∴ (sec x + tan x) . y = sec x + tan x − x + c
∴ y(sec x + tan x) = sec x + tan x − x + c
This is the general solution.

(v) $x \frac{dy}{dx} + 2y = x^2 . \log x$
Solution:

$$x\frac{dy}{dx} + 2y = x^2 \cdot \log x$$

$$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x \cdot \log x \quad \text{.........(1)}$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = \frac{2}{x}$ and $Q = x \cdot \log x$

$$\therefore \text{ I.F.} = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx}$$

$$= e^{2\log x} = e^{\log x^2} = x^2$$

∴ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore y \cdot x^2 = \int (x \log x) \cdot x^2\,dx + c$$

$$\therefore x^2 \cdot y = \int x^3 \cdot \log x\,dx + c$$

$\therefore\ x^2 \cdot y = \int x^3 \cdot \log x\, dx + c$

$$= (\log x) \int x^3\, dx - \int \left[\frac{d}{dx}(\log x) \int x^3\, dx\right] dx + c$$

$$= (\log x) \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx + c$$

$$= \frac{1}{4} x^4 \log x - \frac{1}{4} \int x^3 dx + c$$

$\therefore\ x^2 \cdot y = \frac{1}{4} x^4 \log x - \frac{1}{4} \cdot \frac{x^4}{4} + c$

$\therefore\ x^2 y = \frac{x^4 \log x}{4} - \frac{x^2}{16} + c$

This is the general solution.

(vi) (x + y) $\frac{dy}{dx}$ = 1

Solution:

(x + y) $\frac{dy}{dx}$ = 1

$\therefore \frac{dx}{dy}$ = x + y

$\therefore \frac{dx}{dy}$ − x = y

$\therefore \frac{dx}{dy}$ + (-1) x = y ……….(1)

This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where $P = -1$ and $Q = y$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int -1\,dy} = e^{-y}$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c$

$\therefore\ x \cdot e^{-y} = \int y \cdot e^{-y}\,dy + c$

$$\therefore\ e^{-y} \cdot x = y \int e^{-y}\,dy - \int \left[\frac{d}{dy}(y) \int e^{-y}\,dy \right] dy + c$$

$$= y \cdot \frac{e^{-y}}{-1} - \int 1 \cdot \frac{e^{-y}}{-1}\,dy + c$$

$$= -ye^{-y} + \int e^{-y}\,dy + c$$

$$\therefore\ e^{-y} \cdot x = -ye^{-y} + \frac{e^{-y}}{-1} + c$$

$\therefore\ e^{-y} \cdot x + ye^{-y} + e^{-y} = c$

$\therefore\ e^{-y}(x + y + 1) = c$

$\therefore\ x + y + 1 = ce^{y}$

This is the general solution.

(vii) $(x + a)\frac{dy}{dx} - 3y = (x + a)^5$

Solution:

$$(x + a)\frac{dy}{dx} - 3y = (x + a)^5$$

$$\therefore\ \frac{dy}{dx} - \frac{3y}{x + a} = (x + a)^4$$

$$\therefore \frac{dy}{dx} + \left(\frac{-3}{x+a}\right)y = (x+a)^4 \quad \text{.........(1)}$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = \frac{-3}{x+a}$ and $Q = (x+a)^4$

$$\therefore \text{ I.F.} = e^{\int P\,dx} = e^{\int \frac{-3}{x+a}dx} = e^{-3\int \frac{1}{x+a}dx}$$

$$= e^{-3\log|x+a|} = e^{\log(x+a)^{-3}}$$

$$= (x+a)^{-3} = \frac{1}{(x+a)^3}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore y \cdot \frac{1}{(x+a)^3} = \int (x+a)^4 \cdot \frac{1}{(x+a)^3}dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \int (x+a)\,dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \frac{(x+a)^2}{2} + c$$

$$\therefore 2y = (x+a)^5 + 2c(x+a)^3$$

This is the general solution.

(viii) dr + (2r cot θ + sin 2θ) dθ = 0

Solution:

dr + (2r cot θ + sin 2θ) dθ = 0

$\therefore \frac{dr}{d\theta}$ + (2r cot θ + sin 2θ) = 0

$\therefore \frac{dr}{d\theta}$ + (2 cot θ)r = -sin 2θ ....(1)

This is the linear differential equation of the form dr

$\frac{dr}{d\theta}$ + P . r = Q, where P = 2 cot θ and Q = -sin 2θ

$$\therefore \text{ I.F.} = e^{\int P\,d\theta} = e^{\int 2\cot\theta}\,d\theta$$

$$= e^{2\int \cot\theta\,d\theta} = e^{2\log\sin\theta}$$

$$= e^{\log(\sin^2\theta)} = \sin^2\theta$$

$\therefore$ the solution of (1) is given by

$$r\cdot(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,d\theta + c$$

$$\therefore r\cdot\sin^2\theta = \int -\sin 2\theta\cdot\sin^2\theta\,d\theta + c$$

$$\therefore r\sin^2\theta = \int -2\sin\theta\cos\theta\cdot\sin^2\theta\,d\theta + c$$

$$\therefore r\sin^2\theta = -2\int \sin^3\theta\cos\theta\,d\theta + c$$

Put $\sin\theta = t$ $\quad\therefore \cos\theta\,d\theta = dt$

$$\therefore r\sin^2\theta = -2\int t^3\,dt + c$$

$$\therefore r\sin^2\theta = -2\cdot\frac{t^4}{4} + c$$

$$\therefore r\sin^2\theta = -\frac{1}{2}\sin^4\theta + c$$

$$\therefore r\sin^2\theta + \frac{\sin^4\theta}{2} = c$$

This is the general solution.

(ix) y dx + (x – y^2) dy = 0

Solution:

y dx + (x – y^2) dy = 0

∴ y dx = -(x – y^2) dy

$\therefore \frac{dx}{dy} = -\frac{(x-y^2)}{y} = -\frac{x}{y} + y$

$\therefore \frac{dx}{dy} + \left(\frac{1}{y}\right) \cdot x = y$ ………(1)

This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where $P = \frac{1}{y}$ and $Q = y$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dy + c_1$

$\therefore xy = \int y \cdot y\, dy + c_1$

$\therefore xy = \int y^2\, dy + c_1$

$\therefore xy = \frac{y^3}{3} + c_1$

$\therefore \frac{y^3}{3} = xy + c$, where $c = -c_1$

This is the general solution.

(x) $\left(1-x^{2}\right) \frac{dy}{dx}+2xy=x\left(1-x^{2}\right)^{\frac{1}{2}}$

Solution:

$$(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx}+\left(\frac{2x}{1-x^2}\right)y=\frac{x}{(1-x^2)^{\frac{1}{2}}}$$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{2x}{1-x^2}$ and $Q=\frac{x}{(1-x^2)^{\frac{1}{2}}}$

$$\therefore \text{ I.F.}=e^{\int P\,dx}=e^{\int \frac{2x}{1-x^2}dx}$$

$$=e^{-\int \frac{-2x}{1-x^2}dx}=e^{-\log|1-x^2|}$$

$$=e^{\log\left|\frac{1}{1-x^2}\right|}=\frac{1}{1-x^2}$$

$\therefore$ the solution of (1) is given by

$$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$$

$$\therefore y\cdot\frac{1}{(1-x^2)}=\int\frac{x}{(1-x^2)^{\frac{1}{2}}}\cdot\frac{1}{1-x^2}\,dx+c$$

$$\therefore \frac{y}{(1-x^2)}=\int\frac{x}{(1-x^2)^{\frac{3}{2}}}\,dx+c$$

Put $1-x^2=t$ $\therefore$ $-2x\,dx=dt$

$$\therefore\ x\,dx=-\frac{dt}{2}$$

$$\therefore\ \frac{y}{1-x^2}=\int\frac{1}{t^{\frac{3}{2}}}\cdot\frac{(-dt)}{2}+c$$

$$\therefore\ \frac{y}{1-x^2}=-\frac{1}{2}\int t^{-\frac{3}{2}}dt+c$$

$$\therefore\ \frac{y}{1-x^2}=-\frac{1}{2}\cdot\frac{t^{-\frac{1}{2}}}{(-1/2)}+c$$

$$\therefore\ \frac{y}{1-x^2}=\frac{1}{(1-x^2)^{\frac{1}{2}}}+c$$

$$\therefore\ y=\sqrt{1-x^2}+c(1-x^2)$$

This is the general solution.

(xi) $\left(1+x^2\right)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

Solution:

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$

$$\therefore\ \frac{dy}{dx}+\frac{1}{1+x^2}\cdot y=\frac{e^{\tan^{-1}x}}{1+x^2}\quad\text{........(1)}$$

This is the linear differential equation of the form

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = \frac{1}{1+x^2}$ and $Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx}$

$= e^{\tan^{-1}x}$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \, dx + c$

$\therefore \; y \cdot e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot e^{\tan^{-1}x} dx + c$

$\therefore \; y \cdot e^{\tan^{-1}x} = \int (e^{\tan^{-1}x}) \cdot \frac{e^{\tan^{-1}x}}{1+x^2} \, dx + c$

Put $e^{\tan^{-1}x} = t \quad \therefore \; \frac{e^{\tan^{-1}x}}{1+x^2} dx = dt$

$\therefore \; y \cdot e^{\tan^{-1}x} = \int t \, dt + c$

$\therefore \; y \cdot e^{\tan^{-1}x} = \frac{t^2}{2} + c$

$\therefore \; y \cdot e^{\tan^{-1}x} = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$

$\therefore \; y = \frac{1}{2} e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$

This is the general solution.

Question 2.
Find the equation of the curve which passes through the origin and has the slope x + 3y – 1 at

any point (x, y) on it.
Solution:
Let A(x, y) be the point on the curve y = f(x).
Then slope of the tangent to the curve at the point A is $\frac{dy}{dx}$.
According to the given condition,
$\frac{dy}{dx}$ = x + 3y – 1
$\therefore \frac{dy}{dx}$ – 3y = x – 1 ………(1)
This is the linear differential equation of the form

$$\frac{dy}{dx}+Py=Q, \text{ where } P=-3 \text{ and } Q=x-1$$

$$\therefore \text{ I.F.}=e^{\int P\,dx}=e^{\int -3\,dx}=e^{-3x}$$

$\therefore$ the solution of (1) is given by

$$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$$

$$\therefore y\cdot e^{-3x}=\int (x-1)\cdot e^{-3x}\,dx+c$$

$$\therefore e^{-3x}\cdot y=(x-1)\int e^{-3x}\,dx-\int\left[\frac{d}{dx}(x-1)\cdot\int e^{-3x}\,dx\right]dx+c_1$$

$$\therefore e^{-3x}\cdot y=(x-1)\cdot\frac{e^{-3x}}{-3}-\int 1\cdot\frac{e^{-3x}}{-3}\,dx+c_1$$

$$\therefore e^{-3x}\cdot y=-\frac{1}{3}(x-1)\cdot e^{-3x}+\frac{1}{3}\int e^{-3x}\,dx+c_1$$

$$\therefore e^{-3x}\cdot y=-\frac{1}{3}(x-1)e^{-3x}+\frac{1}{3}\cdot\frac{e^{-3x}}{-3}+c_1$$

$\therefore\ e^{-3x}\cdot y = -\frac{1}{3}(x-1)e^{-3x} - \frac{1}{9}e^{-3x} + c_1$

$\therefore\ 9y = -3(x-1) - 1 + 9c_1 \cdot e^{3x}$

$\therefore\ 9y + 3(x-1) + 1 = 9c_1 \cdot e^{3x}$

$\therefore\ 9y + 3x - 3 + 1 = 9c_1 \cdot e^{3x}$

$\therefore\ 3(x+3y) = 2 + 9c_1 \cdot e^{3x}$

$\therefore\ 3(x+3y) = 2 + c \cdot e^{3x}$, where $c = 9c_1$ ... (2)

This is the general equation of the curve.
But the required curve is passing through the origin (0, 0).
∴ by putting x = 0 and y = 0 in (2), we get
0 = 2 + c
∴ c = -2
∴ from (2), the equation of the required curve is $3(x + 3y) = 2 - 2e^{3x}$ i.e. $3(x + 3y) = 2(1 - e^{3x})$.

Question 3.

Find the equation of the curve passing through the point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ having slope of the tangent to the curve at any point (x, y) is $-\frac{4x}{9y}$.

Solution:
Let A(x, y) be the point on the curve y = f(x).
Then the slope of the tangent to the curve at point A is $\frac{dy}{dx}$.
According to the given condition

$\frac{dy}{dx} = -\frac{4x}{9y}$

∴ y dy = $-\frac{4}{9}$ x dx

Integrating both sides, we get

∫y dy= $-\frac{4}{9}$ ∫x dx

∴ $\frac{y^2}{2} = -\frac{4}{9} \cdot \frac{x^2}{2} + c_1$

∴ $9y^2 = -4x^2 + 18c_1$

∴ $4x^2 + 9y^2 = c$ where $c = 18c_1$

This is the general equation of the curve.

But the required curve is passing through the point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$.

∴ by putting x = $\frac{3}{\sqrt{2}}$ and y = √2 in (1), we get

$4\left(\frac{3}{\sqrt{2}}\right)^2 + 9(\sqrt{2})^2 = c$

∴ 18 + 18 = c

∴ c = 36

∴ from (1), the equation of the required curve is $4x^2 + 9y^2 = 36$.

Question 4.

The curve passes through the point (0, 2). The sum of the coordinates of any point on the curve exceeds the slope of the tangent to the curve at any point by 5. Find the equation of the curve.

Solution:

Let A(x, y) be any point on the curve.

Then slope of the tangent to the curve at the point A is $\frac{dy}{dx}$.

According to the given condition

x + y = $\frac{dy}{dx}$ + 5

∴ $\frac{dy}{dx}$ − y = x − 5 ....(1)

This is the linear differential equation of the form

$\frac{dy}{dx}$ + P . y = Q, where P = -1 and Q = x − 5

∴ I.F. = $e^{\int Pdx} = e^{\int -1dx} = e^{-x}$

∴ the solution of (1) is given by

$$y\cdot(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c$$

$$\therefore\ y\cdot e^{-x} = \int (x-5)e^{-x}\,dx + c$$

$$\therefore\ e^{-x}\cdot y = (x-5)\int e^{-x}\,dx - \int\left[\frac{d}{dx}(x-5)\int e^{-x}\,dx\right]dx + c$$

$$\therefore\ e^{-x}\cdot y = (x-5)\cdot\frac{e^{-x}}{-1} - \int 1\cdot\frac{e^{-x}}{-1}\,dx + c$$

$$\therefore\ e^{-x}\cdot y = -(x-5)\cdot e^{-x} + \int e^{-x}\,dx + c$$

$$\therefore\ e^{-x}\cdot y = -(x-5)e^{-x} + \frac{e^{-x}}{-1} + c$$

$$\therefore\ y = -(x-5) - 1 + ce^{x}$$

$$\therefore\ y = -x + 5 - 1 + ce^{x}$$

$$\therefore\ y = 4 - x + ce^{x} \qquad \ldots (2)$$

This is the general equation of the curve.

But the required curve is passing through the point (0, 2).

∴ by putting x = 0, y = 2 in (2), we get

2 = 4 − 0 + c

∴ c = -2

∴ from (2), the equation of the required curve is $y = 4 - x - 2e^x$.

Question 5.
If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also, the curve passes through the point (0, 1). Find the equation of the curve.
Solution:
Let A(x, y) be the point on the curve y = f(x).
Then slope of the tangent to the curve at the point A is $\frac{dy}{dx}$.
According to the given condition
$\frac{dy}{dx} = x + xy$
∴ $\frac{dy}{dx} - xy = x$ ……….. (1)
This is the linear differential equation of the form
$\frac{dy}{dx} + Py = Q$, where P = -x and Q = x
∴ $\text{I.F.} = e^{\int P dx} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$
∴ the solution of (1) is given by
y . (I.F.) = ∫Q . (I.F.) dx + c

$$\therefore\ y \cdot e^{-\frac{x^2}{2}} = \int x \cdot e^{-\frac{x^2}{2}} dx + c$$

$$\therefore\ e^{-\frac{x^2}{2}} \cdot y = \int x \cdot e^{-\frac{x^2}{2}} dx + c$$

$$\text{Put } -\frac{x^2}{2} = t \qquad \therefore\ -x\,dx = dt$$

$\therefore\ x\,dx = -dt$

$\therefore\ e^{-\frac{x^2}{2}} \cdot y = \int e^t \cdot (-dt) + c$

$\therefore\ e^{-\frac{x^2}{2}} \cdot y = -\int e^t dt + c$

$\therefore\ e^{-\frac{x^2}{2}} \cdot y = -e^t + c$

$\therefore\ e^{-\frac{x^2}{2}} \cdot y = -e^{-\frac{x^2}{2}} + c$

$\therefore\ y = -1 + ce^{\frac{x^2}{2}}$

$\therefore\ 1 + y = ce^{\frac{x^2}{2}}$ ....(2)

This is the general equation of the curve.
But the required curve is passing through the point (0, 1).
∴ by putting x = 0 and y = 1 in (2), we get
1 + 1 = c
∴ c = 2
∴ from (2), the equation of the required curve is 1 + y = $2e^{\frac{x^2}{2}}$.

Ex 6.6

Question 1.
In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

Solution:
Let x be the number of bacteria in the culture at time t.
Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = kx$, where k is a constant

$\therefore \frac{dx}{x} = k\,dt$

On integrating, we get

$\int \frac{dx}{x} = k\int dt + c$

$\therefore \log x = kt + c$

Initially, i.e. when $t = 0$, let $x = x_0$

$\log x_0 = k \times 0 + c$

$\therefore c = \log x_0$

$\therefore \log x = kt + \log x_0$

$\therefore \log x - \log x_0 = kt$

$\therefore \log\left(\frac{x}{x_0}\right) = kt$ ....(1)

Since the number doubles in 4 hours, i.e. when $t = 4$, $x = 2x_0$

$$\therefore \log\left(\frac{2x_0}{x_0}\right) = 4k \qquad \therefore k = \frac{1}{4}\log 2$$

$$\therefore \text{(1) becomes, } \log\left(\frac{x}{x_0}\right) = \frac{t}{4}\log 2$$

When $t = 12$, we get

When $t = 12$, we get

$$\log\left(\frac{x}{x_0}\right) = \frac{12}{4}\log 2 = 3\log 2$$

$$\therefore \log\left(\frac{x}{x_0}\right) = \log 8$$

$$\therefore \frac{x}{x_0} = 8 \qquad \therefore x = 8x_0$$

$\therefore$ the number of bacteria will be 8 times the original number in 12 hours.

Question 2.
If the population of a country doubles in 60 years; in how many years will it be triple (treble) under the assumption that the rate of increase is proportional to the number of inhabitants?
[Given log 2 = 0.6912, log 3 = 1.0986]
Solution:
Let P be the population at time t years.
Then $\frac{dP}{dt}$, the rate of increase of population is proportional to P.
$\therefore \frac{dP}{dt} \propto P$
$\therefore \frac{dP}{dt} = kP$, where k is a constant
$\therefore \frac{dP}{P} = k\,dt$
On integrating, we get
$\int \frac{dP}{P} = k\int dt + c$
$\therefore \log P = kt + c$
Initially i.e. when t = 0, let $P = P_0$
$\therefore \log P_0 = k \times 0 + c$
$\therefore c = \log P_0$
$\therefore \log P = kt + \log P_0$
$\therefore \log P - \log P_0 = kt$
$\therefore \log\left(\frac{P}{P_0}\right) = kt$ ......(1)
Since, the population doubles in 60 years, i.e. when t = 60, $P = 2P_0$

$$\therefore \log\left(\frac{2P_0}{P_0}\right) = 60k \qquad \therefore k = \frac{1}{60}\log 2$$

$$\therefore \text{(1) becomes, } \log\left(\frac{P}{P_0}\right) = \frac{t}{60}\log 2$$

When population becomes triple, i.e. when $P = 3P_0$, we get

$$\log\left(\frac{3P_0}{P_0}\right)=\frac{t}{60}\log 2$$

$$\therefore\ \log 3=\frac{t}{60}\log 2$$

$$\therefore\ t=60\left(\frac{\log 3}{\log 2}\right)=60\left(\frac{1.0986}{0.6912}\right)$$

$$=60\times 1.5894=95.364\approx 95.4 \text{ years}$$

∴ the population becomes triple in 95.4 years (approximately).

Question 3.
If a body cools from 80°C to 50°C at room temperature of 25°C in 30 minutes, find the temperature of the body after 1 hour.
Solution:
Let θ°C be the temperature of the body at time t minutes.
The room temperature is given to be 25°C.
Then by Newton's law of cooling, $\frac{d\theta}{dt}$, the rate of change of temperature, is proportional to (θ – 25).

i.e. $\frac{d\theta}{dt}\propto(\theta-25)$

$$\therefore\ \frac{d\theta}{dt}=-k(\theta-25),\text{ where } k>0$$

$$\therefore\ \frac{d\theta}{\theta-25}=-k\,dt$$

On integrating, we get

$$\int\frac{1}{\theta-25}d\theta=-k\int dt+c$$

$\therefore \log(\theta - 25) = -kt + c$

Initially, i.e. when $t = 0$, $\theta = 80$

$\therefore \log(80 - 25) = -k \times 0 + c \qquad \therefore c = \log 55$

$\therefore \log(\theta - 25) = -kt + \log 55$

$\therefore \log(\theta - 25) - \log 55 = -kt$

$$\log\left(\frac{\theta - 25}{55}\right) = -kt \qquad \text{.........(1)}$$

Now, when $t = 30$, $\theta = 50$

$$\therefore \log\left(\frac{50 - 25}{55}\right) = -30k$$

$$\therefore k = -\frac{1}{30}\log\left(\frac{5}{11}\right)$$

$$\therefore \text{(1) becomes, } \log\left(\frac{\theta - 25}{55}\right) = \frac{t}{30}\log\left(\frac{5}{11}\right)$$

When $t = 1$ hour $= 60$ minutes, then

$$\log\left(\frac{\theta - 25}{55}\right) = \frac{60}{30}\log\left(\frac{5}{11}\right) = 2\log\left(\frac{5}{11}\right)$$

$$\therefore \log\left(\frac{\theta - 25}{55}\right) = \log\left(\frac{5}{11}\right)^2$$

$$\therefore \frac{\theta - 25}{55} = \left(\frac{5}{11}\right)^2 = \frac{25}{121}$$

$$\therefore \theta - 25 = 55 \times \frac{25}{121} = \frac{125}{11}$$

$\therefore\ \theta = 25 + \frac{125}{11} = \frac{400}{11} = 36.36$

∴ the temperature of the body will be 36.36°C after 1 hour.

Question 4.
The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after 2½ hours. [Take √2 = 1.414]
Solution:
Let x be the number of bacteria at time t.
Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.
$\therefore \frac{dx}{dt} \propto x$
$\therefore \frac{dx}{dt} = kx$, where k is a constant
$\therefore \frac{dx}{x} = k\,dt$
On integrating, we get
$\int \frac{dx}{x} = k\int dt + c$
∴ log x = kt + c
Initially, i.e. when t = 0, x = 1000
∴ log 1000 = k × 0 + c
∴ c = log 1000
∴ log x = kt + log 1000
∴ log x – log 1000 = kt
$\therefore \log\left(\frac{x}{1000}\right) = kt$ ……(1)
Now, when t = 1, x = 2 × 1000 = 2000

$\therefore \log(\frac{2000}{1000}) = k$

$\therefore k = \log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{1000}\right) = t \log 2$

If $t = 2\frac{1}{2} = \frac{5}{2}$, then

$$\log\left(\frac{x}{1000}\right) = \frac{5}{2}\log 2 = \log(2)^{\frac{5}{2}}$$

$$\therefore \left(\frac{x}{1000}\right) = (2)^{\frac{5}{2}} = 4\sqrt{2} = 4 \times 1.414 = 5.656$$

$$\therefore x = 5.656 \times 1000 = 5656$$

∴ the number of bacteria after 2½ hours = 5656.

Question 5.

The rate of disintegration of a radioactive element at any time t is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.

Solution:

Let m be the mass of the radioactive element at time t.

Then the rate of disintegration is $\frac{dm}{dt}$ which is proportional to m.

$$\therefore \frac{dm}{dt} \propto m$$

$$\therefore \frac{dm}{dt} = -km, \text{ where } k > 0$$

$\therefore \frac{dm}{m} = -k\,dt$

On integrating, we get

$$\int \frac{1}{m} dm = -k \int dt + c$$

$\therefore \log m = -kt + c$

Initially, i.e. when $t = 0$, $m = 1.5$

$\therefore \log(1.5) = -k \times 0 + c \qquad \therefore c = \log\left(\frac{3}{2}\right)$

$\therefore \log m = -kt + \log\left(\frac{3}{2}\right)$

$\therefore \log m - \log\frac{3}{2} = -kt$

$\therefore \log\left(\frac{2m}{3}\right) = -kt$

When $m = 0.5 = \frac{1}{2}$, then

$$\log\left(\frac{2 \times \frac{1}{2}}{3}\right) = -kt$$

$\therefore \log\left(\frac{1}{3}\right) = -kt$

$\therefore \log(3)^{-1} = -kt$

$\therefore -\log 3 = -kt$

$\therefore t = \frac{1}{k} \log 3$

$\therefore$ t = $\frac{1}{k}$ log 3

$\therefore$ the original mass will disintegrate to 0.5 gm when t = $\frac{1}{k}$ log 3

Question 6.

The rate of decay of certain substances is directly proportional to the amount present at that instant. Initially, there is 25 gm of certain substance and two hours later it is found that 9 gm are left. Find the amount left after one more hour.

Solution:

Let x gm be the amount of the substance left at time t.

Then the rate of decay is $\frac{dx}{dt}$, which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = -kx$, where $k > 0$

$\therefore \frac{1}{x} dx = -k\,dt$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt + c$$

$\therefore \log x = -kt + c$

Initially i.e. when $t = 0$, $x = 25$

$\therefore \log 25 = -k \times 0 + c \qquad \therefore c = \log 25$

$\therefore \log x = -kt + \log 25$

$\therefore \log x - \log 25 = -kt$

$\therefore \log\left(\frac{x}{25}\right) = -kt$ ....(1)

Now, when $t = 2$, $x = 9$

$\therefore \log\left(\frac{9}{25}\right) = -2k$

$\therefore -2k = \log\left(\frac{3}{5}\right)^2 = 2\log\left(\frac{3}{5}\right)$

$\therefore k = -\log\left(\frac{3}{5}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{x}{25}\right) = t\log\left(\frac{3}{5}\right)$

When $t = 3$, then

$\log\left(\frac{x}{25}\right) = 3\log\left(\frac{3}{5}\right) = \log\left(\frac{3}{5}\right)^3$

$\therefore \frac{x}{25} = \frac{27}{125}$

$\therefore x = \frac{27}{5}$

$\therefore$ the amount left after 3 hours $\frac{27}{5}$ gm.

Question 7.

Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.

Solution:

Let P be the population of the city at time t.
Then $\frac{dP}{dt}$, the rate of increase of population is proportional to P.
∴ $\frac{dP}{dt} \propto P$
∴ $\frac{dP}{dt}$ = kP, where k is a constant.
∴ $\frac{dP}{P}$ = k dt
On integrating, we get
$\int \frac{1}{P} dP$ = k∫dt + c
∴ log P = kt + c
Initially, i.e. when t = 0, P = 30000
∴ log 30000 = k × 0 + c
∴ c = log 30000
∴ log P = kt + log 30000
∴ log P – log 30000 = kt
∴ log($\frac{P}{30000}$) = kt …….(1)
Now, when t = 40, P = 40000

$$\therefore \log\left(\frac{40000}{30000}\right) = k \times 40$$

$$\therefore k = \frac{1}{40}\log\left(\frac{4}{3}\right)$$

$$\therefore \text{(1) becomes, } \log\left(\frac{P}{30000}\right) = \frac{t}{40}\log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore \ P = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

∴ the population of the city at time t = $30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$.

Question 8.
A body cools according to Newton's law from 100°C to 60°C in 20 minutes. The temperature of the surroundings is 20°C. How long will it take to cool down to 30°C?
Solution:
Let θ°C be the temperature of the body at time t.
The temperature of the surrounding is given to be 20°C.
According to Newton's law of cooling

$$\frac{d\theta}{dt} \propto \theta - 20$$

$$\therefore \ \frac{d\theta}{dt} = -k(\theta - 20), \text{ where } k > 0$$

$$\therefore \ \frac{d\theta}{\theta - 20} = -k\,dt$$

On integrating, we get

$$\int \frac{1}{\theta - 20}\,d\theta = -k\int dt + c$$

$$\therefore \ \log(\theta - 20) = -kt + c$$

Initially, i.e. when $t = 0$, $\theta = 100$

$$\therefore \ \log(100 - 20) = -k \times 0 + c \qquad \therefore \ c = \log 80$$

$\therefore \ \log(\theta - 20) = -kt + \log 80$

$\therefore \ \log(\theta - 20) - \log 80 = -kt$

$\therefore \ \log\left(\frac{\theta - 20}{80}\right) = -kt \quad$ ....(1)

Now, when $t = 20$, $\theta = 60$

$\therefore \ \log\left(\frac{60 - 20}{80}\right) = -k \times 20$

$\therefore \ \log\left(\frac{40}{80}\right) = -20k \qquad \therefore \ k = -\frac{1}{20}\log\left(\frac{1}{2}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{\theta - 20}{80}\right) = \frac{t}{20}\log\left(\frac{1}{2}\right)$

When $\theta = 30$, then

$\log\left(\frac{30 - 20}{80}\right) = \frac{t}{20}\log\left(\frac{1}{2}\right)$

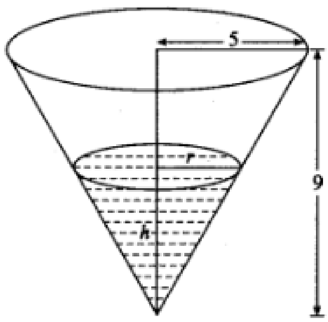
$\therefore \ \log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2}\right)^{\frac{t}{20}}$

$\therefore \ \left(\frac{1}{2}\right)^{\frac{t}{20}} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$

$\therefore \ \frac{t}{20} = 3 \qquad \therefore \ t = 60$

∴ the body will cool down to 30°C in 60 minutes, i.e. in 1 hour.

Question 9.
A right circular cone has a height of 9 cm and a radius of the base of 5 cm. It is inverted and water is poured into it. If at any instant the water level rises at the rate of $\left(\frac{\pi}{A}\right)$ cm/sec, where A is the area of the water surface
at that instant, show that the vessel will be full in 75 seconds.
Solution:



Let r be the radius of the water surface and h be the height of the water at time t.
∴ area of the water surface A = πr^2 sq cm.
Since height of the right circular cone is 9 cm and radius of the base is 5 cm.
$\frac{r}{h}=\frac{5}{9}$
∴ r = $\frac{5}{9}h$
∴ area of water surface, i.e. A = $\pi\left(\frac{5}{9}h\right)^2$
∴ A = $\frac{25\pi h^2}{81}$ ……..(1)
The water level, i.e. the rate of change of h is $\frac{dh}{dt}$ rises at the rate of $\left(\frac{\pi}{A}\right)$ cm/sec.

$\therefore \frac{dh}{dt} = \frac{\pi}{A} = \frac{\pi \times 81}{25\pi h^2}$

$\therefore \frac{dh}{dt} = \frac{81}{25h^2}$

$\therefore h^2 dh = \frac{81}{25} dt$

On integrating, we get

$\int h^2 \, dh = \frac{81}{25} \int dt + c$

$\therefore \frac{h^3}{3} = \frac{81}{25} \cdot t + c$

Initially, i.e. when $t = 0$, $h = 0$

$\therefore 0 = 0 + c \qquad \therefore c = 0$

$\therefore \frac{h^3}{3} = \frac{81}{25} t$

When the vessel will be full, $h = 9$

$\therefore \frac{(9)^3}{3} = \frac{81}{25} \times t$

$\therefore t = \frac{81 \times 9 \times 25}{3 \times 81} = 75$

Hence, the vessel will be full in 75 seconds.

Question 10.
Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the raindrop at any time t.
Solution:
Let r be the radius, V be the volume and S be the surface area of the spherical raindrop at time t.
Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$
The rate at which the raindrop evaporates is $\frac{dV}{dt}$ which is proportional to the surface area.
$\therefore \frac{dV}{dt} \propto S$
$\therefore \frac{dV}{dt} = -kS$, where $k > 0$ ....(1)

Now, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$$\therefore \frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore \text{(1) becomes, } 4\pi r^2 \frac{dr}{dt} = -k(4\pi r^2)$$

$$\therefore \frac{dr}{dt} = -k$$

$$\therefore dr = -k\,dt$$

On integrating, we get
$\int dr = -k\int dt + c$
$\therefore r = -kt + c$
Initially, i.e. when t = 0, r = 3
$\therefore 3 = -k \times 0 + c$
$\therefore c = 3$

∴ $r = -kt + 3$
When $t = 1, r = 2$
∴ $2 = -k \times 1 + 3$
∴ $k = 1$
∴ $r = -t + 3$
∴ $r = 3 - t$, where $0 \le t \le 3$.
This is the required expression for the radius of the raindrop at any time t.

Question 11.
The rate of growth of the population of a city at any time t is proportional to the size of the population. For a certain city, it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years, if the initial population is 10,000. [Take e = 2.7182]
Solution:
Let P be the population of the city at time t.
Then the rate of growth of population is $\frac{dP}{dt}$ which is proportional to P.
∴ $\frac{dP}{dt} \propto P$
∴ $\frac{dP}{dt} = kP$, where $k = 0.04$
∴ $\frac{dP}{dt} = (0.04)P$
∴ $\frac{1}{P}\, dP = (0.04)dt$
On integrating, we get
$\int \frac{1}{P} dP = (0.04) \int dt + c$
∴ $\log P = (0.04)t + c$
Initially, i.e., when $t = 0, P = 10000$
∴ $\log 10000 = (0.04) \times 0 + c$
∴ $c = \log 10000$
∴ $\log P = (0.04)t + \log 10000$

∴ log P = (0.04)t + log 10000
∴ log P – log 10000 = (0.04)t
∴ log($\frac{P}{10000}$) = (0.04)t
When t = 25, then
∴ log($\frac{P}{10000}$) = 0.04 × 25 = 1
∴ log($\frac{P}{10000}$) = log e ……[∵ log e = 1]
∴ $\frac{P}{10000}$ = e = 2.7182
∴ P = 2.7182 × 10000 = 27182
∴ the population of the city after 25 years will be 27,182.

Question 12.
Radium decomposes at a rate proportional to the amount present at any time. If p percent of the amount disappears in one year, what percent of the amount of radium will be left after 2 years?
Solution:
Let x be the amount of the radium at time t.
Then the rate of decomposition is $\frac{dx}{dt}$ which is proportional to x.

$$\therefore \frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = -kx, \text{ where } k > 0$$

$$\therefore \frac{1}{x}dx = -k\,dt$$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt$$

$\therefore \log x = -kt + c$

Let the original amount be x_0, i.e. $x = x_0$, when $t = 0$.

$\therefore \log x_0 = -k \times 0 + c \qquad \therefore c = \log x_0$

$\therefore \log x = -kt + \log x_0$

$\therefore \log x - \log x_0 = -kt$

$$\therefore \log\left(\frac{x}{x_0}\right) = -kt \qquad \ldots (1)$$

But $p\%$ of the amount disappears in one year,

$\therefore$ when $t = 1$, $x = x_0 - p\%$ of x_0, i.e. $x = x_0 - \frac{px_0}{100}$

$$\therefore \log\left(\frac{x_0 - \frac{px_0}{100}}{x_0}\right) = -k \times 1$$

$$\therefore k = -\log\left(1 - \frac{p}{100}\right) = -\log\left(\frac{100-p}{100}\right)$$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = t \log\left(\frac{100-p}{100}\right)$

When $t = 2$, then

$$\log\left(\frac{x}{x_0}\right) = 2\log\left(\frac{100-p}{100}\right) = \log\left(\frac{100-p}{100}\right)^2$$

$$\therefore \frac{x}{x_0}=\left(\frac{100-p}{100}\right)^2$$

$$\therefore x=\left(\frac{100-p}{100}\right)^2 x_0=\left(1-\frac{p}{100}\right)^2 x_0$$

$$\therefore \% \text{ left after 2 years}=\frac{100\times\left(1-\frac{p}{100}\right)^2 x_0}{x_0}$$

$$=100\left(1-\frac{p}{100}\right)^2=\left[10\left(1-\frac{p}{100}\right)\right]^2$$

$$=\left(10-\frac{p}{10}\right)^2$$

Hence, $\left(10-\frac{p}{10}\right)^2\%$ of the amount will be left after 2 years.

Maharashtra Board Solutions Class 12 Arts & Science Maths (Part 2)

- Chapter 1- Differentiation
- Chapter 2- Applications of Derivatives
- Chapter 3- Indefinite Integration
- Chapter 4- Definite Integration
- Chapter 5- Application of Definite Integration
- Chapter 6- Differential Equations
- Chapter 7- Probability Distributions
- Chapter 8- Binomial Distribution

About About Maharashtra State Board (MSBSHSE)

The Maharashtra State Board of Secondary and Higher Secondary Education or MSBSHSE (Marathi: महाराष्ट्र राज्य माध्यमिक आणि उच्च माध्यमिक शिक्षण मंडळ), is an **autonomous and statutory body established in 1965**. The board was amended in the year 1977 under the provisions of the Maharashtra Act No. 41 of 1965.

The Maharashtra State Board of Secondary & Higher Secondary Education (MSBSHSE), Pune is an independent body of the Maharashtra Government. There are more than 1.4 million students that appear in the examination every year. The Maha State Board conducts the board examination twice a year. This board conducts the examination for SSC and HSC.

The Maharashtra government established the Maharashtra State Bureau of Textbook Production and Curriculum Research, also commonly referred to as Ebalbharati, in 1967 to take up the responsibility of providing quality textbooks to students from all classes studying under the Maharashtra State Board. MSBHSE prepares and updates the curriculum to provide holistic development for students. It is designed to tackle the difficulty in understanding the concepts with simple language with simple illustrations. Every year around 10 lakh students are enrolled in schools that are affiliated with the Maharashtra State Board.

FAQs

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As of now, the MSCERT and Balbharti are responsible for the syllabus and textbooks of Classes 1 to 8, while Classes 9 and 10 are under the Maharashtra State Board of Secondary and Higher Secondary Education (MSBSHSE).

How many state boards are there in Maharashtra?

The Maharashtra State Board of Secondary & Higher Secondary Education, conducts the HSC and SSC Examinations in the state of Maharashtra through its nine Divisional Boards located at Pune, Mumbai, Aurangabad, Nasik, Kolhapur, Amravati, Latur, Nagpur and Ratnagiri.

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