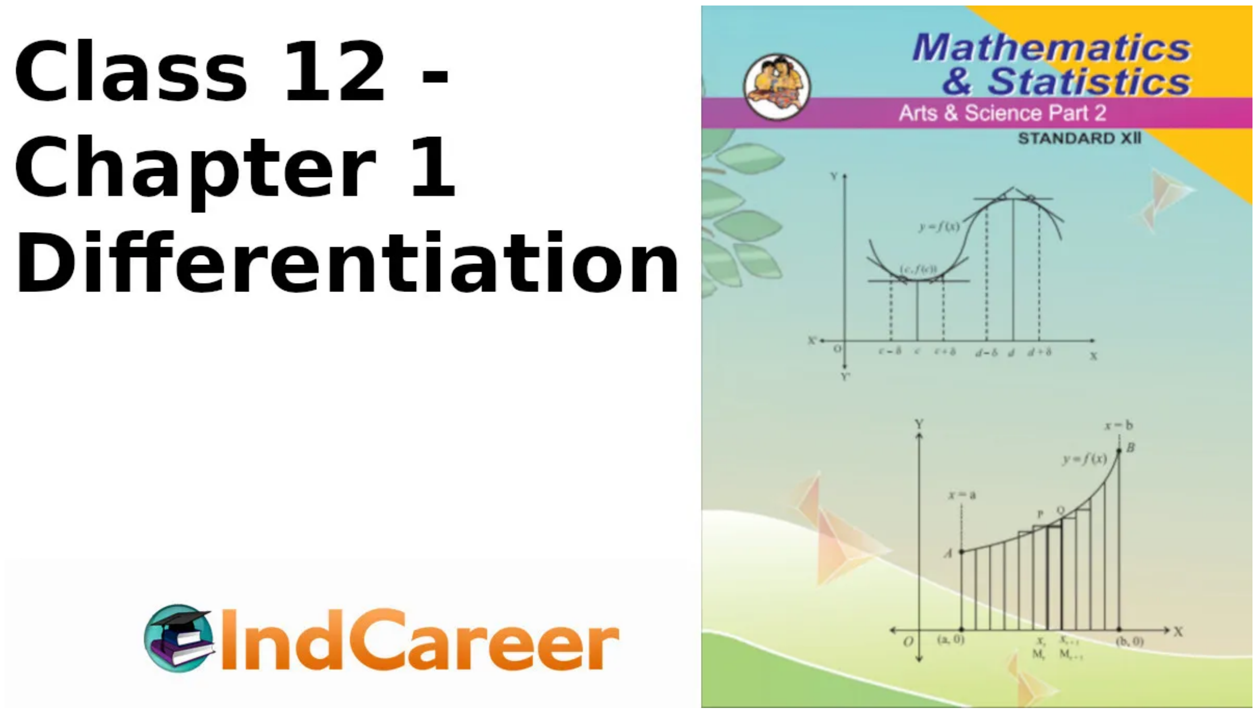
Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 1- Differentiation



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Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 1- Differentiation

Class 12: Maths Chapter 1 solutions. Complete Class 12 Maths Chapter 1 Notes.

Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 2): Chapter 1- Differentiation

Maharashtra Board 12th Maths Chapter 1, Class 12 Maths Chapter 1 solutions

Ex 1.1

Question 1.

Differentiate the following w.r.t. x :

(i) $(x^3 - 2x - 1)^5$

Solution:

Method 1:

Let $y = (x^3 - 2x - 1)^5$

Put $u = x^3 - 2x - 1$. Then $y = u^5$

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^5) = 5u^4$$
$$= 5(x^3 - 2x - 1)^4$$
$$\text{and } \frac{du}{dx} = \frac{d}{dx}(x^3 - 2x - 1)$$
$$= 3x^2 - 2 \times 1 - 0 = 3x^2 - 2$$
$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 5(x^3 - 2x - 1)^4 (3x^2 - 2)$$
$$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$$

Method 2:

Let y = $(x^3 - 2x - 1)^5$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x^3-2x-1)^5$$

$$=5(x^3-2x-1)^4\times\frac{d}{dx}(x^3-2x-1)$$

$$=5(x^3-2x-1)^4\times(3x^2-2\times1-0)$$

$$=5(3x^2-2)(x^3-2x-1)^4.$$

(ii) $\left(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5\right)^{\frac{5}{2}}$

Solution:

Let y = $\left(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5\right)^{\frac{5}{2}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)^{\frac{5}{2}}$$

$$=\frac{5}{2}(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)^{\frac{5}{2}-1}\times\frac{d}{dx}(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)$$

$$=\frac{5}{2}(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)^{\frac{3}{2}}\times(2\times\frac{3}{2}x^{\frac{3}{2}-1}-3\times\frac{4}{3}x^{\frac{4}{3}-1}-0)$$

$$=\frac{5}{2}(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)^{\frac{3}{2}}(3x^{\frac{1}{2}}-4x^{\frac{1}{3}})$$

$$=\frac{5}{2}(3\sqrt{x}-4\sqrt[3]{x})(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5)^{\frac{3}{2}}.$$

(iii) $\sqrt{x^2+4x-7}$

Solution:

$$y=\sqrt{x^2+4x-7}\left[\sqrt{x}=\frac{1}{2\sqrt{x}}\right]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x^2+4x-7}}\cdot\frac{d}{dx}\left(x^2+4x-7\right)$$

$$= \frac{1}{2\sqrt{x^2+4x-7}}\left(\frac{d}{dx}x^2 + \frac{d}{dx}4x - \frac{d}{dx}7\right)$$

$$= \frac{1}{2\sqrt{x^2+4x-7}}.(2x+4-0)$$

$$= \frac{2(x+2)}{2\sqrt{x^2+4x-7}}$$

$$= \frac{(x+2)}{\sqrt{x^2+4x-7}}.$$

(iv) $\sqrt{x^2+\sqrt{x^2+1}}$

Solution:

Let y = $\sqrt{x^2+\sqrt{x^2+1}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2+\sqrt{x^2+1})^{\frac{1}{2}}$$

$$= \frac{1}{2}(x^2+\sqrt{x^2+1})^{-\frac{1}{2}}.\frac{d}{dx}(x^2+\sqrt{x^2+1})$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(\sqrt{x^2+1})\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[2x+\frac{1}{2\sqrt{x^2+1}}.\frac{d}{dx}(x^2+1)\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[2x + \frac{1}{2\sqrt{x^2+1}}(2x+0)\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[2x + \frac{x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}} \cdot \left[\frac{2x\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{x(2\sqrt{x^2+1}+1)}{2\sqrt{x^2+1} \cdot \sqrt{x^2+\sqrt{x^2+1}}}.$$

(v) $\frac{3}{5\sqrt[3]{(2x^2-7x-5)^5}}$

Solution:

Let y = $\frac{3}{5\sqrt[3]{(2x^2-7x-5)^5}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{3}{5}\frac{d}{dx}(2x^2-7x-5)^{-\frac{5}{3}}$$

$$= \frac{3}{5} \times \left(-\frac{5}{3}\right)(2x^2-7x-5)^{-\frac{5}{3}-1} \cdot \frac{d}{dx}(2x^2-7x-5)$$

$$= -(2x^2-7x-5)^{-\frac{8}{3}} \cdot (2 \times 2x - 7 \times 1 - 0)$$

$$= -\frac{4x-7}{(2x^2-7x-5)^{\frac{8}{3}}}.$$

(vi) $\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$

Solution:

Let y = $\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\left[\frac{d}{dx}(3x-5)^{\frac{1}{2}}-\frac{d}{dx}(3x-5)^{-\frac{1}{2}}\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times$$

$$\left[\frac{1}{2}(3x-5)^{-\frac{1}{2}}\cdot\frac{d}{dx}(3x-5)-\left(-\frac{1}{2}\right)(3x-5)^{-\frac{3}{2}}\cdot\frac{d}{dx}(3x-5)\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times$$

$$\left[\frac{1}{2\sqrt{3x-5}}\cdot(3\times1-0)+\frac{1}{2(3x-5)^{\frac{3}{2}}}\cdot(3\times1-0)\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\left[\frac{3}{2\sqrt{3x-5}}+\frac{3}{2(3x-5)^{\frac{3}{2}}}\right]$$

$$= \frac{15}{2}\left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}}\right)^4 \cdot \left[\frac{3x-5+1}{(3x-5)^{\frac{3}{2}}}\right]$$

$$= \frac{15(3x-4)}{2(3x-5)^{\frac{3}{2}}}\left(\sqrt{3x-5} - \frac{1}{\sqrt{3x-5}}\right)^4.$$

Question 2.
Diffrentiate the following w.r.t. x
(i) $\cos(x^2 + a^2)$
Solution:
Let $y = \cos(x^2 + a^2)$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}[\cos(x^2 + a^2)]$
$= -\sin(x^2 + a^2) \cdot \frac{d}{dx} x^2 + a^2)$
$= -\sin(x^2 + a^2) \cdot (2x + 0)$
$= -2x\sin(x^2 + a^2)$

(ii) $\sqrt{e^{(3x+2)} + 5}$
Solution:
Let $y = \sqrt{e^{(3x+2)} + 5}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[e^{(3x+2)} + 5]^{\frac{1}{2}}$$

$$= \frac{1}{2}[e^{(3x+2)}+5]^{-\frac{1}{2}} \cdot \frac{d}{dx}[e^{(3x+2)}+5]$$

$$= \frac{1}{2\sqrt{e^{(3x+2)}+5}} \cdot [e^{(3x+2)} \cdot \frac{d}{dx}(3x+2)+0]$$

$$= \frac{1}{2\sqrt{e^{(3x+2)}+5}} \cdot [e^{(3x+2)} \cdot (3\times 1+0)]$$

$$= \frac{3e^{(3x+2)}}{2\sqrt{e^{(3x+2)}+5}}.$$

(iii) $\log[\tan(\frac{x}{2})]$

Solution:

Let y = $\log[\tan(\frac{x}{2})]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\log\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)} \cdot \frac{d}{dx}\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)} \cdot \sec^2\left(\frac{x}{2}\right) \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{1}{\cos^2\left(\frac{x}{2}\right)} \cdot \frac{1}{2} \times 1$$

$$= \frac{1}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$= \frac{1}{\sin x} = \operatorname{cosec} x.$$

(iv) $\sqrt{\tan \sqrt{x}}$

Solution:

Let y = $\sqrt{\tan \sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{\tan\sqrt{x}})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$\sec^2\sqrt{x}$

$$= \frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

(v) $\cot^3[\log(x^3)]$

Solution:

Let $y = \cot^3[\log(x^3)]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\cot(\log x^3)]^3$$

$$= 3[\cot(\log x^3)]^2 \cdot \frac{d}{dx}[\cot(\log x^3)]$$

$$= 3\cot^2[\log(x^3)] \cdot [-\operatorname{cosec}^2(\log x^3)] \cdot \frac{d}{dx}(\log x^3)$$

$$= -3\cot^2[\log(x^3)] \cdot \operatorname{cosec}^2[\log(x^3)] \cdot 3\frac{d}{dx}(\log x)$$

$$= -3\cot^2[\log(x^3)] \cdot \operatorname{cosec}^2[\log(x^3)] \cdot 3 \times \frac{1}{x}$$

$$= \frac{-9\operatorname{cosec}^2[\log(x^3)] \cdot \cot^2[\log(x^3)]}{x}.$$

(vi) $5^{\sin^3 x + 3}$

Solution:

Let $y = 5^{\sin^3 x + 3}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(5^{\sin^3 x + 3})$$

$$= 5^{\sin^3 x + 3} \cdot \log 5 \cdot \frac{d}{dx}(\sin^3 x + 3)$$

$$= 5^{\sin^3 x + 3} \cdot \log 5 \cdot [3 \sin^2 x \cdot \frac{d}{dx}(\sin x) + 0]$$

$$= 5^{\sin^3 x + 3} \cdot \log 5 \cdot [3 \sin^2 x \cos x]$$

$$= 3 \sin^2 x \cos x \cdot 5^{\sin^3 x + 3} \cdot \log 5.$$

(vii) cosec ($\sqrt{\cos X}$)

Solution:

Let y = cosec ($\sqrt{\cos X}$)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\text{cosec}(\sqrt{\cos x})]$$

$$= -\text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{d}{dx}\sqrt{\cos x}$$

$$= -\text{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{1}{2\sqrt{\cos x}} \cdot \frac{d}{dx}(\cos x)$$

$$= -\operatorname{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x}) \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$$

$$= \frac{\sin x \cdot \operatorname{cosec}(\sqrt{\cos x}) \cdot \cot(\sqrt{\cos x})}{2\sqrt{\cos x}}.$$

(viii) $\log[\cos(x^3 - 5)]$
Solution:
Let $y = \log[\cos(x^3 - 5)]$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\log[\cos(x^3 - 5)]\}$$

$$= \frac{1}{\cos(x^3 - 5)} \cdot \frac{d}{dx}[\cos(x^3 - 5)]$$

$$= \frac{1}{\cos(x^3 - 5)} \cdot [-\sin(x^3 - 5)] \cdot \frac{d}{dx}(x^3 - 5)$$

$$= -\tan(x^3 - 5) \times (3x^2 - 0)$$

$$= -3x^2 \tan(x^3 - 5).$$

(ix) $e^{3\sin^2 x - 2\cos^2 x}$
Solution:
Let $y = e^{3\sin^2 x - 2\cos^2 x}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[e^{3\sin^2 x - 2\cos^2 x}]$$

$$= e^{3\sin^2 x - 2\cos^2 x} \cdot \frac{d}{dx}(3\sin^2 x - 2\cos^2 x)$$

$$= e^{3\sin^2 x - 2\cos^2 x} \cdot \left[3\frac{d}{dx}(\sin x)^2 - 2\frac{d}{dx}(\cos x)^2\right]$$

$$= e^{3\sin^2 x - 2\cos^2 x} \cdot \left[3 \times 2\sin x \cdot \frac{d}{dx}(\sin x) - 2 \times 2\cos x \cdot \frac{d}{dx}(\cos x)\right]$$

$$= e^{3\sin^2 x - 2\cos^2 x} \cdot [6\sin x\cos x - 4\cos x(-\sin x)]$$

$$= e^{3\sin^2 x - 2\cos^2 x} \cdot (10\sin x\cos x)$$

$$= 5(2\sin x\cos x) \cdot e^{3\sin^2 x - 2\cos^2 x}$$

$$= 5\sin 2x \cdot e^{3\sin^2 x - 2\cos^2 x}.$$

(x) $\cos^2[\log(x^2 + 7)]$
Solution:
Let $y = \cos^2[\log(x^2 + 7)]$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos[\log(x^2+7)]\}^2$$

$$= 2\cos[\log(x^2+7)] \cdot \frac{d}{dx}\{\cos[\log(x^2+7)]\}$$

$$= 2\cos[\log(x^2+7)] \cdot \{-\sin[\log(x^2+7)]\} \cdot \frac{d}{dx}[\log(x^2+7)]$$

$$= -2\sin[\log(x^2+7)] \cdot \cos[\log(x^2+7)] \times \frac{1}{x^2+7} \cdot \frac{d}{dx}(x^2+7)$$

$$= -\sin[2\log(x^2+7)] \times \frac{1}{x^2+7} \cdot (2x+0)$$

$$\ldots [\because 2\sin x \cdot \cos x = \sin 2x]$$

$$= \frac{-2x \cdot \sin[2\log(x^2+7)]}{x^2+7}.$$

(xi) tan[cos (sinx)]

Solution:

Let y = tan[cos (sinx)]

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\tan[\cos(\sin x)]\}$$

$$= \sec^2[\cos(\sin x)] \cdot \frac{d}{dx}[\cos(\sin x)]$$

$$= \sec^2[\cos(\sin x)] \cdot [-\sin(\sin x)] \cdot \frac{d}{dx}(\sin x)$$

$$= -\sec^2[\cos(\sin x)] \cdot \sin(\sin x) \cdot \cos x.$$

(xii) sec[tan (x^4 + 4)]

Solution:

Let y = sec[tan (x^4 + 4)]

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\sec[\tan(x^4+4)]\}$$

$$= \sec[\tan(x^4+4)] \cdot \tan[\tan(x^4+4)] \cdot \frac{d}{dx}[\tan(x^4+4)]$$

$= \sec[\tan(x^4+4)] \cdot \tan[\tan(x^4+4)] \cdot \sec^2(x^4+4) \cdot \frac{d}{dx}(x^4+4)$

$= \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)] \cdot \sec^2(x^4 + 4)(4x^3 + 0)$

$= 4x^3\sec^2(x^4 + 4) \cdot \sec[\tan(x^4 + 4)] \cdot \tan[\tan(x^4 + 4)]$.

(xiii) $e^{\log[(\log x)^2 - \log x^2]}$

Solution:

Let $y = e^{\log[(\log x)^2 - \log x^2]}$

$= (\log x)^2 - \log x^2$ …[$\because e^{\log x} = x$]

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}[(\log x)^2 - 2\log x]$

$= \frac{d}{dx}(\log x)^2 - 2\frac{d}{dx}(\log x)$

$= 2\log x \cdot \frac{d}{dx}(\log x) - 2 \times \frac{1}{x}$

$= 2\log x \times \frac{1}{x} - \frac{2}{x}$

$= \frac{2\log x}{x} - \frac{2}{x}$.

(xiv) $\sin\sqrt{\sin\sqrt{x}}$

Solution:

Let $y = \sin\sqrt{\sin\sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin\sqrt{\sin\sqrt{x}})$$

$$=\cos\sqrt{\sin\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{\sin\sqrt{x}})$$

$$=\cos\sqrt{\sin\sqrt{x}}\times\frac{1}{2\sqrt{\sin\sqrt{x}}}\cdot\frac{d}{dx}(\sin\sqrt{x})$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}}{2\sqrt{\sin\sqrt{x}}}\times\cos\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}\cdot\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}\cdot\cos\sqrt{x}}{4\sqrt{x}\cdot\sqrt{\sin\sqrt{x}}}.$$

(xv) $\log[\sec(e^{x^2})]$

Solution:

Let $y = \log[\sec(e^{x^2})]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\sec e^{x^2})]$$

$$=\frac{1}{\sec(e^{x^2})}\cdot\frac{d}{dx}[\sec(e^{x^2})]$$

$$= \frac{1}{\sec(e^{x^2})} \cdot \sec(e^{x^2})\tan(e^{x^2}) \cdot \frac{d}{dx}(e^{x^2})$$

$$= \tan(e^{x^2}) \cdot e^{x^2} \cdot \frac{d}{dx}(x^2)$$

$$= \tan(e^{x^2}) \cdot e^{x^2} \cdot 2x$$

$$= 2x \cdot e^{x^2} \tan(e^{x^2}).$$

(xvi) $\log_{e^2}(\log x)$

Solution:

Let $y = \log_{e^2}(\log x) = \frac{\log(\log x)}{\log e^2}$

$$= \frac{\log(\log x)}{2\log e} = \frac{\log(\log x)}{2} \quad \ldots [\because \log e = 1]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}[\log(\log x)]$$

$$= \frac{1}{2} \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{2\log x} \times \frac{1}{x} = \frac{1}{2x\log x}.$$

(xvii) $[\log\{\log(\log x)\}]^2$

Solution:

let y = [log{log(logx)}]^2
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log\{\log(\log x)\}]^2$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{d}{dx}[\log\{\log(\log x)\}]$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\cdot\frac{d}{dx}[\log(\log x)]$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\times\frac{1}{\log x}\times\frac{d}{dx}(\log x)$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\times\frac{1}{\log x}\times\frac{1}{x}$$

$$=2\cdot\left[\frac{\log\{\log(\log x)\}}{x\cdot\log x\cdot\log(\log x)}\right].$$

(xviii) $\sin^2 x^2 - \cos^2 x^2$
Solution:
Let y = $\sin^2 x^2 - \cos^2 x^2$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\sin^2 x^2-\cos^2 x^2]$$

$$=\frac{d}{dx}(\sin x^2)^2-\frac{d}{dx}(\cos x^2)^2$$

$$= 2\sin x^2 \cdot \frac{d}{dx}(\sin x^2) - 2\cos x^2 \cdot \frac{d}{dx}(\cos x^2)$$

$$= 2\sin x^2 \cdot \cos x^2 \cdot \frac{d}{dx}(x^2) - 2\cos x^2 \cdot (-\sin x^2) \cdot \frac{d}{dx}(x^2)$$

$= 2\sin x^2 \cdot \cos x^2 \times 2x + 2\sin x^2 \cdot \cos x^2 \times 2x$
$= 4x(2\sin x^2 \cdot \cos x^2)$
$= 4x\sin(2x^2)$.

Question 3.
Diffrentiate the following w.r.t. x
(i) $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$
Solution:
Let $y = (x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}[(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4]$
$= \frac{d}{dx} = (x^2 + 4x + 1)^3 + \frac{d}{dx}(x^3 - 5x - 2)^4$
$= 3(x^2 + 4x + 1)^2 \cdot \frac{d}{dx}(x^2 + 4x + 1) + 4(x^3 - 5x - 2)^4 \cdot \frac{d}{dx}(x^3 - 5x - 2)$
$= 3(x^2 + 4x + 1)^3 \cdot (2x + 4 \times 1 + 0) + 4(x^3 - 5x - 2)^3 \cdot (3x^2 - 5 \times 1 - 0)$
$= 6(x + 2)(x^2 + 4x + 1)^2 + 4(3x^2 - 5)(x^3 - 5x - 2)^3$.
(ii) $(1 + 4x)^5(3 + x - x^2)^8$
Solution:
Let $y = (1 + 4x)^5(3 + x - x^2)^8$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(1 + 4x)^5(3 + x - x^2)^8]$$

$$= (1 + 4x)^5 \cdot \frac{d}{dx}(3 + x - x^2)^8 + (3 + x - x^2)^8 \cdot \frac{d}{dx}(1 + 4x)^5$$

$$= (1+4x)^5 \times 8(3+x-x^2)^7 \cdot \frac{d}{dx}(3+x-x^2) + (3+x-x^2)^8 \times 5(1+4x)^4 \cdot \frac{d}{dx}(1+4x)$$

$= 8(1 + 4x)^5 (3 + x - x^2)^7 \cdot (0 + 1 - 2x) + 5(1 + 4x)^4 (3 + x - x^2)^8 \cdot (0 + 4 \times 1)$

$= 8(1 - 2x)(1 + 4x)^5(3 + x - x^2)^7 + 20(1 + 4x)^4(3 + x - x^2)^8$.

(iii) $\frac{x}{\sqrt{7-3x}}$

Solution:

Let y = $\frac{x}{\sqrt{7-3x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\sqrt{7-3x}}\right) = \frac{\sqrt{7-3x} \cdot \frac{d}{dx}(x) - x\frac{d}{dx}(\sqrt{7-3x})}{(\sqrt{7-3x})^2}$$

$$= \frac{\sqrt{7-3x} \times 1 - x \times \frac{1}{2\sqrt{7-3x}} \cdot \frac{d}{dx}(7-3x)}{7-3x}$$

$$= \frac{\sqrt{7-3x} - \frac{x}{2\sqrt{7-3x}}(0 - 3 \times 1)}{7-3x}$$

$$= \frac{2(7-3x)+3x}{2(7-3x)^{\frac{3}{2}}} = \frac{14-6x+3x}{2(7-3x)^{\frac{3}{2}}} = \frac{14-3x}{2(7-3x)^{\frac{3}{2}}}$$

(iv) $\frac{(x^3-5)^5}{(x^3+3)^3}$

Solution:

Let y = $\frac{(x^3-5)^5}{(x^3+3)^3}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(x^3-5)^5}{(x^3+3)^3}\right]$$

$$= \frac{(x^3+3)^3 \cdot \frac{d}{dx}(x^3-5)^5 - (x^3-5)^5 \cdot \frac{d}{dx}(x^3+3)^3}{[(x^3+3)^3]^2}$$

$$= \frac{(x^3+3)^3 \times 5(x^3-5)^4 \cdot \frac{d}{dx}(x^3-5) - (x^3-5)^5 \times 3(x^3+3)^2 \cdot \frac{d}{dx}(x^3+3)}{(x^3+3)^6}$$

$$= \frac{5(x^3+3)^3(x^3-5)^4 \cdot (3x^2-0) - 3(x^3-5)^5(x^3+3)^2 \cdot (3x^2+0)}{(x^3+3)^6}$$

$$= \frac{3x^2(x^3+3)^2(x^3-5)^4[5(x^3+3) - 3(x^3-5)]}{(x^3+3)^6}$$

$$= \frac{3x^2(x^3-5)^4(5x^3+15-3x^3+15)}{(x^3+3)^4}$$

$$= \frac{3x^2(x^3-5)^4(2x^3+30)}{(x^3+3)^4}$$

$$= \frac{6x^2(x^3+15)(x^3-5)^4}{(x^3+3)^4}$$

(v) $(1 + \sin^2 x)^2(1 + \cos^2 x)^3$

Solution:
Let y = $(1 + \sin^2 x)^2(1 + \cos^2 x)^3$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(1+\sin^2 x)^2(1+\cos^2 x)^3]$$

$$= (1+\sin^2 x)^2 \cdot \frac{d}{dx}(1+\cos^2 x)^3 + (1+\cos^2 x)^3 \cdot \frac{d}{dx}(1+\sin^2 x)^2$$

$$= (1+\sin^2 x)^2 \times 3(1+\cos^2 x)^2 \cdot \frac{d}{dx}(1+\cos^2 x) + (1+\cos^2 x)^3 \times 2(1+\sin^2 x) \cdot \frac{d}{dx}(1+\sin^2 x)$$

$$= 3(1+\sin^2 x)^2(1+\cos^2 x)^2 \cdot \left[0 + 2\cos x \cdot \frac{d}{dx}(\cos x)\right] + 2(1+\sin^2 x)(1+\cos^2 x)^3 \cdot \left[0 + 2\sin x \cdot \frac{d}{dx}(\sin x)\right]$$

= $3(1 + \sin^2 x)^2 (1 + \cos^2 x)^2 \cdot [2\cos x(-\sin x)] + 2 (1 + \sin^2 x)(1 + \cos^2 x)^3 \cdot [2\sin x \cdot \cos x]$
= $3 (1 + \sin^2 x)^2 (1 + \cos^2 x)^2 (-\sin 2x) + 2(1 + \sin^2 x)(1 + \cos^2 x)^3(\sin 2x)$
= $\sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 [-3(1 + \sin^2 x) + 2(1 + \cos^2 x)]$
= $\sin 2x (1 + \sin^2 x)(1 + \cos^2 x)^2(-3 - 3\sin^2 x + 2 + 2\cos^2 x)$
= $\sin 2x (1 + \sin^2 x)(1 + \cos^2 x)^2 [-1 - 3 \sin^2 x + 2 (1 - \sin^2 x)]$
= $\sin 2x(1 + \sin^2 x)(1 + \cos^2 x)^2 (-1 - 3 \sin^2 x + 2 - 2 \sin^2 x)$
= $\sin 2x (1 + \sin^2 x)(1 + \cos^2 x)^2(1 - 5 \sin^2 x)$.

(vi) $\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}$
Solution:
Let y = $\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sqrt{\cos x} + \sqrt{\cos \sqrt{x}}\right]$$

$$= \frac{d}{dx}(\cos x)^{\frac{1}{2}} + \frac{d}{dx}(\cos \sqrt{x})^{\frac{1}{2}}$$

$$= \frac{1}{2}(\cos x)^{-\frac{1}{2}} \cdot \frac{d}{dx}(\cos x) + \frac{1}{2}(\cos\sqrt{x})^{-\frac{1}{2}} \cdot \frac{d}{dx}(\cos\sqrt{x})$$

$$= \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x) + \frac{1}{2\sqrt{\cos\sqrt{x}}} \times (-\sin\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin\sqrt{x}}{2\sqrt{\cos\sqrt{x}}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin x}{2\sqrt{\cos x}} - \frac{\sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}.$$

(vii) log(sec 3x+ tan 3x)
Solution:
Let y = log(sec 3x+ tan 3x)
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\sec 3x + \tan 3x)]$$

$$= \frac{1}{\sec 3x + \tan 3x} \cdot \frac{d}{dx}(\sec 3x + \tan 3x)$$

$$= \frac{1}{\sec 3x + \tan 3x} \times \left[\frac{d}{dx}(\sec 3x) + \frac{d}{dx}(\tan 3x)\right]$$

$$= \frac{1}{\sec 3x + \tan 3x} \times \left[\sec 3x \tan 3x \cdot \frac{d}{dx}(3x) + \sec^2 3x \cdot \frac{d}{dx}(3x)\right]$$

$$= \frac{1}{\sec 3x + \tan 3x} \times [\sec 3x \tan 3x \times 3 + \sec^2 3x \times 3]$$

$$= \frac{3\sec 3x(\tan 3x + \sec 3x)}{\sec 3x + \tan 3x} = 3\sec 3x.$$

(viii) $\frac{1+\sin x^\circ}{1-\sin x^\circ}$

Solution:

$$\text{Let } y = \frac{1+\sin x^\circ}{1-\sin x^\circ} = \frac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)} \quad \ldots \left[\because x^\circ = \left(\frac{\pi x}{180}\right)^c\right]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)}\right]$$

$$= \frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]\cdot\frac{d}{dx}\left[1+\sin\left(\frac{\pi x}{180}\right)\right] - \left[1+\sin\left(\frac{\pi x}{180}\right)\right]\cdot\frac{d}{dx}\left[1-\sin\left(\frac{\pi x}{180}\right)\right]}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$= \frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]\cdot\left[0+\cos\left(\frac{\pi x}{180}\right)\cdot\frac{d}{dx}\left(\frac{\pi x}{180}\right)\right] - \left[1+\sin\left(\frac{\pi x}{180}\right)\right]\cdot\left[0-\cos\left(\frac{\pi x}{180}\right)\cdot\frac{d}{dx}\left(\frac{\pi x}{180}\right)\right]}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$= \frac{(1-\sin x^\circ)\left[(\cos x^\circ)\times\frac{\pi}{180}\times 1\right] - (1+\sin x^\circ)\left[(-\cos x^\circ)\times\frac{\pi}{180}\times 1\right]}{(1-\sin x^\circ)^2}$$

$$= \frac{\frac{\pi}{180}\cos x^\circ(1-\sin x^\circ+1+\sin x^\circ)}{(1-\sin x^\circ)^2}$$

$$= \frac{\pi \cos x^\circ}{90(1-\sin x^\circ)^2}.$$

(ix) $\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$

Solution:

Let $y = \cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)\right]$$

$$= \frac{d}{dx}\left[\cot\left(\frac{\log x}{2}\right)\right] - \frac{d}{dx}\left[\log\left(\frac{\cot x}{2}\right)\right]$$

$$= -\operatorname{cosec}^2\left(\frac{\log x}{2}\right)\cdot\frac{d}{dx}\left(\frac{\log x}{2}\right) - \frac{1}{\left(\frac{\cot x}{2}\right)}\cdot\frac{d}{dx}\left(\frac{\cot x}{2}\right)$$

$$= -\operatorname{cosec}^2\left(\frac{\log x}{2}\right)\times\frac{1}{2}\times\frac{1}{x} - \frac{2}{\cot x}\times\frac{1}{2}\times(-\operatorname{cosec}^2 x)$$

$$\operatorname{cosec}^2\left(\frac{\log x}{2}\right)$$

IndCareer

$$= -\frac{\operatorname{cosec}^2\left(\frac{\log x}{2}\right)}{2x} + \tan x \cdot \operatorname{cosec}^2 x.$$

(x) $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution:

$$\text{Let } y = \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} = \frac{e^{2x}-\frac{1}{e^{2x}}}{e^{2x}+\frac{1}{e^{2x}}} = \frac{e^{4x}-1}{e^{4x}+1}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^{4x}-1}{e^{4x}+1}\right)$$

$$= \frac{(e^{4x}+1)\cdot\frac{d}{dx}(e^{4x}-1) - (e^{4x}-1)\cdot\frac{d}{dx}(e^{4x}+1)}{(e^{4x}+1)^2}$$

$$= \frac{(e^{4x}+1)\left[e^{4x}\cdot\frac{d}{dx}(4x) - 0\right] - (e^{4x}-1)\left[e^{4x}\cdot\frac{d}{dx}(4x) + 0\right]}{(e^{4x}+1)^2}$$

$$= \frac{(e^{4x}+1)\cdot e^{4x}\times 4 - (e^{4x}-1)\cdot e^{4x}\times 4}{(e^{4x}+1)^2}$$

$$= \frac{4e^{4x}(e^{4x}+1-e^{4x}+1)}{(e^{4x}+1)^2} = \frac{8e^{4x}}{(e^{4x}+1)^2}.$$

(xi) $\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$
Solution:
let y = $\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}\right)$$

$$=\frac{(e^{\sqrt{x}}-1)\frac{d}{dx}(e^{\sqrt{x}}+1)-(e^{\sqrt{x}}+1)\frac{d}{dx}(e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{(e^{\sqrt{x}}-1)\left[e^{\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{x})+0\right]-(e^{\sqrt{x}}+1)\left[e^{\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{x})-0\right]}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{(e^{\sqrt{x}}-1)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]-(e^{\sqrt{x}}+1)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(e^{\sqrt{x}}-1-e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{-e^{\sqrt{x}}}{\sqrt{x}(e^{\sqrt{x}}-1)^2}.$$

(xii) $\log[\tan^3x\cdot\sin^4x\cdot(x^2+7)^7]$
Solution:

Let y = log [$\tan^3 x \cdot \sin^4 x \cdot (x^2 + 7)^7$]
= log $\tan^3 x$ + log $\sin^4 x$ + log $(x^2 + 7)^7$
= 3 log tan x + 4 log sin x + 7 log $(x^2 + 7)$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[3\log\tan x+4\log\sin x+7\log(x^2+7)]$$

$$=3\frac{d}{dx}(\log\tan x)+4\frac{d}{dx}(\log\sin x)+7\frac{d}{dx}[\log(x^2+7)]$$

$$=3\times\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)+4\times\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)+$$

$$7\times\frac{1}{x^2+7}\cdot\frac{d}{dx}(x^2+7)$$

$$=3\times\frac{1}{\tan x}\cdot\sec^2 x+4\times\frac{1}{\sin x}\cdot\cos x+7\times\frac{1}{x^2+7}\cdot(2x+0)$$

$$=3\times\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}+4\cot x+\frac{14x}{x^2+7}$$

$$=\frac{6}{2\sin x\cos x}+4\cot x+\frac{14x}{x^2+7}$$

$$=\frac{6}{\sin 2x}+4\cot x+\frac{14x}{x^2+7}$$

= 6cosec2x + 4 cotx + $\frac{14x}{x^2+7}$

(xiii) $\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$

Solution:

$$\text{Let } y=\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$$

$$=\log\left(\sqrt{\frac{2\sin^2\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}}\right)$$

$$=\log\tan\left(\frac{3x}{2}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\tan\left(\frac{3x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{3x}{2}\right)}\times\frac{d}{dx}\left[\tan\left(\frac{3x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{3x}{2}\right)}\times\sec^2\left(\frac{3x}{2}\right)\cdot\frac{d}{dx}\left(\frac{3x}{2}\right)$$

$$=\frac{\cos\left(\frac{3x}{2}\right)}{\sin\left(\frac{3x}{2}\right)}\times\frac{1}{\cos^2\left(\frac{3x}{2}\right)}\times\frac{3}{2}\times 1$$

$$= 3 \times \frac{1}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}$$

$$= 3 \times \frac{1}{\sin 3x} = 3\operatorname{cosec} 3x.$$

(xiv) $\log\left(\sqrt{\frac{1+\cos\left(\frac{5x}{2}\right)}{1-\cos\left(\frac{5x}{2}\right)}}\right)$

Solution:

Using $\log\left(\frac{a}{b}\right)$ = log a – log b

log a^b = b log a

$$y = \log\left(\sqrt{1+\cos\frac{5x}{2}}\right) - \log\left(\sqrt{1-\cos\left(\frac{5x}{2}\right)}\right)$$

$$y = \log\left(1+\cos\left(\frac{x}{2}\right)^{\frac{1}{2}} - \log\left(1-\cos\left(\frac{5x}{2}\right)\right)^{\frac{1}{2}}\right.$$

$$y = \frac{1}{2}\log\left[1+\cos\left(\frac{5x}{2}\right)\right] - \frac{1}{2}\log\left[\left(1-\cos\left(\frac{5x}{2}\right)\right]\right.$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{1+\cos\left(\frac{5x}{2}\right)}\frac{d}{dx}\left(1+\cos\frac{5x}{2}\right) - \frac{1}{2} \times \frac{1}{1-\cos\left(\frac{5x}{2}\right)}\frac{d}{dx}\left(1-\cos\frac{5x}{2}\right)$$

$$= \frac{1}{2\left(1+\cos\left(\frac{5x}{2}\right)\right)}\left(-\sin\left(\frac{5x}{2}\right).\frac{5}{2} - \frac{1}{2\left(1-\cos\left(\frac{5x}{2}\right)\right)}\left(\sin\left(\frac{5x}{2}\right).\frac{5}{2}\right.\right.$$

$= \frac{-5\sin\left(\frac{5x}{2}\right)}{4\left(1+\cos\left(\frac{5x}{2}\right)\right)} - \frac{5\sin\left(\frac{5x}{2}\right)}{4\left(1-\cos\left(\frac{5x}{2}\right)\right)}$

$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right)\left[\frac{1}{1+\cos\left(\frac{5x}{2}\right)} + \frac{1}{1-\cos\left(\frac{5x}{2}\right)}\right]$

$= \frac{\frac{-5}{2}\sin\left(\frac{5x}{2}\right)\left[1-\cos\left(\frac{5x}{2}\right)+1+\cos\ \frac{5x}{2}\right]}{\left[1-\cos^2\left(\frac{5x}{2}\right)\right]}$

$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right) \times \frac{2}{\sin^2\left(\frac{5x}{2}\right)}$ ...[∵ $1 - \cos^2 x = \sin^2 x$]

$= \frac{-5}{4}\frac{1}{\sin\left(\frac{5x}{2}\right)}$

$-\frac{5}{2} \times$ cosecx

$-\frac{5}{2}\text{cosec}\left(\frac{5x}{2}\right)$

(xv) $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

Solution:

$\text{Let } y = \log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

$= \log\left(\sqrt{\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}}\right)$

$$= \log\left(\sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}}\right)$$

$$= \log\left(\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}\right)$$

$$= \log\left(\frac{1-\sin x}{\cos x}\right)$$

$$= \log\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \log(\sec x - \tan x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\sec x - \tan x)]$$

$$= \frac{1}{\sec x - \tan x} \cdot \frac{d}{dx}(\sec x - \tan x)$$

$$= \frac{1}{\sec x - \tan x} \times (\sec x \tan x - \sec^2 x)$$

$$= \frac{-\sec x(\sec x - \tan x)}{\sec x - \tan x}$$

$$= -\sec x.$$

(xvi) $\log\left[4^{2x}\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$

Solution:

$$\text{Let } y = \log\left[4^{2x}\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$$

$$= \log 4^{2x} + \log\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}$$

$$= 2x\log 4 + \frac{3}{2}\log\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)$$

$$= 2x\log 4 + \frac{3}{2}[\log(x^2+5) - \log(2x^3-4)^{\frac{1}{2}}]$$

$$= 2x\log 4 + \frac{3}{2}[\log(x^2+5) - \frac{1}{2}\log(2x^3-4)]$$

$$= 2x\log 4 + \frac{3}{2}\log(x^2+5) - \frac{3}{4}\log(2x^3-4)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[2x\log 4 + \frac{3}{2}\log(x^2+5) - \frac{3}{4}\log(2x^3-4)]$$

$$= (2\log 4)\frac{d}{dx}(x) + \frac{3}{2}\frac{d}{dx}[\log(x^2+5)] - \frac{3}{4}\frac{d}{dx}[\log(2x^3-4)]$$

$$= (2\log 4)\times 1 + \frac{3}{2}\times\frac{1}{x^2+5}\cdot\frac{d}{dx}(x^2+5) -$$

$$\frac{3}{4}\times\frac{1}{2x^3-4}\cdot\frac{d}{dx}(2x^3-4)$$

$$=2\log 4+\frac{3}{2(x^2+5)}\times(2x+0)-\frac{3}{4(2x^3-4)}\times(2\times 3x^2-0)$$

$$=2\log 4+\frac{3x}{x^2+5}-\frac{9x^2}{2(2x^3-4)}.$$

(xvii) $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

Solution:

Let y = $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

Using

log(A.B) = logA + logB

y = $\log e^{x^2}+\log\left(\frac{(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right)$

= $\log e^{x^2}+\log(5-4x)^{\frac{3}{2}}-\log\left(\sqrt[3]{7-6x}\right)$

= $x^2\log e+\frac{3}{2}\log(5-4x)-\log(7-6x)^{\frac{1}{3}}$

= $x^2+\frac{3}{2}\log(5-4x)-\frac{1}{3}\log(7-6x)$

Now,

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{3}{2}\frac{d}{dx}\log(5-4x) - \frac{1}{3}\frac{d}{dx}\log(7-6x)$$

$$= 2x + \frac{3}{2}\frac{1}{5-4x}(-4) - \frac{1}{3}\frac{1}{(7-6x)}x(-6)$$

$$= 2x - \frac{6}{(5-4x)} + \frac{2}{(7-6x)}$$

$$2x - \frac{6}{5-4x} + \frac{2}{7-6x}.$$

(xviii) $\log\left[\frac{a^{\cos x}}{(x^2-3)^3 \log x}\right]$

Solution:

$$\text{Let } y = \log\left[\frac{a^{\cos x}}{(x^2-3)^3 \log x}\right]$$

$$= \log a^{\cos x} - \log(x^2-3)^3 - \log(\log x)$$

$$= (\cos x)(\log a) - 3\log(x^2-3) - \log(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\cos x)(\log a) - 3\log(x^2-3) - \log(\log x)]$$

$$= (\log a)\cdot\frac{d}{dx}(\cos x) - 3\frac{d}{dx}[\log(x^2-3)] - \frac{d}{dx}[\log(\log x)]$$

$$= (\log a)(-\sin x) - 3\times\frac{1}{x^2-3}\cdot\frac{d}{dx}(x^2-3) - \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$= -(\sin x)(\log a) - \frac{3}{x^2-3} \times (2x-0) - \frac{1}{\log x} \times \frac{1}{x}$$

$$= -(\sin x)(\log a) - \frac{6x}{x^2-3} - \frac{1}{x \log x}.$$

(xix) y= $(25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

Solution:

y = $(25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$

= $5^{2\log_5(\sec x)} - 4^{2\log_4(\tan x)}$

= $5^{\log_5(\sec^5 x)} - 4^{\log_4(\tan^2 x)}$

= $\sec^2 x - \tan^2 x$ … [∵ = x]

∴ y = 1

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(1) = 0$

(xx) $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

Solution:

Let y = $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(x^2+2)^4}{\sqrt{x^2+5}}\right]$$

$$= \frac{\sqrt{x^2+5}\cdot\frac{d}{dx}(x^2+2)^4 - (x^2+2)^4\cdot\frac{d}{dx}(\sqrt{x^2+5})}{(\sqrt{x^2+5})^2}$$

$$= \frac{\sqrt{x^2+5}\times 4(x^2+2)^3\cdot\frac{d}{dx}(x^2+2) - (x^2+2)^4\times\frac{1}{2\sqrt{x^2+5}}\cdot\frac{d}{dx}(x^2+5)}{x^2+5}$$

$$= \frac{\sqrt{x^2+5}\times 4(x^2+2)^3\cdot(2x+0) - \frac{(x^2+2)^4}{2\sqrt{x^2+5}}\times(2x+0)}{x^2+5}$$

$$= \frac{8x(x^2+5)(x^2+2)^3 - x(x^2+2)^4}{(x^2+5)^{\frac{3}{2}}}$$

$$= \frac{x(x^2+2)^3[8(x^2+5)-(x^2+2)]}{(x^2+5)^{\frac{3}{2}}}$$

$$= \frac{x(x^2+2)^3(8x^2+40-x^2-2)}{(x^2+5)^{\frac{3}{2}}}$$

$$= \frac{x(x^2+2)^3(7x^2+38)}{(x^2+5)^{\frac{3}{2}}}.$$

Question 4.
A table of values of f, g, f ‘ and g’ is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	−3	4
4	3	4	5	−6
6	5	2	−4	7

(i) If r(x) = f [g(x)] find r’ (2).
Solution:
r(x) = f[g(x)]
$\therefore$ r'(x) = $\frac{d}{dx}$f[g(x)]
= f'[g(x)]·$\frac{d}{dx}$[g(x)]
= f'[g(x)·[g'(x)]
$\therefore$ r'(2) = f'[g(2)]·g'(2)
= f'(6)·g'(2) … [$\because$ g(x) = 6, when x = 2]
= -4 × 4 … [From the table]
= -16.

(ii) If R(x) = g[3 + f(x)] find R’ (4).
Solution:
R(x) = g[3 + f(x)]
$\therefore$ R'(x) = $\frac{d}{dx}${g[3+f(x)]}
= g'[3 + f(x)]·$\frac{d}{dx}$[3 + f(x)]
= g'[3 +f(x)]·[0 + f'(x)]
= g'[3 + f(x)]·f'(x)

= g'[3 + f(x)]·f'(x)

∴ R'(4) = g'[3 + f(4)]·f'(4)

= g'[3 + 3]·f'(4) … [∵ f(x) = 3, when x = 4]

= g'(6)·f'(4)

= 7 × 5 … [From the table]

= 35.

(iii) If s(x) = f[9– f(x)] find s' (4).

Solution:

s(x) = f[9– f(x)]

∴ s'(x) = $\frac{d}{dx}$ {f[9 – f(x)]}

= f'[9 – f(x)]· $\frac{d}{dx}$ [0 – f(x)]

= f'[9 – f(x)]·[0 – f'(x)]

= -f'[9 – f(x)] – f'(x)

∴ s'(4) = -f'[9 – f(4)] – f'(4)

= -f'[9 – 3] – f'(4) … [∵ f(x) = 3, when x = 4]

= -f'(6) – f'(4)

= -(-4)(5) … [From the table]

= 20.

(iv) If S(x) = g[g(x)] find S' (6)

Solution:

S(x) = g[g(x)]

∴ S'(x) = $\frac{d}{dx}$ g[g(x)]

= g'[g(x)]· $\frac{d}{dx}$ [g(x)]

$= g'[g(x)] \cdot \frac{d}{dx}[g(x)]$
$= g'[g(x)] \cdot g'(x)$
$\therefore S'(6) = g'[g'(6)] \cdot g'(6)$
$= g'(2) \cdot g'(6)$ … [∵ g (x) = 2, when x = 6]
= 4 × 7 … [From the table]
= 28.

Question 5.

Assume that f '(3) = -1, g'(2) = 5, g(2) = 3 and y = f[g(x)] then $\left[\frac{dy}{dx}\right]_{x=2}$ = ?

Solution:

y = f[g(x)]

$\therefore \frac{dy}{dx} = \frac{d}{dx}\{[g(x)]\}$

$= f'[g(x)] \cdot \frac{d}{dx}[g(x)]$

$= f'[g(x)] \cdot g'(x)$

$\therefore \left[\frac{dy}{dx}\right]_{x=2} = f'[g(2)] \cdot g'(2)$

$= f'(3) \cdot g'(2)$ … [∵ $g(2) = 3$]

$= -1 \times 5$ … (Given)

$= -5.$

Question 6.

If h(x) = $\sqrt{4f(x) + 3g(x)}$, f(1) = 4, g(1) = 3, f '(1) = 3, g'(1) = 4 find h'(1).

Given f(1) = 4, g(1) = 3, f '(1) = 3, g'(1) = 4 .....(1)

Now, $h(x)=\sqrt{4f(x)+3g(x)}$

$$\therefore\ h'(x)=\frac{d}{dx}[\sqrt{4f(x)+3g(x)}]$$

$$=\frac{1}{2\sqrt{4f(x)+3g(x)}}\cdot\frac{d}{dx}[4f(x)+3g(x)]$$

$$=\frac{1}{2\sqrt{4f(x)+3g(x)}}\times[4f'(x)+3g'(x)]$$

$$\therefore\ h'(1)=\frac{1}{2\sqrt{4f(1)+3g(1)}}\times[4f'(1)+3g'(1)]$$

$$=\frac{1}{2\sqrt{4\times4+3\times3}}\times[4\times3+3\times4]\quad\ldots\text{[By (1)]}$$

$$=\frac{1}{2\sqrt{25}}\times24$$

$$=\frac{1}{2\times5}\times24=\frac{12}{5}.$$

Question 7.

Find the x co-ordinates of all the points on the curve y = sin 2x – 2 sin x, 0 ≤ x < 2π where $\frac{dy}{dx}$ = 0.

Solution:

y = sin 2x – 2 sin x, 0 ≤ x < 2π

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\sin 2x-2\sin x)$$

$$= \frac{d}{dx}(\sin 2x) - 2\frac{d}{dx}(\sin x)$$

$$= \cos 2x \cdot \frac{d}{dx}(2x) - 2\cos x$$

= cos2x × 2 – 2cosx
= 2 (2 $\cos^2$x – 1) – 2 cosx
= 4 $\cos^2$x – 2 – 2 cos x
= 4 $\cos^2$x – 2 cos x – 2
If $\frac{dy}{dx}$ = 0, then 4 $\cos^2$x – 2 cos x – 2 = 0
∴ 4$\cos^2$x – 4cosx + 2cosx – 2 = 0
∴ 4 cosx (cosx – 1) + 2 (cosx – 1) = 0
∴ (cosx – 1)(4cosx + 2) = 0
∴ cosx – 1 = 0 or 4cosx + 2 = 0
∴ cos x = 1 or cos x = $-\frac{1}{2}$
∴ cos x = cos 0

$$\text{or} \quad \cos x = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \frac{\cos 2\pi}{3}$$

$$\text{or} \quad \cos x = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{4\pi}{3}$$

... [∵ $0 \leqslant x < 2\pi$]

$$\therefore\ x = 0 \quad \text{or} \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}.$$

$$\therefore\ x = 0 \quad \text{or} \quad \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3}.$$

Question 8.
Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]
"Let f (x) = x^2 + 5 and g(x) = e^x + 3 then
f [g(x)] = _________ and g [f(x)] =_________.
Now f '(x) = _________ and g'(x) = _________.
The derivative off [g (x)] w. r. t. x in terms of f and g is _________.
Therefore $\frac{d}{dx}$[f[g(x)]] = __________ and [$\frac{d}{dx}$[f[g(x)]]]$_{x=0}$ = ____________.
The derivative of g[f(x)] w. r. t. x in terms of f and g is _____________.
Therefore $\frac{d}{dx}$[g[f(x)]] = __________ and [$\frac{d}{dx}$[g[f(x)]]]$_{x=1}$ = ____________."
Hint basket : { f '[g(x)]·g'(x), $2e^{2x} + 6e^x$, 8, g'[f(x)]·f '(x), $2xe^{x^2+5}$, $-2e^6$, $e^{2x} + 6e^x + 14$, $e^{x^2+5} + 3$, 2x, e^x}
Solution:
f[g(x)] = $e^{2x} + 6e^x + 14$
g[f(x)] = $e^{x^2+5} + 3$
f'(x) = 2x, g'f(x) = e^x
The derivative of f[g(x)] w.r.t. x in terms of and g is f'[g(x)]·g'(x).
∴ $\frac{d}{dx}$\{f[g(x)]\} = $2e^{2x} + 6e^x$ and $\frac{d}{dx}$\{f[g(x)]\}$_{x=0}$ = 8
The derivative of g[f(x)] w.r.t. x in terms of f and g is g'f(x)]·f'(x).
∴ $\frac{d}{dx}$\{g[(f(x)]\} = $2xe^{x^2+5}$ and
$\frac{d}{dx}$\{g[(f(x)]\}$_{x=-1}$ = $-2e^6$.

Ex 1.2

Question 1.

Find the derivative of the function y = f (x) using the derivative of the inverse function x = f^{-1}(y) in the following

(i) y = x−−√

Solution:

y = x−−√ … (1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$y^2 = x \therefore x = y^2$

$$\therefore\ x = f^{-1}(y) = y^2$$

$$\therefore\ \frac{dx}{dy} = \frac{d}{dy}(y^2) = 2y$$

$$= 2\sqrt{x} \qquad \text{... [By (1)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{2\sqrt{x}}.$$

$= -4\sqrt{x}\sqrt{2-\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = -\frac{1}{4\sqrt{x}\sqrt{2-\sqrt{x}}}.$

(iii) y = $\sqrt[3]{x-2}$

Solution:

y = $\sqrt[3]{x-2}$ ….(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$y^3 = x - 2 \qquad \therefore x = y^3 + 2$

$\therefore x = f^{-1}(y) = y^3 + 2$

$\therefore \frac{dx}{dy} = \frac{d}{dy}(y^3 + 2)$

$= 3y^2 + 0 = 3y^2$

$= 3(\sqrt[3]{(x-2)})^2 \qquad \ldots$ [By (1)]

$= 3(x-2)^{\frac{2}{3}} = 3\cdot(\sqrt[3]{(x-2)^2})$

$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{3\sqrt[3]{(x-2)^2)}}, x > 2.$

(iv) y = log (2x – 1)

Solution:
y = log (2x – 1) …(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),

$2x-1=e^y \qquad \therefore 2x=e^y+1$

$\therefore x=f^{-1}(y)=\frac{1}{2}(e^y+1)$

$\therefore \frac{dx}{dy}=\frac{1}{2}\frac{d}{dy}(e^y+1)$

$=\frac{1}{2}(e^y+0)=\frac{1}{2}e^y$

$=\frac{1}{2}e^{\log(2x-1)} \qquad \ldots [\text{By } (1)]$

$=\frac{1}{2}(2x-1) \qquad \ldots [\because e^{\log x}=x]$

$\therefore \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{2}{2x-1}.$

(v) y = 2x + 3
Solution:
y = 2x + 3 ….(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),

$2x = y - 3 \qquad \therefore\ x = \frac{y-3}{2}$

$$\therefore\ x = f^{-1}(y) = \frac{y-3}{2}$$

$$\therefore\ \frac{dx}{dy} = \frac{1}{2}\frac{d}{dy}(y-3)$$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2.$$

(vi) $y = e^x - 3$

Solution:

$y = e^x - 3$ ….(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$e^x = y + 3$

$\therefore x = \log(y + 3)$

$\therefore x = f^{-1}(y) = \log(y + 3)$

$$\therefore\ \frac{dx}{dy} = \frac{d}{dy}[\log(y+3)]$$

$$= \frac{1}{y+3}\cdot\frac{d}{dy}(y+3)$$

$$= \frac{1}{y+3} \cdot (1+0) = \frac{1}{y+3}$$

$$= \frac{1}{e^x - 3 + 3} \quad \text{... [By (1)]}$$

$$= \frac{1}{e^x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\left(\frac{1}{e^x}\right)} = e^x.$$

(vii) $y = e^{2x-3}$

Solution:

$y = e^{2x-3}$ ....(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$2x - 3 = \log y \therefore 2x = \log y + 3$

$$\therefore x = f^{-1}(y) = \frac{1}{2}(\log y + 3)$$

$$\therefore \frac{dx}{dy} = \frac{1}{2}\frac{d}{dy}(\log y + 3)$$

$$= \frac{1}{2}\left(\frac{1}{y} + 0\right) = \frac{1}{2y}$$

$$\therefore \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{1}{\left(\frac{1}{2e^{2x-3}}\right)}=2e^{2x-3}.$$

(viii) $y = \log_2\left(\frac{x}{2}\right)$

Solution:

$y = \log_2\left(\frac{x}{2}\right)$ ...(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$\frac{x}{2} = 2^y \therefore x = 2 \cdot 2^y = 2^{y+1}$

$\therefore x = f^{-1}(y) = 2^{y+1}$

$$\therefore \frac{dx}{dy}=\frac{d}{dy}(2^{y+1})$$

$$=2^{y+1}\cdot\log 2\cdot\frac{d}{dy}(y+1)$$

$$=2^{y+1}\cdot\log 2\cdot(1+0)$$

$$=2^{y+1}\cdot\log 2=2^{\log_2\left(\frac{x}{2}\right)+1}\cdot\log 2 \quad \ldots \text{[By (1)]}$$

$$=2^{\log_2\left(\frac{x}{2}\right)+\log_2 2}\cdot\log 2$$

$$=2^{\log_2\left(\frac{x}{2}\times 2\right)}\cdot\log 2=2^{\log_2 x}\cdot\log 2$$

$= x\log 2$ ... $[\because a^{\log_a x} = x]$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{x\log 2}.$$

Question 2.
Find the derivative of the inverse function of the following
(i) $y = x^2 \cdot e^x$
Solution:
$y = x^2 \cdot e^x$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cdot e^x)$$

$$= x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$= x^2 \cdot e^x + e^x \times 2x$$

$$= xe^x(x+2)$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{xe^x(x+2)}.$$

(ii) y = x cos x

Solution:

y = x cos x

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos x)$$

$$=x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)$$

$$=x(-\sin x)+\cos x\times 1$$

$$=\cos x-x\sin x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\cos x-x\sin x}.$$

(iii) $y = x{\cdot}7^x$

Solution:

$y = x{\cdot}7^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot 7^x)$$

$$=x\frac{d}{dx}(7^x)+7^x\frac{d}{dx}(x)$$

$= x \cdot 7^x \log 7 + 7^x \times 1$

$= 7^x (x \log 7 + 1)$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{7^x (x \log 7 + 1)}.$$

(iv) $y = x^2 + \log x$

Solution:

$y = x^2 + \log x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \log x)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\log x)$$

$$= 2x + \frac{1}{x} = \frac{2x^2 + 1}{x}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{\left(\frac{2x^2+1}{x}\right)} = \frac{x}{2x^2+1}.$$

(v) y = x logx
Solution:
y = x logx
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\log x)$$
$$=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$
$$=x\times\frac{1}{x}+(\log x)\times 1$$
$$=1+\log x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{1+\log x}.$$

Question 3.
Find the derivative of the inverse of the following functions, and also fid their value at the points indicated against them.
(i) $y = x^5 + 2x^3 + 3x$, at x = 1
Solution:
$y = x^5 + 2x^3 + 3x$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^5 + 2x^3 + 3x)$

$= 5x^4 + 2 \times 3x^2 + 3 \times 1$
$= 5x^4 + 6x^2 + 3$
The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{5x^4 + 6x^2 + 3}$$

$$\text{At } x = 1, \frac{dx}{dy} = \frac{1}{(5x^4 + 6x^2 + 3)_{\text{at } x=1}}$$

$$= \frac{1}{5(1)^4 + 6(1)^2 + 3}$$

$$= \frac{1}{5 + 6 + 3} = \frac{1}{14}.$$

(ii) $y = e^x + 3x + 2$, at $x = 0$
Solution:
$y = e^x + 3x + 2$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(e^x + 3x + 2)$
The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{e^x + 3}$$

$$\text{At } x = 0, \frac{dx}{dy} = \frac{1}{(e^x + 3)_{\text{at } x=0}}$$

$$= \frac{1}{e^0 + 3} = \frac{1}{1+3} = \frac{1}{4}.$$

(iii) $y = 3x^2 + 2 \log x^3$, at $x = 1$

Solution:

$y = 3x^2 + 2 \log x^3$

$= 3x^2 + 6 \log x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2 + 6 \log x)$$

$$= 3\frac{d}{dx}(x^2) + 6\frac{d}{dx}(\log x)$$

$$= 3 \times 2x + 6 \times \frac{1}{x} = 6x + \frac{6}{x}$$

$$= \frac{6x^2 + 6}{x}$$

The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{\left(\frac{6x^2+6}{x}\right)}$$

$$= \frac{x}{6x^2 + 6}$$

At $x=1$, $\frac{dx}{dy}=\left(\frac{x}{6x^2+6}\right)_{\text{at } x=1}$

$$=\frac{1}{6(1)^2+6}=\frac{1}{12}.$$

(iv) $y = \sin(x - 2) + x^2$, at $x = 2$

Solution:

$y = \sin(x - 2) + x^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\sin(x-2)+x^2]$$

$$=\frac{d}{dx}[\sin(x-2)]+\frac{d}{dx}(x^2)$$

$$=\cos(x-2).\frac{d}{dx}(x-2)+2x$$

$$=\cos(x-2).(1-0)+2x$$

$$=\cos(x-2)+2x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\cos(x-2)+2x}$$

At $x=2$, $\frac{dx}{dy}=\left(\frac{1}{[\cos(x-2)+2x]}\right)_{\text{at } x=2}$

$$= \frac{1}{\cos 0 + 2(2)} = \frac{1}{1+4} = \frac{1}{5}.$$

Question 4.
If f(x) = x^3 + x – 2, find $(f^{-1})'$ (0).
Question is modified.
If f(x) = x^3 + x – 2, find $(f^{-1})'$ (-2).
Solution:
f(x) = x^3 + x – 2 ….(1)
Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(x^3 + x - 2)$$

$$= 3x^2 + 1 - 0 = 3x^2 + 1$$

We know that

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \dots (2)$$

From (1), $y = f(x) = -2$, when $x = 0$

$$\therefore \text{ from (2), } (f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{(3x^2+1)_{\text{at } x=0}}$$

$$= \frac{1}{3(0)+1} = 1.$$

Question 5.

Using derivative prove

(i) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

Solution:

let $f(x) = \tan^{-1}x + \cot^{-1}x$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(\tan^{-1}x + \cot^{-1}x)$$

$$= \frac{d}{dx}(\tan^{-1}x) + \frac{d}{dx}(\cot^{-1}x)$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Since, f'(x) = 0, f(x) is a constant function.

Let f(x) = k.

For any value of x, f(x) = k

Let x = 0.

Then f(0) = k ….(2)

From (1), $f(0) = \tan^{-1}(0) + \cot^{-1}(0)$

$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$$\therefore k = \frac{\pi}{2} \quad \text{… [By (2)]}$$

$$\therefore f(x) = k = \frac{\pi}{2}$$

Hence, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$. … [By (1)]

(ii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ . . . [for $|x| \geq 1$]

Let f(x) = $\sec^{-1}x + \text{cosec}^{-1}x$ for |x| ≥ 1 ....(1)

Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(\sec^{-1}x + \text{cosec}^{-1}x)$$

$$= \frac{d}{dx}(\sec^{-1}x) + \frac{d}{dx}(\text{cosec}^{-1}x)$$

$$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} = 0.$$

Since, f'(x) = 0, f(x) is a constant function.

Let f(x) = k.

For any value of x, f(x) = k, where |x| > 1

Let x = 2.

Then, f(2) = k ......(2)

From (1), $f(2) = \sec^{-1}(2) + \text{cosec}^{-1}(2) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

$\therefore\ k = \frac{\pi}{2}$ ... [By (2)]

$\therefore\ f(x) = k = \frac{\pi}{2}$

Hence, $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$. ... [By (1)]

Question 6.

Diffrentiate the following w. r. t. x.
(i) $\tan^{-1}(\log x)$
Solution:
Let $y = \tan^{-1}(\log x)$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\log x)]$$

$$= \frac{1}{1+(\log x)^2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{1+(\log x)^2} \times \frac{1}{x}$$

$$= \frac{1}{x[1+(\log x)^2]}.$$

(ii) $\operatorname{cosec}^{-1}(e^{-x})$
Solution:
Let $y = \operatorname{cosec}^{-1}(e^{-x})$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\operatorname{cosec}^{-1}(e^{-x})]$$

$$= \frac{-1}{e^{-x}\sqrt{(e^{-x})^2 - 1}} \cdot \frac{d}{dx}(e^{-x})$$

$$= \frac{-1}{e^{-x}\sqrt{e^{-2x} - 1}} \times e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{-1}{\sqrt{e^{-2x}-1}} \times -1$$

$$= \frac{1}{\sqrt{\frac{1}{e^{2x}}-1}}$$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}.$$

(iii) $\cot^{-1}(x^3)$

Solution:

Let $y = \cot^{-1}(x^3)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(x^3)]$$

$$= \frac{-1}{1+(x^3)^2} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{-1}{1+x^6} \times 3x^2$$

$$= \frac{-3x^2}{1+x^6}.$$

(iv) $\cot^{-1}(4^x$

Solution:

Let y = $\cot^{-1}(4^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cot^{-1}(4^x)]$$

$$=\frac{-1}{1+(4^x)^2}\cdot\frac{d}{dx}(4^x)$$

$$=\frac{-1}{1+4^{2x}}\times 4^x\log 4$$

$$=-\frac{4^x\log 4}{1+4^{2x}}.$$

(v) $\tan^{-1}(\sqrt{x})$

Solution:

Let y = $\tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}(1+x)}.$$

(vi) $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Solution:

Let $y = \sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x^2}{2}}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{1+x^2}{2}\right)}} \times \frac{1}{\sqrt{2}}\frac{d}{dx}(\sqrt{1+x^2})$$

$$= \frac{\sqrt{2}}{\sqrt{2-1-x^2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx}(1+x^2)$$

$$= \frac{1}{\sqrt{1-x^2}} \times \frac{1}{2\sqrt{1+x^2}} \cdot (0+2x)$$

$$= \frac{x}{\sqrt{(1-x^2)(1+x^2)}} = \frac{x}{\sqrt{1-x^4}}.$$

(vii) $\cos^{-1}(1-x^2)$

Solution:

Let y = $\cos^{-1}(1 - x^2)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\cos^{-1}(1 - x^2)]$$

$$= \frac{-1}{\sqrt{1-(1-x^2)^2}} \cdot \frac{d}{dx}(1 - x^2)$$

$$= \frac{-1}{\sqrt{1-(1-2x^2+x^4)}} \cdot (0 - 2x)$$

$$= \frac{2x}{\sqrt{2x^2 - x^4}}$$

$$= \frac{2x}{x\sqrt{2-x^2}} = \frac{2}{\sqrt{2-x^2}}.$$

(viii) $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Solution:

Let y = $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(x^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} x^{\frac{1}{2}}$

$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$.

(ix) $\cos^3[\cos^{-1}(x^3)]$
Solution:
Let y = $\cos^3[\cos^{-1}(x^3)]$
= $[\cos(\cos^{-1}x^3)]^3$
= $(x^3)^3 = x^9$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^9) = 9x^8$.

(x) $\sin^4[\sin^{-1}(\sqrt{x})]$
Solution:
Let y = $\sin^4[\sin^{-1}(\sqrt{x})]$
= $\{\sin[\sin^{-1}(\sqrt{x})]\}^8$
= $(\sqrt{x})^4 = x^2$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$.

Question 7.
Diffrentiate the following w. r. t. x.
(i) $\cot^{-1}[\cot(e^{x^2})]$
Solution:

Let $y = \cot^{-1}[\cot(e^{x^2})] = e^{x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot \frac{d}{dx}(x^2)$$

$$= e^{x^2} \times 2x = 2xe^{x^2}.$$

(ii) $\text{cosec}^{-1}\left(\frac{1}{\cos(5^x)}\right)$

Solution:

$$\text{Let } y = \text{cosec}^{-1}\left[\frac{1}{\cos(5^x)}\right]$$

$$= \text{cosec}^{-1}[\sec(5^x)]$$

$$= \text{cosec}^{-1}\left[\text{cosec}\left(\frac{\pi}{2} - 5^x\right)\right]$$

$$= \frac{\pi}{2} - 5^x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 5^x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(5^x)$$

$$= 0 - 5^x \cdot \log 5$$

$$= -5^x \cdot \log 5.$$

(iii) $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$

Solution:

$$\text{Let } y=\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$$

$$=\cos^{-1}\left(\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2}}\right)$$

$$=\cos^{-1}\left[\cos\left(\frac{x}{2}\right)\right]$$

$$=\frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{1}{2}\frac{d}{dx}(x)$$

$$=\frac{1}{2}\times 1=\frac{1}{2}.$$

(iv) $\cos^{-1}\left(\sqrt{\frac{1-\cos(x^2)}{2}}\right)$

Solution:

$$\text{Let } y=\cos^{-1}\left(\sqrt{\frac{1-\cos(x^2)}{2}}\right)$$

$$= \cos^{-1}\left(\sqrt{\frac{2\sin^2\left(\frac{x^2}{2}\right)}{2}}\right)$$

$$= \cos^{-1}\left[\sin\left(\frac{x^2}{2}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - \frac{x^2}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x^2}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x^2}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{1}{2}\frac{d}{dx}(x^2)$$

$$= 0 - \frac{1}{2} \times 2x = -x.$$

(v) $\tan^{-1}\left(\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right)$

Solution:

$$\text{Let } y = \tan^{-1}\left[\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\cdot\tan\left(\frac{x}{2}\right)}\right] \quad \ldots \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{1}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{1}{2} \times 1 = -\frac{1}{2}.$$

(vi) $\operatorname{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$

Solution:

Let $y = \operatorname{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\cos 6x}\right) \quad \ldots [\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$= \text{cosec}^{-1}(\sec 6x)$

$= \text{cosec}^{-1}\left[\text{cosec}\left(\frac{\pi}{2} - 6x\right)\right]$

$= \frac{\pi}{2} - 6x$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 6x\right)$

$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 6\frac{d}{dx}(x)$

$= 0 - 6 \times 1 = -6.$

(vii) $\tan^{-1}\left(\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right)$

Solution:

Let $y = \tan^{-1}\left[\frac{1 + \cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right]$

$= \tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{6}\right)}{2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)}\right]$

$$= \tan^{-1}\left[\cot\left(\frac{x}{6}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{6}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x}{6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{6}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{1}{6}\frac{d}{dx}(x)$$

$$= 0 - \frac{1}{6} \times 1 = -\frac{1}{6}.$$

(viii) $\cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

Solution:

Let $y = \cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

$$= \cot^{-1}\left[\frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\tan\left(\frac{3x}{2}\right)\right]$$

$$= \cot^{-1}\left[\cot\left(\frac{\pi}{2} - \frac{3x}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{3x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{3x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{3}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{3}{2} \times 1 = -\frac{3}{2}.$$

(ix) $\tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$

Solution:

$$\text{Let } y = \tan^{-1}\left(\frac{\cos 7x}{1 + \sin 7x}\right)$$

$$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2} - 7x\right)}{1 + \cos\left(\frac{\pi}{2} - 7x\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{7x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{7x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{7x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{7x}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{7x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}-\frac{7x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{7}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{7}{2}\times 1 = -\frac{7}{2}.$$

(x) $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

Solution:

Let $y = \tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$= \tan^{-1}\left[\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}}\right]$$

$= \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$

$= \frac{\pi}{2}-\frac{x}{2}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)$

$=\frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{d}{dx}(x)$

$=0-\frac{1}{2}\times 1=-\frac{1}{2}.$

(xi) $\tan^{-1}(\text{cosec } x + \cot x)$

Solution:

Let $y = \tan^{-1}(\text{cosec } x + \cot x)$

$= \tan^{-1}\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)$

$= \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2}-\frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{d}{dx}(x)$$

$$= 0-\frac{1}{2}\times 1 = -\frac{1}{2}.$$

(xii) $\cot^{-1}\left(\frac{\sqrt{1+\sin\left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}\right)$

Solution:

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin\left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}\right]$

$$1+\sin\left(\frac{4x}{3}\right)=1+\cos\left(\frac{\pi}{2}-\frac{4x}{3}\right)=2\cos^2\left(\frac{\pi}{4}-\frac{2x}{3}\right)$$

$$\therefore \sqrt{1+\sin\left(\frac{4x}{3}\right)}=\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)$$

Also, $1-\sin\left(\frac{4x}{3}\right)=1-\cos\left(\frac{\pi}{2}-\frac{4x}{3}\right)=2\sin^2\left(\frac{\pi}{4}-\frac{2x}{3}\right)$

$$\therefore \sqrt{1-\sin\left(\frac{4x}{3}\right)}=\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)$$

$$\therefore \frac{\sqrt{1+\sin\left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}$$

$$=\frac{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$=\frac{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$=\frac{1+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots \left[\text{Dividing by } \cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)\right]$$

$$=\frac{\tan\frac{\pi}{4}+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\frac{\pi}{4}\cdot\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots \left[\because \tan\frac{\pi}{4}=1\right]$$

$$=\tan\left[\frac{\pi}{4}+\frac{\pi}{4}-\frac{2x}{3}\right]=\tan\left(\frac{\pi}{2}-\frac{2x}{3}\right)$$

$$=\cot\left(\frac{2x}{3}\right)$$

$$\therefore\ y=\cot^{-1}\left[\cot\left(\frac{2x}{3}\right)\right]=\frac{2x}{3}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{3}\right)=\frac{2}{3}\frac{d}{dx}(x)$$

$$=\frac{2}{3}\times 1=\frac{2}{3}.$$

Question 8.

(i) $\sin^{-1}\left(\frac{4\sin x + 5\cos x}{\sqrt{41}}\right)$

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{4\sin x + 5\cos x}{\sqrt{41}}\right)$$

$$= \sin^{-1}\left[(\sin x)\left(\frac{4}{\sqrt{41}}\right) + (\cos x)\left(\frac{5}{\sqrt{41}}\right)\right]$$

Since, $\left(\frac{4}{\sqrt{41}}\right)^2 + \left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41} + \frac{25}{41} = 1,$

we can write, $\frac{4}{\sqrt{41}} = \cos\alpha$ and $\frac{5}{\sqrt{41}} = \sin\alpha.$

$$\therefore\ y = \sin^{-1}(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$= \sin^{-1}[\sin(x+\alpha)]$$

$$= x + \alpha, \quad \text{where } \alpha \text{ is a constant}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x+\alpha)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(\alpha)$$

$$= 1 + 0 = 1.$$

(ii) $\cos^{-1}\left(\frac{\sqrt{3}\cos x-\sin x}{2}\right)$

Solution:

$$\text{Let } y=\cos^{-1}\left(\frac{\sqrt{3}\cos x-\sin x}{2}\right)$$

$$=\cos^{-1}\left[(\cos x)\left(\frac{\sqrt{3}}{2}\right)-(\sin x)\left(\frac{1}{2}\right)\right]$$

$$=\cos^{-1}\left(\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}\right)$$

$$\dots\left[\because\ \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2},\ \sin\frac{\pi}{6}=\frac{1}{2}\right]$$

$$=\cos^{-1}\left[\cos\left(x+\frac{\pi}{6}\right)\right]$$

$$=x+\frac{\pi}{6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+\frac{\pi}{6}\right)$$

$$=\frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{\pi}{6}\right)$$

$$=1+0=1.$$

(iii) $\sin^{-1}\left(\frac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

Solution:

y = $\sin^{-1}\left(\frac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

= $\sin^{-1}\left(\frac{1}{\sqrt{2}}\cos\sqrt{x}+\frac{1}{\sqrt{2}}\sin\sqrt{x}\right)$

Put,

$\frac{1}{\sqrt{2}}$ = sinx

$\frac{1}{\sqrt{2}}$ = cosα

Also,

$\sin^2\alpha + \cos^2\alpha = \left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2 = 1$

And,

tanα = 1

∴ $\alpha = \tan^{-1}1$

y = $\sin^{-1}(\sin\alpha.\cos\sqrt{x}+\cos\alpha.\sin(x)$

= $\sin^{-1}(\sin(\alpha+\sqrt{x}))$

y = $\alpha+\sqrt{x}$

y = $\tan^{-1}(1)+\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}+\sqrt{x})$$

$$= 0 + \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

(iv) $\cos^{-1}\left(\frac{3\cos 3x - 4\sin 3x}{5}\right)$

Solution:

$$\text{Let } y = \cos^{-1}\left(\frac{3\cos 3x - 4\sin 3x}{5}\right)$$

$$= \cos^{-1}\left[(\cos 3x)\left(\frac{3}{5}\right) - (\sin 3x)\left(\frac{4}{5}\right)\right]$$

Since, $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1,$

we can write, $\frac{3}{5} = \cos\alpha$ and $\frac{4}{5} = \sin\alpha.$

$$\therefore\ y = \cos^{-1}(\cos 3x\cos\alpha - \sin 3x\sin\alpha)$$

$$= \cos^{-1}[\cos(3x+\alpha)]$$

$$= 3x + \alpha, \quad \text{where } \alpha \text{ is a constant.}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(3x + \alpha)$$

$$= 3\frac{d}{dx}(x) + \frac{d}{dx}(\alpha)$$

$$= 3 \times 1 + 0 = 3.$$

(v) $\cos^{-1}\left(\frac{3\cos(e^x) + 2\sin(e^x)}{\sqrt{13}}\right)$

Solution:

$y = \cos^{-1}\left(\frac{3\cos(e^x) + 2\sin(e^x)}{\sqrt{13}}\right)$

$= \cos^{-1}\left(\cos(e^x).\frac{3}{\sqrt{13}} + \sin(e^x)\frac{2}{\sqrt{13}}\right)$

Put,

$\frac{3}{\sqrt{13}}$ = cosx

$\frac{2}{\sqrt{3}}$ = sinx

Also,

$\sin^2\alpha + \cos^2\alpha = \frac{9}{13} + \frac{4}{13} = 1$

And,

$\tan\alpha = \frac{\sin x}{\cos\alpha} = \frac{2}{3}$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$

$y = \cos^{-1}(\cos e^x . \cos\alpha + \sin e^x . \cos\alpha)$

$y = \cos^{-1}(\cos e^x - \alpha))$ $\quad \because \cos^{-1}x . (\cos x) = x$

$y = e^x - \alpha$

$= e^x = \tan^{-1}\left(\frac{2}{3}\right)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= e^x - 0$

$= e^x$.

(vi) $\operatorname{cosec}^{-1}\left(\frac{10}{6\sin(2^x) - 8\cos(2^x)}\right)$

Solution:

Let $y = \operatorname{cosec}^{-1}\left[\frac{10}{6\sin(2^x) - 8\cos(2^x)}\right]$

$= \sin^{-1}\left[\frac{6\sin(2^x) - 8\cos(2^x)}{10}\right]$

$$\dots \left[\because \operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \sin^{-1}\left[\{\sin(2^x)\}\left(\frac{6}{10}\right) - \{\cos(2^x)\}\left(\frac{8}{10}\right)\right]$$

Since, $\left(\frac{6}{10}\right)^2 + \left(\frac{8}{10}\right)^2 = \frac{36}{100} + \frac{64}{100} = 1,$

we can write, $\frac{6}{10} = \cos\alpha$ and $\frac{8}{10} = \sin\alpha.$

y = $\sin^{-1}[\sin(2^x)\cdot\cos a - \cos(2^x)\cdot\sin a]$

= $\sin^{-}[\sin(2^x - a)]$

= $2^x - a$, where a is a constant

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(2^x - a)$

= $\frac{d}{dx}(2^x) - \frac{d}{dx}(a)$

= $2^x\cdot\log 2 - 0$

= $2^x\cdot\log 2$

Question 9.

Diffrentiate the following w. r. t. x.

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$

$\therefore\ y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x) = 2\frac{d}{dx}(\tan^{-1}x)$$

$$= 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

(ii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution:

$$\text{Let } y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Put $x = \tan\theta$. Then $\theta = \tan^{-1}x$

$$\therefore\ y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x) = 2\frac{d}{dx}(\tan^{-1}x)$$

$$= 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

(iii) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$

$$\therefore \; y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\tan^{-1} x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 2\frac{d}{dx}(\tan^{-1} x)$$

$$= 0 - 2 \times \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

(iv) $\sin^{-1}(2x\sqrt{1-x^2})$

Solution:

$$\text{Let } y = \sin^{-1}(2x\sqrt{1-x^2})$$

Put $x = \sin\theta$. Then $\theta = \sin^{-1}x$

$$\therefore\ y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$
$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta = 2\sin^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin^{-1}x) = 2\frac{d}{dx}(\sin^{-1}x)$$
$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}.$$

We can also put $x = \cos\theta$. Then $\theta = \cos^{-1}x$

$$\therefore\ y = \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$
$$= \sin^{-1}(2\cos\theta\sin\theta) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta = 2\cos^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1}x) = 2\frac{d}{dx}(\cos^{-1}x)$$
$$= 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx} = \pm\frac{2}{\sqrt{1-x^2}}$.

(v) $\cos^{-1}(3x - 4x^3)$

(v) $\cos^{-1}(3x - 4x^3)$

Solution:

Let $y = \cos^{-1}(3x - 4x^3)$

Put $x = \sin\theta$. Then $\theta = \sin^{-1}x$

$\therefore\ y = \cos^{-1}(3\sin\theta - 4\sin^3\theta)$

$$= \cos^{-1}(\sin 3\theta) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 3\theta\right)\right]$$

$$= \frac{\pi}{2} - 3\theta = \frac{\pi}{2} - 3\sin^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 3\sin^{-1}x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 3\frac{d}{dx}(\sin^{-1}x)$$

$$= 0 - 3 \times \frac{1}{\sqrt{1-x^2}} = \frac{-3}{\sqrt{1-x^2}}.$$

(vi) $\cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

Solution:

Let $y = \cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$$= \cos^{-1}\left[\frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}\right]$$

$$= \cos^{-1}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right)$$

Put $e^x = \tan\theta$. Then $\theta = \tan^{-1}(e^x)$

$$\therefore\ y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) = \cos^{-1}\left[-\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)\right]$$

$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta = \pi - 2\tan^{-1}(e^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\pi - 2\tan^{-1}(e^x)]$$

$$= \frac{d}{dx}(\pi) - 2\frac{d}{dx}[\tan^{-1}(e^x)]$$

$$= 0 - 2 \times \frac{1}{1 + (e^x)^2} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{-2}{1 + e^{2x}} \times e^x = -\frac{2e^x}{1 + e^{2x}}.$$

(vii) $\cos^{-1}\left(\frac{1-9^x}{1+9^x}\right)$

Solution:

$$\text{Let } y=\cos^{-1}\left(\frac{1-9^x}{1+9^x}\right)=\cos^{-1}\left[\frac{1-(3^x)^2}{1+(3^x)^2}\right]$$

Put $3^x=\tan\theta$. Then $\theta=\tan^{-1}(3^x)$

$$\therefore\ y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)$$

$$=2\theta=2\tan^{-1}(3^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[2\tan^{-1}(3^x)]=2\frac{d}{dx}[\tan^{-1}(3^x)]$$

$$=2\times\frac{1}{1+(3^x)^2}\cdot\frac{d}{dx}(3^x)$$

$$=\frac{2}{1+3^{2x}}\times 3^x\log 3$$

$$=\frac{2\cdot 3^x\log 3}{1+3^{2x}}.$$

(viii) $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$$

$$= \sin^{-1}\left[\frac{4^x \cdot 4^{\frac{1}{2}}}{1+(2^2)^{2x}}\right]$$

$$= \sin^{-1}\left(\frac{2\cdot 4^x}{1+4^{2x}}\right)$$

Put $4^x = \tan\theta$, Then $\theta = \tan^{-1}(4^x)$

$$\therefore\ y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}(4^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}(4^x)] = 2\frac{d}{dx}[\tan^{-1}(4^x)]$$

$$= 2 \times \frac{1}{1+(4^x)^2} \cdot \frac{d}{dx}(4^x)$$

$$= \frac{2}{1+4^{2x}} \times 4^x \log 4$$

$$= \frac{2\cdot 4^x \log 4}{1+4^{2x}}.$$

Note : The answer can also be written as :

$$\frac{dy}{dx}=\frac{4^{\frac{1}{2}}\cdot 4^{x}\log 4}{1+4^{2x}}=\frac{4^{x+\frac{1}{2}}\cdot\log 4}{1+4^{2x}}.$$

(ix) $\sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right)$

Solution:

$$\text{Let } y=\sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right)=\sin^{-1}\left[\frac{1-(5x)^2}{1+(5x)^2}\right]$$

Put $5x=\tan\theta$. Then $\theta=\tan^{-1}(5x)$

$$\therefore\ y=\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\cos 2\theta)$$

$$=\sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]=\frac{\pi}{2}-2\theta$$

$$=\frac{\pi}{2}-2\tan^{-1}(5x)$$

Differentiating w.r.t. x, we get

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left[\frac{\pi}{2}-2\tan^{-1}(5x)\right]$$

$$=\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}[\tan^{-1}(5x)]$$

$$=0-2\times\frac{1}{1+(5x)^2}\cdot\frac{d}{dx}(5x)$$

$$=\frac{-2}{1+25x^2}\times 5=\frac{-10}{1+25x^2}.$$

(x) $\sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$

Solution:

$$\text{Let } y=\sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$$

$$=\sin^{-1}\left[\frac{1-\left(x^{\frac{3}{2}}\right)^2}{1+\left(x^{\frac{3}{2}}\right)^2}\right]$$

Put $x^{\frac{3}{2}}=\tan\theta$. Then $\theta=\tan^{-1}\left(x^{\frac{3}{2}}\right)$

$$\therefore\ y=\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\cos 2\theta)$$

$$=\sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]=\frac{\pi}{2}-2\theta$$

$$=\frac{\pi}{2}-2\tan^{-1}(x^{\frac{3}{2}})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{\pi}{2}-2\tan^{-1}(x^{\frac{3}{2}})\right]$$

$$=\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}[\tan^{-1}(x^{\frac{3}{2}})]$$

$$=0-2\times\frac{1}{1+\left(x^{\frac{3}{2}}\right)^2}\cdot\frac{d}{dx}(x^{\frac{3}{2}})$$

$$= -\frac{2}{1+x^3} \times \frac{3}{2} x^{\frac{1}{2}}$$

$$= -\frac{3\sqrt{x}}{1+x^3}.$$

(xi) $\tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$

$$\text{Let } y = \tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$$

Put $x^{\frac{5}{2}} = \tan\theta$. Then $\theta = \tan^{-1}(x^{\frac{5}{2}})$

$$\therefore \ y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta = 2\tan^{-1}(x^{\frac{5}{2}})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}(x^{\frac{5}{2}})]$$

$$= 2\frac{d}{dx}[\tan^{-1}(x^{\frac{5}{2}})]$$

$$= 2 \times \frac{1}{1+\left(x^{\frac{5}{2}}\right)^2} \cdot \frac{d}{dx}(x^{\frac{5}{2}})$$

$$= \frac{2}{1+x^5} \times \frac{5}{2} x^{\frac{3}{2}}$$

$$= \frac{5x\sqrt{x}}{1+x^5}.$$

(xii) $\cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

Solution:

Let $y = \cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

$$= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \quad \ldots \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-1 \times \sqrt{x}}\right)$$

$$= \tan^{-1}(1) + \tan^{-1}(\sqrt{x})$$

$$\ldots \left[\because \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} x + \tan^{-1} y\right]$$

$$= \frac{\pi}{4} + \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{4} + \tan^{-1}(\sqrt{x})\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$= 0 + \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1+x)}.$$

Question 10.
Diffrentiate the following w. r. t. x.
(i) $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$
Solution:
Let y = $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

$$= \tan^{-1}\left[\frac{5x+3x}{1-(5x)(3x)}\right]$$

$$= \tan^{-1}(5x) + \tan^{-1}(3x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(5x) + \tan^{-1}(3x)]$$

$$= \frac{d}{dx}[\tan^{-1}(5x)] + \frac{d}{dx}[\tan^{-1}(3x)]$$

$$= \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x) + \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x)$$

$$=\frac{1}{1+25x^2}\times 5+\frac{1}{1+9x^2}\times 3$$

$$=\frac{5}{1+25x^2}+\frac{3}{1+9x^2}.$$

(ii) $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

$$=\tan^{-1}\left(\frac{2x}{1+35x^2}\right) \quad \dots \left[\because \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$=\tan^{-1}\left[\frac{7x-5x}{1+(7x)(5x)}\right]$$

$$=\tan^{-1}(7x)-\tan^{-1}(5x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(7x)-\tan^{-1}(5x)]$$

$$=\frac{d}{dx}[\tan^{-1}(7x)]-\frac{d}{dx}[\tan^{-1}(5x)]$$

$$=\frac{1}{1+(7x)^2}\cdot\frac{d}{dx}(7x)-\frac{1}{1+(5x)^2}\cdot\frac{d}{dx}(5x)$$

$$= \frac{1}{1+49x^2} \times 7 - \frac{1}{1+25x^2} \times 5$$

$$= \frac{7}{1+49x^2} - \frac{5}{1+25x^2}.$$

(iii) $\tan^{-1}\left(\frac{2\sqrt{x}}{1+3x}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{2\sqrt{x}}{1+3x}\right)$

$$= \tan^{-1}\left[\frac{3\sqrt{x}-\sqrt{x}}{1+(3\sqrt{x})(\sqrt{x})}\right]$$

$$= \tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})]$$

$$= \frac{d}{dx}[\tan^{-1}(3\sqrt{x})] - \frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$= \frac{1}{1+(3\sqrt{x})^2} \cdot \frac{d}{dx}(3\sqrt{x}) - \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+9x} \times 3 \times \frac{1}{2\sqrt{x}} - \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\left[\frac{3}{1+9x}-\frac{1}{1+x}\right].$$

(iv) $\tan^{-1}\left(\frac{2^{x+2}}{1-3(4^x)}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{2^{x+2}}{1-3(4^x)}\right)$

$$=\tan^{-1}\left[\frac{2^2\cdot 2^x}{1-3(4^x)}\right]$$

$$=\tan^{-1}\left[\frac{4\cdot 2^x}{1-3(4^x)}\right]$$

$$=\tan^{-1}\left[\frac{3\cdot 2^x+2^x}{1-(3\cdot 2^x\times 2^x)}\right]$$

$$=\tan^{-1}(3\cdot 2^x)+\tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(3\cdot 2^x)+\tan^{-1}(2^x)]$$

$$=\frac{d}{dx}[\tan^{-1}(3\cdot 2^x)]+\frac{d}{dx}[\tan^{-1}(2^x)]$$

$$=\frac{1}{1+(3\cdot 2^x)^2}\cdot\frac{d}{dx}(3\cdot 2^x)+\frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$=\frac{1}{1+9(2^{2x})}\times 3\times 2^x\log 2+\frac{1}{1+2^{2x}}\times 2^x\log 2$$

$$= 2^x \log 2\left[\frac{3}{1+9(2^{2x})} + \frac{1}{1+2^{2x}}\right].$$

(v) $\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

$$= \tan^{-1}\left[\frac{2 \cdot 2^x - 2^x}{1+(2 \cdot 2^x)(2^x)}\right]$$

$$= \tan^{-1}(2 \cdot 2^x) - \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(2 \cdot 2^x) - \tan^{-1}(2^x)]$$

$$= \frac{d}{dx}[\tan^{-1}(2 \cdot 2^x)] - \frac{d}{dx}[\tan^{-1}(2^x)]$$

$$= \frac{1}{1+(2 \cdot 2^x)^2} \cdot \frac{d}{dx}(2 \cdot 2^x) - \frac{1}{1+(2^x)^2} \cdot \frac{d}{dx}(2^x)$$

$$= \frac{1}{1+4(2^{2x})} \times 2 \times 2^x \log 2 - \frac{1}{1+2^{2x}} \times 2^x \log 2$$

$$= 2^x \log 2\left[\frac{2}{1+4(2^{2x})} - \frac{1}{1+2^{2x}}\right].$$

(vi) $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

Solution:

Let $y = \cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

$$= \tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right) \quad \ldots \left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{5\left(\frac{x}{a}\right)}{1-6\left(\frac{x}{a}\right)^2}\right] \quad \ldots \text{[Dividing by } a^2\text{]}$$

$$= \tan^{-1}\left[\frac{3\left(\frac{x}{a}\right)+2\left(\frac{x}{a}\right)}{1-3\left(\frac{x}{a}\right)\times 2\left(\frac{x}{a}\right)}\right]$$

$$= \tan^{-1}\left(\frac{3x}{a}\right)+\tan^{-1}\left(\frac{2x}{a}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{a}\right)+\tan^{-1}\left(\frac{2x}{a}\right)\right]$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{a}\right)\right]+\frac{d}{dx}\left[\tan^{-1}\left(\frac{2x}{a}\right)\right]$$

$$= \frac{1}{1+\left(\frac{3x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{3x}{a}\right)+\frac{1}{1+\left(\frac{2x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{a}\right)$$

$$= \frac{1}{1+\left(\frac{9x^2}{a^2}\right)} \times \frac{3}{a} \times 1 + \frac{1}{1+\left(\frac{4x^2}{a^2}\right)} \times \frac{2}{a} \times 1$$

$$= \frac{a^2}{a^2+9x^2} \times \frac{3}{a} + \frac{a^2}{a^2+4x^2} \times \frac{2}{a}$$

$$= \frac{3a}{a^2+9x^2} + \frac{2a}{a^2+4x^2}.$$

(vii) $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

Solution:

Let $y = \tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

$$= \tan^{-1}\left[\frac{\frac{a}{b}+\tan x}{1-\frac{a}{b}\cdot\tan x}\right]$$

$$= \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(\tan x)$$

$$= \tan^{-1}\left(\frac{a}{b}\right) + x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right) + x\right]$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right)\right] + \frac{d}{dx}(x)$$

$= 0 + 1 = 1.$

(viii) $\tan^{-1}\left(\frac{5-x}{6x^2-5x-3}\right)$

Solution:

Let $y = \tan^{-1}\left(\frac{5-x}{6x^2-5x-3}\right)$

$= \tan^{-1}\left[\frac{5-x}{1+(6x^2-5x-4)}\right]$

$$= \tan^{-1}\left[\frac{(2x+1)-(3x-4)}{1+(2x+1)(3x-4)}\right]$$

$$= \tan^{-1}(2x+1) - \tan^{-1}(3x-4)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(2x+1)] - \frac{d}{dx}[\tan^{-1}(3x-4)]$$

$$= \frac{1}{1+(2x+1)^2}\cdot\frac{d}{dx}(2x+1) - \frac{1}{1+(3x-4)^2}\cdot\frac{d}{dx}(3x-4)$$

$$= \frac{1}{1+(2x+1)^2}\cdot(2\times 1+0) - \frac{1}{1+(3x-4)^2}\cdot(3\times 1-0)$$

$$= \frac{2}{1+(2x+1)^2} - \frac{3}{1+(3x-4)^2}.$$

(ix) $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$

$$= \tan^{-1}\left(\frac{3x+2}{4-x-2x^2}\right) \ldots \left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{3x+2}{1-(2x^2+x-3)}\right]$$

$$= \tan^{-1}\left[\frac{(2x+3)+(x-1)}{1-(2x+3)(x-1)}\right]$$

$$= \tan^{-1}(2x+3) + \tan^{-1}(x-1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(2x+3) + \tan^{-1}(x-1)]$$

$$= \frac{d}{dx}[\tan^{-1}(2x+3)] + \frac{d}{dx}[\tan^{-1}(x-1)]$$

$$= \frac{1}{1+(2x+3)^2} \cdot \frac{d}{dx}(2x+3) + \frac{1}{1+(x-1)^2} \cdot \frac{d}{dx}(x-1)$$

$$= \frac{1}{1+(2x+3)^2} \cdot (2 \times 1 + 0) + \frac{1}{1+(x-1)^2} \cdot (1-0)$$

$$= \frac{2}{1+(2x+3)^2} + \frac{1}{1+(x-1)^2}.$$

Ex 1.3

Question 1.

Differentiate the following w.r.t. x:

(i) $\frac{(x+1)^{2}}{(x+2)^{3}(x+3)^{4}}$

Solution:

Let y = $\frac{(x+1)^{2}}{(x+2)^{3}(x+3)^{4}}$

Then, log y = log [latex]\frac{(x+1)^{2}}{(x+2)^{3}(x+3)^{4}}[/latex]

= log $(x + 1)^2$ – log $(x + 2)^3$ – log $(x + 3)^4$

= 2 log (x +1) – 3 log (x + 2) – 4 log (x + 3)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=2\frac{d}{dx}[\log(x+1)]-3\frac{d}{dx}[\log(x+2)]-4\frac{d}{dx}[\log(x+3)]$$

$$=2\times\frac{1}{x+1}\cdot\frac{d}{dx}(x+1)-3\times\frac{1}{x+2}\cdot\frac{d}{dx}(x+2)-4\times\frac{1}{x+3}\cdot\frac{d}{dx}(x+3)$$

$$=\frac{2}{x+1}\cdot(1+0)-\frac{3}{x+2}\cdot(1+0)-\frac{4}{x+3}\cdot(1+0)$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{2}{x+1}-\frac{3}{x+2}-\frac{4}{x+3}\right]$$

$$=\frac{(x+1)^2}{(x+2)^2(x+3)^4}\cdot\left[\frac{2}{x+1}-\frac{3}{x+2}-\frac{4}{x+3}\right].$$

(ii) $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

Solution:

Let y = $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

Then log y = log [latex]\sqrt[3]{\frac{4 x-1}{(2 x+3)(5-2 x)^{2}}}[/latex]

$$=\frac{1}{3}\log\left[\frac{4x-1}{(2x+3)(5-2x)^2}\right]$$

$$=\frac{1}{3}[\log(4x-1)-\log(2x+3)(5-2x)^2]$$

$$=\frac{1}{3}\log(4x-1)-\frac{1}{3}\log(2x+3)-\frac{2}{3}\log(5-2x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{3}\frac{d}{dx}[\log(4x-1)]-\frac{1}{3}\frac{d}{dx}[\log(2x+3)]-\frac{2}{3}\frac{d}{dx}[\log(5-2x)]$$

$$=\frac{1}{3}\times\frac{1}{4x-1}\cdot\frac{d}{dx}(4x-1)-\frac{1}{3}\times\frac{1}{2x+3}\cdot\frac{d}{dx}(2x+3)-\frac{2}{3}\times\frac{1}{5-2x}\cdot\frac{d}{dx}(5-2x)$$

$$=\frac{1}{3(4x-1)}\cdot(4\times1-0)-\frac{1}{3(2x+3)}\cdot(2\times1+0)-$$

$$\frac{2}{3(5-2x)}\cdot(0-2\times 1)$$

$$\therefore \frac{dy}{dx}=y\left[\frac{4}{3(4x-1)}-\frac{2}{3(2x+3)}+\frac{4}{3(5-2x)}\right]$$

$$=\sqrt[3]{\frac{(4x-1)}{(2x+3)(5-2x)^2}}\left[\frac{4}{3(4x-1)}-\frac{2}{3(2x+3)}+\frac{4}{3(5-2x)}\right].$$

(iii) $\left(x^{2}+3\right)^{\frac{3}{2}} \cdot \sin ^{3} 2 x \cdot 2^{x^{2}}$

Solution:

Let y = $\left(x^{2}+3\right)^{\frac{3}{2}} \cdot \sin ^{3} 2 x \cdot 2^{x^{2}}$

Then log y = log [latex]\left(x^{2}+3\right)^{\frac{3}{2}} \cdot \sin ^{3} 2 x \cdot 2^{x^{2}}[/latex]

$$=\log(x^2+3)^{\frac{3}{2}}+\log\sin^3 2x+\log 2^{x^2}$$

$$=\frac{3}{2}\log(x^2+3)+3\log(\sin 2x)+x^2\cdot\log 2$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{3}{2}\frac{d}{dx}[\log(x^2+3)]+3\frac{d}{dx}[\log(\sin 2x)]+\log 2\cdot\frac{d}{dx}(x^2)$$

$$=\frac{3}{2}\times\frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3)+3\times\frac{1}{\sin 2x}\cdot\frac{d}{dx}(\sin 2x)+\log 2\times 2x$$

$$=\frac{3}{2(x^2+3)}\cdot(2x+0)+\frac{3}{\sin 2x}\times\cos 2x\cdot\frac{d}{dx}(2x)+2x\log 2$$

$$=\frac{6x}{2(x^2+3)}+3\cot 2x\times 2+2x\log 2$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{3x}{x^2+3}+6\cot 2x+2x\log 2\right]$$

$$=(x^2+3)^{\frac{3}{2}}\cdot\sin^3 2x\cdot 2^{x^2}\left[\frac{3x}{x^2+3}+6\cot 2x+2x\log 2\right].$$

(iv) $\frac{\left(x^{2}+2x+2\right)^{\frac{3}{2}}}{(\sqrt{x}+3)^{3}(\cos x)^{x}}$

Solution:

Let y = $\frac{\left(x^{2}+2x+2\right)^{\frac{3}{2}}}{(\sqrt{x}+3)^{3}(\cos x)^{x}}$

Then log y = log [latex]\frac{\left(x^{2}+2 x+2\right)^{\frac{3}{2}}}{(\sqrt{x}+3)^{3}(\cos x)^{x}}
[/latex]

$$=\log(x^2+2x+2)^{\frac{3}{2}}-\log(\sqrt{x}+3)^3(\cos x)^x$$

$$=\frac{3}{2}\log(x^2+2x+2)-3\log(\sqrt{x}+3)-x\log(\cos x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{3}{2}\frac{d}{dx}[\log(x^2+2x+2)]-3\frac{d}{dx}[\log(\sqrt{x}+3)]$$

$$-\frac{d}{dx}[x\log(\cos x)]$$

$$= \frac{3}{2} \times \frac{1}{x^2+2x+2} \cdot \frac{d}{dx}(x^2+2x+2)$$

$$- 3 \times \frac{1}{\sqrt{x}+3} \cdot \frac{d}{dx}(\sqrt{x}+3)$$

$$- \left\{ x \frac{d}{dx}[\log(\cos x)] + \log(\cos x) \cdot \frac{d}{dx}(x) \right\}$$

$$= \frac{3}{2(x^2+2x+2)} \times (2x+2\times 1+0) - \frac{3}{\sqrt{x}+3} \times \left(\frac{1}{2\sqrt{x}}+0\right) -$$

$$\left\{ x \times \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \log(\cos x) \times 1 \right\}$$

$$= \frac{3(2x+2)}{2(x^2+2x+2)} - \frac{3}{2\sqrt{x}(\sqrt{x}+3)} -$$

$$\left\{ x \times \frac{1}{\cos x} \cdot (-\sin x) + \log(\cos x) \right\}$$

$$\therefore \frac{dy}{dx} =$$

$$y\left[\frac{3(x+1)}{x^2+2x+2} - \frac{3}{2\sqrt{x}(\sqrt{x}+3)} + x\tan x - \log(\cos x)\right]$$

$$= \frac{(x^2+2x+2)^{\frac{3}{2}}}{(\sqrt{x}+3)^3(\cos x)^x}\left[\frac{3(x+1)}{x^2+2x+2} - \frac{3}{2\sqrt{x}(\sqrt{x}+3)} +\right.$$

$$\left. x\tan x - \log(\cos x)\right].$$

(v) $\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$

Solution:

Let y = $\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$

Then log y = log [latex]\frac{x^{5} \cdot \tan ^{3} 4 x}{\sin ^{2} 3 x}[/latex]

= log x^5 + log $\tan^3 4x$ – log $\sin^2 3x$

= 5 log x+ 3 log (tan 4x) – 2 log (sin 3x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 5\frac{d}{dx}(\log x) + 3\frac{d}{dx}[\log(\tan 4x)] - 2\frac{d}{dx}[\log(\sin 3x)]$$

$$= 5 \times \frac{1}{x} + 3 \times \frac{1}{\tan 4x} \cdot \frac{d}{dx}(\tan 4x) - 2 \times \frac{1}{\sin 3x} \cdot \frac{d}{dx}(\sin 3x)$$

$$= \frac{5}{x} + 3 \times \frac{1}{\tan 4x} \times \sec^2 4x \cdot \frac{d}{dx}(4x) - 2 \times \frac{1}{\sin 3x} \times \cos 3x \cdot \frac{d}{dx}(3x)$$

$$= \frac{5}{x} + 3 \cdot \frac{\cos 4x}{\sin 4x} \times \frac{1}{\cos^2 4x} \times 4 - 2\cot 3x \times 3$$

$$= \frac{5}{x} + \frac{24}{2\sin 4x \cdot \cos 4x} - 6\cot 3x$$

$$\therefore \frac{dy}{dx} = y\left[\frac{5}{x} + \frac{24}{\sin 8x} - 6\cot 3x\right]$$

$$= \frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}\left[\frac{5}{x} + 24 \operatorname{cosec} 8x - 6 \cot 3x\right].$$

(vi) $x^{\tan^{-1} x}$

Solution:

Let y = $x^{\tan^{-1} x}$

Then log y = log ($x^{\tan^{-1} x}$) = ($\tan^{-1}$ x)(log x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}[(\tan^{-1} x)(\log x)]$$

$$= (\tan^{-1} x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$= (\tan^{-1} x) \times \frac{1}{x} + (\log x) \times \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right]$$

$$= x^{\tan^{-1} x}\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right].$$

(vii) $(\sin x)^x$

Solution:

Let y = $(\sin x)^x$

Then log y = log $(\sin x)^x$ = x . log (sin x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}[x\cdot\log(\sin x)]$$

$$=x\cdot\frac{d}{dx}[\log(\sin x)]+\log(\sin x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)+\log(\sin x)\times 1$$

$$\therefore\ \frac{dy}{dx}=y\left[x\times\frac{1}{\sin x}\cdot\cos x+\log(\sin x)\right]$$

$$=(\sin x)^x[x\cot x+\log(\sin x)].$$

(viii) sin x^x

Solution:

Let y = (sin x^x)

Then $\frac{dy}{dx}=\frac{d}{dx}[(\sin x^x)]$

$\frac{dy}{dx}=\cos(x^x)\cdot\frac{d}{dx}(x^x)$ ……. (1)

Let u = x^x

Then log u = log x^x = x . log x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\cdot\log x)$$

$$=x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x)$$

From (1) and (2), we get

$$\frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \log x).$$

Question 2.
Differentiate the following w.r.t. x:
(i) $x^e + x^x + e^x + e^e$
Solution:
Let $y = x^e + x^x + e^x + e^e$
Let $u = x^x$
Then $\log u = \log x^x = x \log x$
Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x)(1)$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \quad \text{... (1)}$$

Now, $y = x^e + u + e^x + e^e$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^e) + \frac{du}{dx} + \frac{d}{dx}(e^x) + \frac{d}{dx}(e^e)$$

$$= ex^{e-1} + x^x(1 + \log x) + e^x + 0 \quad \text{... [By (1)]}$$

$$= ex^{e-1} + x^x(1 + \log x) + e^x$$

$$= ex^{e-1} + e^x + x^x(1 + \log x).$$

(ii) $x^{x^x} + e^{x^x}$

Solution:

Let y = $x^{x^x} + e^{x^x}$

Put u = x^{x^x} and v = e^{x^x}

Then y = u + v

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Take u = x^{x^x}

log u = log x^{x^x} = x^x . log x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x^x \cdot \log x)$$

$$= x^x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^x)$$

$$= x^x \times \frac{1}{x} + (\log x)\cdot\frac{d}{dx}(x^x) \qquad \ldots (2)$$

To find $\frac{d}{dx}(x^x)$

Let $w = x^x$. Then $\log w = x\log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{w}\cdot\frac{dw}{dx} = \frac{d}{dx}(x\log x)$$

$$= x\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{dw}{dx} = w(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x) \qquad \ldots (3)$$

$\therefore$ from (2),

$$\frac{1}{u}\cdot\frac{du}{dx} = x^x \times \frac{1}{x} + (\log x)\cdot x^x(1 + \log x)$$

$$\therefore \frac{du}{dx} = y\left[x^x \times \frac{1}{x} + (\log x)\cdot x^x(1 + \log x)\right]$$

$$= x^{x^x}\cdot x^x\left[\frac{1}{x} + (\log x)\cdot(1 + \log x)\right]$$

$$= x^{x^x} \cdot x^x . \log x\left[1 + \log x + \frac{1}{x \log x}\right]. \quad \dots (4)$$

Also, $v = e^{x^x}$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(e^{x^x})$$

$$= e^{x^x} \cdot \frac{d}{dx}(e^{x^x})$$

$$= e^{x^x} \cdot x^x (1 + \log x) \quad \dots (5)$$

... [By (3)]

From (1), (4) and (5), we get

$$\frac{dy}{dx} = x^{x^x} \cdot x^x \cdot \log x\left[1 + \log x + \frac{1}{x \log x}\right] + e^{x^x} \cdot x^x (1 + \log x).$$

(iii) $(\log x)^x - (\cos x)^{\cot x}$

Solution:

Let $y = (\log x)^x - (\cos x)^{\cot x}$

Put $u = (\log x)^x$ and $v = (\cos x)^{\cot x}$

Then $y = u - v$

$\therefore \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ……..(1)

Take $u = (\log x)^x$

$\therefore \log u = \log (\log x)^x = x . \log (\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}[x \cdot \log(\log x)]$$

$$= x\frac{d}{dx}[\log(\log x)] + \log(\log x)\cdot\frac{d}{dx}(x)$$

$$= x \times \frac{1}{\log x}\cdot\frac{d}{dx}(\log x) + \log(\log x) \times 1$$

$$= x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x)$$

$$\therefore \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$= (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \quad \ldots (2)$$

Also, $v = (\cos x)^{\cot x}$

$$\therefore \log v = \log(\cos x)^{\cot x} = (\cot x)\cdot(\log \cos x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{d}{dx}[(\cot x)\cdot\log(\cos x)]$$

$$= (\cot x)\cdot\frac{d}{dx}(\log \cos x) + (\log \cos x)\cdot\frac{d}{dx}(\cot x)$$

$$= \cot x \times \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + (\log \cos x)(-\operatorname{cosec}^2 x)$$

$$= \cot x \times \frac{1}{\cos x} \times (-\sin x) - (\operatorname{cosec}^2 x)(\log \cos x)$$

$$\therefore \frac{dv}{dx} = v\left[\frac{1}{\tan x} \times (-\tan x) - (\operatorname{cosec}^2 x)(\log \cos x)\right]$$

$$= -(\cos x)^{\cot x}[1 + (\text{cosec}^2 x)(\log \cos x)]$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + (\cos x)^{\cot x}[1 + (\text{cosec}^2 x)(\log \cos x)].$$

(iv) $x^{e^x} + (\log x)^{\sin x}$

Solution:

Let y = $x^{e^x} + (\log x)^{\sin x}$

Put u = x^{e^x} and v = $(\log x)^{\sin x}$

Then y = u + v

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ……….(1)

Take u = x^{e^x}

$\therefore$ log u = log x^{e^x} = e^x . log x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{d}{dx}(e^x \log x)$$

$$= e^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(e^x)$$

$$= e^x \cdot \frac{1}{x} + (\log x)(e^x)$$

$$\therefore \frac{du}{dx} = y\left[\frac{e^x}{x} + e^x \cdot \log x\right]$$

$= e^x \cdot x^{e^x}\left[\frac{1}{x} + \log x\right]$. ... (2)

Also, v = $(\log x)^{\sin x}$

∴ log v = log $(\log x)^{\sin x}$ = (sin x) . (log log x)

Differentiating both sides w.r.t. x, we get

$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[(\sin x) \cdot (\log \log x)]$

= $(\sin x) \cdot \frac{d}{dx}\left[(\log \log x) + (\log \log x) \cdot \frac{d}{dx}(\sin x)\right]$

$= \sin x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) + (\log \log x) \cdot (\cos x)$

$\therefore \frac{dv}{dx} = v\left[\frac{\sin x}{\log x} \times \frac{1}{x} + (\cos x)(\log \log x)\right]$

$= (\log x)^{\sin x}\left[\frac{\sin x}{x \log x} + (\cos x)(\log \log x)\right]$ ... (2)

From (1), (2) and (3), we get

$$\frac{dy}{dx} = e^x \cdot x^{e^x}\left[\frac{1}{x} + \log x\right] + (\log x)^{\sin x}\left[\frac{\sin x}{x \log x} + (\cos x)(\log \log x)\right].$$

(v) $e^{\tan x} + (\log x)^{\tan x}$

Solution:

Let y = $e^{\tan x} + (\log x)^{\tan x}$

Put u = $(\log x)^{\tan x}$

∴ log u = $\log(\log x)^{\tan x}$ = (tan x).(log log x)

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}[(\tan x)\cdot(\log\log x)]$$

$$=(\tan x)\cdot\frac{d}{dx}(\log\log x)+(\log\log x)\cdot\frac{d}{dx}(\tan x)$$

$$=\tan x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)+(\log\log x)(\sec^2 x)$$

$$=\tan x\times\frac{1}{\log x}\times\frac{1}{x}+(\log\log x)(\sec^2 x)$$

$$\therefore\ \frac{du}{dx}=u\left[\frac{\tan x}{x\log x}+(\log\log x)(\sec^2 x)\right]$$

$$=(\log x)^{\tan x}\left[\frac{\tan x}{x\log x}+(\log\log x)(\sec^2 x)\right]$$

Now, $y=e^{\tan x}+u$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(e^{\tan x})+\frac{du}{dx}$$

$$=e^{\tan x}\cdot\frac{d}{dx}(\tan x)+\frac{du}{dx}$$

$$=e^{\tan x}\cdot\sec^2 x+(\log x)^{\tan x}\left[\frac{\tan x}{x\log x}+(\log\log x)(\sec^2 x)\right].$$

(vi) $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

Solution:

Let $y = (\sin x)^{\tan x} + (\cos x)^{\cot x}$

Put $u = (\sin x)^{\tan x}$ and $v = (\cos x)^{\cot x}$

Then $y = u + v$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ....(1)

Take $u = (\sin x)^{\tan x}$

$\therefore \log u = \log (\sin x)^{\tan x} = (\tan x) . (\log \sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}[(\tan x)(\log \sin x)]$$

$$= (\tan x) \cdot \frac{d}{dx}(\log \sin x) + (\log \sin x) \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{\tan x}{\sin x} \cdot \frac{d}{dx}(\sin x) + (\log \sin x)(\sec^2 x)$$

$$= \frac{(\sin x)/(\cos x)}{\sin x} \cdot \cos x + (\sec^2 x)(\log \sin x)$$

$$= 1 + (\sec^2 x)(\log \sin x)$$

$$\therefore \frac{du}{dx} = u[1 + (\sec^2 x)(\log \sin x)]$$

$$= (\sin x)^{\tan x}[1 + (\sec^2 x)(\log \sin x)] \quad \ldots (2)$$

Also, $v = (\cos x)^{\cot x}$

$$\therefore \log v = \log (\cos x)^{\cot x} = (\cot x) \cdot (\log \cos x)$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[(\cot x) \cdot (\log \cos x)]$$

$$= (\cot x)\cdot\frac{d}{dx}(\log\cos x) + (\log\cos x)\cdot\frac{d}{dx}(\cot x)$$

$$= \cot x \times \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + (\log\cos x)\cdot(-\operatorname{cosec}^2 x)$$

$$= \cot x \times \frac{1}{\cos x} \times (-\sin x) - (\operatorname{cosec}^2 x)(\log\cos x)$$

$$\therefore \frac{dv}{dx} = v\left[\frac{1}{\tan x} \times (-\tan x) - (\operatorname{cosec}^2 x)(\log\cos x)\right]$$

$$= -(\cos x)^{\cot x}[1 + (\operatorname{cosec}^2 x)(\log\cos x)] \quad \ldots (3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + (\sec^2 x)(\log\sin x)] -$$

$$(\cos x)^{\cot x}[1 + (\operatorname{cosec}^2 x)(\log\cos x)].$$

(vii) $10^{x^x} + x^{x^{10}} + x^{10^x}$

Solution:

Let y = $10^{x^x} + x^{x^{10}} + x^{10^x}$

Put u = 10^{x^x}, v = $x^{x^{10}}$ and w = x^{10^x}

Then y = u + v + w

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$ ....(1)

Take, $u = 10^{x^x}$

$$\therefore \frac{du}{dx} = \frac{d}{dx}(10^{x^x}) = 10^{x^x}\cdot\log 10\cdot\frac{d}{dx}(x^x)$$

To find $\frac{d}{dx}(x^x)$

$\therefore \log z = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{z} \cdot \frac{dz}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x)(1)$$

$$\therefore \frac{dz}{dx} = z(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x)$$

$$\therefore \frac{du}{dx} = 10^{x^x} \cdot \log 10 \cdot x^x (1 + \log x) \quad \ldots (2)$$

Take, $v = x^{x^{10}}$

$\therefore \log v = \log x^{x^{10}} = x^{10} \cdot \log x$

Differentiating with sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}(x^{10} \log x)$$

$$= x^{10} \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^{10})$$

$$= x^{10} \times \frac{1}{x} + (\log x)(10x^9)$$

$$\therefore \frac{dv}{dx} = v[x^9 + 10x^9 \log x]$$

$\therefore \frac{dv}{dx} = x^{x^{10}} \cdot x^9 (1 + 10 \log x)$ ... (3)

Also, $w = x^{10^x}$

$\therefore \log w = \log x^{10^x} = 10^x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{w} \cdot \frac{dw}{dx} = \frac{d}{dx}(10^x \cdot \log x)$$

$$= 10^x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(10^x)$$

$$= 10^x \times \frac{1}{x} + (\log x)(10^x \cdot \log 10)$$

$$\therefore \frac{dw}{dx} = w\left[\frac{10^x}{x} + 10^x \cdot (\log x)(\log 10)\right]$$

$$\therefore \frac{dw}{dx} = x^{10^x} \cdot 10^x \left[\frac{1}{x} + (\log x)(\log 10)\right] \quad \text{... (4)}$$

From (1), (2), (3) and (4), we get

$$\frac{dy}{dx} = 10^{x^x} \cdot \log 10 \cdot x^x (1 + \log x) + x^{x^{10}} \cdot x^9 (1 + 10 \log x) + x^{10^x} \cdot 10^x \left[\frac{1}{x} + (\log x)(\log 10)\right].$$

(viii) $\left[(\tan x)^{\tan x}\right]^{\tan x}$ at x = $\frac{\pi}{4}$

Solution:

Let y = $\left[(\tan x)^{\tan x}\right]^{\tan x}$

∴ log y = log [latex]\left[(\tan x)^{\tan x}\right]^{\tan x}[/latex]
= tan x . log(tan x)$^{\tan x}$
= tan x . tan x log (tan x)
= (tan x)^2 . log (tan x)
Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}[(\tan x)^2.\log(\tan x)]$$

$$=(\tan x)^2\cdot\frac{d}{dx}(\log\tan x)+(\log\tan x)\cdot\frac{d}{dx}(\tan x)^2$$

$$=(\tan x)^2\times\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)+(\log\tan x)\times 2\tan x\cdot\frac{d}{dx}(\tan x)$$

$$=(\tan x)^2\times\frac{1}{\tan x}\cdot\sec^2 x+(\log\tan x)\times 2\tan x\sec^2 x$$

$$\therefore\ \frac{dy}{dx}=y[(\tan x)(\sec^2 x)+(\log\tan x)(2\tan x\sec^2 x)]$$

$$=[(\tan x)^{\tan x}]^{\tan x}\cdot(\tan x\sec^2 x)[1+2\log\tan x]$$

If $x=\frac{\pi}{4}$, then

$$\frac{dy}{dx}=\left[\left(\tan\frac{\pi}{4}\right)^{\tan\frac{\pi}{4}}\right]^{\tan\frac{\pi}{4}}\cdot\left(\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}\right)\left[1+2\log\tan\frac{\pi}{4}\right]$$

$$=[(1)^1]^1\cdot[1(\sqrt{2})^2][1+2\log 1]$$

$$=1\times 2\times 1 \qquad \ldots [\because\ \log 1=0]$$

$$=2.$$

Question 3.
Find $\frac{dy}{dx}$ if
(i) √x + √y = √a
Solution:
√x + √y = √a
Differentiating both sides w.r.t. x, we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\therefore \frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx}=-\sqrt{\frac{y}{x}}.$$

(ii) x√x + y√y = a√a
Solution:
x√x + y√y = a√a
$\therefore x^{\frac{3}{2}}+y^{\frac{3}{2}}=a^{\frac{3}{2}}$
Differentiating both sides w.r.t. x, we get

$$\frac{3}{2}\cdot x^{\frac{1}{2}}+\frac{3}{2}\cdot y^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\therefore \frac{3}{2}\cdot y^{\frac{1}{2}}\frac{dy}{dx}=-\frac{3}{2}x^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{-x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=-\sqrt{\frac{x}{y}}.$$

(iii) $x + \sqrt{xy} + y = 1$

Solution:

$x + \sqrt{xy} + y = 1$

Differentiating both sides w.r.t. x, we get

$$1+\frac{1}{2\sqrt{xy}}\cdot\frac{d}{dx}(xy)+\frac{dy}{dx}=0$$

$$\therefore\ 1+\frac{1}{2\sqrt{xy}}\cdot\left[x\frac{dy}{dx}+y\times 1\right]+\frac{dy}{dx}=0$$

$$\therefore\ 1+\frac{1}{2}\sqrt{\frac{x}{y}}\frac{dy}{dx}+\frac{1}{2}\sqrt{\frac{y}{x}}+\frac{dy}{dx}=0$$

$$\therefore\ \left(\frac{1}{2}\sqrt{\frac{x}{y}}+1\right)\frac{dy}{dx}=-\frac{1}{2}\sqrt{\frac{y}{x}}-1$$

$$\therefore\ \left(\frac{\sqrt{x}+2\sqrt{y}}{2\sqrt{y}}\right)\frac{dy}{dx}=\frac{-\sqrt{y}-2\sqrt{x}}{2\sqrt{x}}$$

$$\therefore\ \frac{dy}{dx}=\frac{-\sqrt{y}(2\sqrt{x}+\sqrt{y})}{\sqrt{x}(\sqrt{x}+2\sqrt{y})}.$$

(iv) $x^3 + x^2y + xy^2 + y^3 = 81$

Solution:

$x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. x, we get

$$3x^2 + x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) + x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ 3x^2 + x^2\frac{dy}{dx} + y \times 2x + x \times 2y\frac{dy}{dx} + y^2 \times 1 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ 3x^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\therefore\ \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

(v) $x^2y^2 - \tan^{-1}(\sqrt{x^2 + y^2}) = \cot^{-1}(\sqrt{x^2 + y^2})$

Solution:

$x^2y^2 - \tan^{-1}(\sqrt{x^2 + y^2}) = \cot^{-1}(\sqrt{x^2 + y^2})$

$\therefore x^2y^2 = \tan^{-1}(\sqrt{x^2 + y^2}) + \cot^{-1}(\sqrt{x^2 + y^2})$

$\therefore x^2y^2 = \frac{\pi}{2}$ ....[$\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$]

Differentiating both sides w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore\ x^2 \times 2y\frac{dy}{dx} + y^2 \times 2x = 0$$

$$\therefore\ 2x^2y\frac{dy}{dx} = -2xy^2$$

$\therefore x\frac{dy}{dx} = -y$

$\therefore \frac{dy}{dx} = -\frac{y}{x}$.

(vi) $xe^y + ye^x = 1$

Solution:

$xe^y + ye^x = 1$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(xe^y) + \frac{d}{dx}(ye^x) = 0$$

$$\therefore x \cdot \frac{d}{dx}(e^y) + e^y \cdot \frac{d}{dx}(x) + y \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{dy}{dx} = 0$$

$$\therefore x \cdot e^y \frac{dy}{dx} + e^y \times 1 + y \times e^x + e^x \frac{dy}{dx} = 0$$

$$\therefore (e^x + xe^y)\frac{dy}{dx} = -e^y - ye^x$$

$$\therefore \frac{dy}{dx} = -\left(\frac{e^y + ye^x}{e^x + xe^y}\right).$$

(vii) $e^{x+y} = \cos(x - y)$

Solution:

$e^{x+y} = \cos(x - y)$

Differentiating both sides w.r.t. x, we get

$$e^{x+y} \cdot \frac{d}{dx}(x + y) = -\sin(x - y) \cdot \frac{d}{dx}(x - y)$$

$$\therefore\ e^{x+y}\left(1+\frac{dy}{dx}\right)=-\sin(x-y)\left(1-\frac{dy}{dx}\right)$$

$$\therefore\ e^{x+y}+e^{x+y}.\frac{dy}{dx}=-\sin(x-y)+\sin(x-y)\frac{dy}{dx}$$

$$\therefore\ [e^{x+y}-\sin(x-y)]\frac{dy}{dx}=-\sin(x-y)-e^{x+y}$$

$$\therefore\ \frac{dy}{dx}=-\left[\frac{\sin(x-y)+e^{x+y}}{e^{x+y}-\sin(x-y)}\right]=\frac{\sin(x-y)+e^{x+y}}{\sin(x-y)-e^{x+y}}.$$

(viii) cos (xy) = x + y

Solution:

cos (xy) = x + y

Differentiating both sides w.r.t. x, we get

$$-\sin(xy)\cdot\frac{d}{dx}(xy)=1+\frac{dy}{dx}$$

$$\therefore\ -\sin(xy)\left[x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)\right]=1+\frac{dy}{dx}$$

$$\therefore\ -\sin(xy)\left[x\frac{dy}{dx}+y\times 1\right]=1+\frac{dy}{dx}$$

$$\therefore\ -x\sin(xy)\frac{dy}{dx}-y\sin(xy)=1+\frac{dy}{dx}$$

$$\therefore\ -\frac{dy}{dx}-x\sin(xy)\frac{dy}{dx}=1+y\sin(xy)$$

$$\therefore\ -[1+x\sin(xy)]\frac{dy}{dx}=1+y\sin(xy)$$

$$\therefore \frac{dy}{dx} = \frac{-[1 + y\sin(xy)]}{1 + x\sin(xy)}.$$

(ix) $e^{e^{x-y}} = \frac{x}{y}$

Solution:

$e^{e^{x-y}} = \frac{x}{y}$

∴ $e^{x-y} = \log(\frac{x}{y})$ ....[$e^x = y \Rightarrow x = \log y$]

∴ $e^{x-y} = \log x - \log y$

Differentiating both sides w.r.t. x, we get

$$e^{x-y} \cdot \frac{d}{dx}(x - y) = \frac{1}{x} - \frac{1}{y}\frac{dy}{dx}$$

$$\therefore e^{x-y}\left(1 - \frac{dy}{dx}\right) = \frac{1}{x} - \frac{1}{y}\frac{dy}{dx}$$

$$\therefore e^{x-y} - e^{x-y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{y}\frac{dy}{dx}$$

$$\therefore \left(\frac{1}{y} - e^{x-y}\right)\frac{dy}{dx} = \frac{1}{x} - e^{x-y}$$

$$\left(\frac{1 - ye^{x-y}}{y}\right)\frac{dy}{dx} = \frac{1 - xe^{x-y}}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(1 - xe^{x-y})}{x(1 - ye^{x-y})}.$$

Question 4.

Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following, where a and p are constants.

(i) $x^7y^5 = (x + y)^{12}$

Solution:

$x^7y^5 = (x + y)^{12}$

$(\log x^7y^5) = \log(x + y)^{12}$

$\log x^7 + \log y^5 = \log(x + y)^{12}$

$7 \log x + 5 \log y = 12 \log (x + y)$

Differentiating both sides w.r.t. x, we get

$$7 \times \frac{1}{x} + 5 \times \frac{1}{y} \cdot \frac{dy}{dx} = 12 \times \frac{1}{x+y} \cdot \frac{d}{dx}(x+y)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{dy}{dx} = \frac{12}{x+y} \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{dy}{dx} = \frac{12}{x+y} + \frac{12}{x+y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{5}{y} - \frac{12}{x+y}\right)\frac{dy}{dx} = \frac{12}{x+y} - \frac{7}{x}$$

$$\therefore \left[\frac{5x + 5y - 12y}{y(x+y)}\right]\frac{dy}{dx} = \frac{12x - 7x - 7y}{x(x+y)}$$

$$\therefore \left[\frac{5x - 7y}{y(x+y)}\right]\frac{dy}{dx} = \frac{5x - 7y}{x(x+y)}$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \qquad \therefore \frac{dy}{dx} = \frac{y}{x}.$$

(ii) $x^py^4 = (x + y)^{p+4}$, $p \in N$

Solution:

$x^p y^4 = (x + y)^{p+4}$

Taking log

$\log (x^p y^4) = \log(x + y)^{p+4}$

$\log x^p + \log y^4 = (p + 4) \log(x + y)$

$p \log x + 4 \log y = (p + 4) \log(x + y)$

Differentiating both sides w.r.t. x, we get

$$p.\frac{d}{dx}\log x + 4.\frac{d}{dx}\log y = (p+4)\frac{d}{dx}\log(x+y)$$

$$\frac{p}{x} + 4\frac{1}{y}\frac{dy}{dx} = (p+4)\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{p}{4} + \frac{4}{y}\frac{dy}{dx} = \frac{(p+4)}{(x+y)} + \frac{p+4}{(x+y)}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\frac{4}{y} - \frac{(p+4)}{(x+y)}\right] = \frac{p+4}{x+y} - \frac{p}{x}$$

$$\frac{dy}{dx}\left[\frac{4(x+y) - y(p+4)}{y(x+y)}\right] = \frac{x(p+4) - p(x+y)}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{4x + 4y - py - 4y}{y(x+y)}\right] = \frac{px + 4x - px - py}{x(x+y)}$$

$$\frac{dy}{dx}\left[\frac{4x - py}{y}\right] = \frac{4x - py}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

(iii) $\sec\left(\frac{x^5+y^5}{x^5-y^5}\right) = a^2$

Solution:

$\sec\left(\frac{x^5+y^5}{x^5-y^5}\right) = a^2$

$\therefore \frac{x^5+y^5}{x^5-y^5} = \sec^{-1}(a^2) = k$ ... (Say)

$\therefore x^5 + y^5 = kx^5 - ky^5$

$\therefore (1+k)y^5 = (k-1)x^5$

$\therefore \frac{y^5}{x^5} = \frac{k-1}{k+1}$

$\therefore \frac{y}{x} = \left(\frac{k-1}{k+1}\right)^{\frac{1}{5}}$, a constant

Differentiating both sides w.r.t. x, we get

$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$

$\therefore \frac{x \cdot \frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$

$\therefore x\frac{dy}{dx} - y \times 1 = 0$

$\therefore \frac{dy}{dx} = \frac{y}{x}.$

Alternative Method :

$\sec\left(\frac{x^5+y^5}{x^5-y^5}\right) = a^2$

$$\therefore \frac{x^5+y^5}{x^5-y^5} = \sec^{-1}a^2 = k \quad \text{... (Say)}$$

$$\therefore x^5+y^5 = kx^5 - ky^5$$

$$\therefore (1+k)y^5 = (k-1)x^5$$

$$\therefore \frac{y^5}{x^5} = \frac{k-1}{k+1} \quad \text{... (1)}$$

$$\therefore y^5 = k'x^5, \text{ where } k' = \frac{k-1}{k+1}$$

Differentiating both sides w.r.t. x, we get

$$5y^4\frac{dy}{dx} = k' \times 5x^4$$

$$\therefore \frac{dy}{dx} = k' \cdot \frac{x^4}{y^4}$$

$$\therefore \frac{dy}{dx} = \left(\frac{k-1}{k+1}\right) \cdot \frac{x^4}{y^4}$$

$$= \frac{y^5}{x^5} \times \frac{x^4}{y^4} \quad \text{... [By (1)]}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

(iv) $\tan^{-1}\left(\frac{3x^2-4y^2}{3x^2+4y^2}\right) = a^2$

Solution:

$\tan^{-1}\left(\frac{3x^2-4y^2}{3x^2+4y^2}\right) = a^2$

$\therefore \frac{3x^2-4y^2}{3x^2+4y^2}=\tan a^2=k$ ... (Say)

$\therefore 3x^2-4y^2=3kx^2+4ky^2$

$\therefore (4k+4)y^2=(3-3k)x^2$

$\therefore \frac{y^2}{x^2}=\frac{3-3k}{4k+4}$

$\therefore \frac{y}{x}=\sqrt{\frac{3-3k}{4k+4}}$, a constant

Differentiating both sides w.r.t. x, we get

$\frac{d}{dx}\left(\frac{y}{x}\right)=0$

$\therefore \frac{x\frac{dy}{dx}-y\cdot\frac{d}{dx}(x)}{x^2}=0$

$\therefore x\frac{dy}{dx}-y\times 1=0$

$\therefore x\cdot\frac{dy}{dx}=y$

$\therefore \frac{dy}{dx}=\frac{y}{x}.$

(v) $\cos^{-1}\left(\frac{7x^4+5y^4}{7x^4-5y^4}\right)=\tan^{-1}a$

Solution:

$$\cos^{-1}\left(\frac{7x^4+5y^4}{7x^4-5y^4}\right)=\tan^{-1}a$$

$\frac{7x^4+5y^4}{7x^4-5y^4}$ = $\cos(\tan^{-1} a)$ = b

$\frac{7x^4+5y^4}{7x^4-5y^4}$ = b

$7x^4 + 5y^4 = b(7x^4 - 5y^4)$

$7x^4 + 5y^4 = 7bx^4 - 5by^4$

$5y^4 + 5by^4 = 7bx^4 - 7x^4$

$5y^4(1 + b) = 7x^4(b - 1)$

$$\frac{5y^4}{7x^4}=\frac{b-1}{1+b}$$

$\frac{y^4}{x^4}=\frac{7(b-1)}{5(1+b)}$ = x

$\frac{y^4}{x^4}$ = c....(1)

$y^4 = cx^4$

Differentiating both sides w.r.t. x, we get

$4.y^3\frac{dy}{dx}$ = $c.4x^3$

$$\frac{dy}{dx} = \frac{c.4x^3}{4y^3}$$

$$\frac{dy}{dx} = \frac{c.x^3}{y^3}$$

$$\frac{dy}{dx} = \frac{y^4}{x^4} \cdot \frac{x^3}{y^3} \quad \text{...from..(1)}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

(vi) $\log\left(\frac{x^{20}-y^{20}}{x^{20}+y^{20}}\right) = 20$

Solution:

$$\log\left(\frac{x^{20}-y^{20}}{x^{20}+y^{20}}\right) = 20$$

$$\therefore \frac{x^{20}-y^{20}}{x^{20}+y^{20}} = e^{20} = k \quad \text{... (Say)}$$

$$\therefore x^{20} - y^{20} = kx^{20} + ky^{20}$$

$$\therefore (1+k)y^{20} = (1-k)x^{20}$$

$$\therefore \frac{y^{20}}{x^{20}} = \frac{1-k}{1+k}$$

$$\therefore \frac{y}{x} = \left(\frac{1-k}{1+k}\right)^{\frac{1}{20}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right)=0$$

$$\therefore \frac{x\frac{dy}{dx}-y\cdot\frac{d}{dx}(x)}{x^2}=0$$

$$\therefore x\frac{dy}{dx}-y\times 1=0$$

$$\therefore x\frac{dy}{dx}=y$$

$$\therefore \frac{dy}{dx}=\frac{y}{x}.$$

(vii) $e^{\frac{x^7-y^7}{x^7+y^7}}=a$

Solution:

$$e^{\frac{x^7-y^7}{x^7+y^7}}=a$$

$$\therefore \frac{x^7-y^7}{x^7+y^7}=\log a=k \qquad \text{... (Say)}$$

$$\therefore x^7-y^7=kx^7+ky^7$$

$$\therefore (1+k)y^7=(1-k)x^7$$

$$\therefore \frac{y^7}{x^7}=\frac{1-k}{1+k}$$

$$\therefore \frac{y}{x} = \left(\frac{1-k}{1+k}\right)^{\frac{1}{7}}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

$$\therefore \frac{x\frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2} = 0$$

$$\therefore x\frac{dy}{dx} - y \times 1 = 0$$

$$\therefore x\frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

(viii) $\sin\left(\frac{x^3-y^3}{x^3+y^3}\right) = a^3$

Solution:

$$\sin\left(\frac{x^3-y^3}{x^3+y^3}\right) = a^3$$

$$\frac{x^3-y^3}{x^3+y^3} = \sin a^3 = b$$

$$\frac{x^3 - y^3}{x^3 + y^3} = b$$

$x^3 - y^3 = b(x^3 + y^3)$

$x^3 - y^3 = bx^3 + by^3$

$x^3 - bx^3 = by^3 + y^3$

$x^3(1 - b) = y^3(b + 1)$

$$\frac{y^3}{x^3} = \frac{1-b}{1+b} = e$$

$$\frac{y^3}{x^3} = c \quad .....(1)$$

$y^3 = cx^3$

Differentiating both sides w.r.t. x, we get

$$3y^2\frac{dy}{dx} = c.3x^2$$

$$\frac{y^2 dy}{dx} = cx^2$$

$$\frac{dy}{dx} c\frac{x^2}{y^2}$$

$$\frac{dy}{dx} = \frac{y^3}{x^3} \cdot \frac{x^2}{y^2} \quad ....\text{from}(1)$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

Question 5.

(i) If log (x + y) = log (xy) + p, where p is a constant, then prove that $\frac{dy}{dx} = -\frac{y^2}{x^2}$.

Solution:

log (x + y) = log (xy) + p

∴ log (x + y) = log x + log y + p

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x+y} \cdot \frac{d}{dx}(x+y) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \frac{1}{x+y} + \frac{1}{x+y} \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{1}{x+y} - \frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+y}$$

$$\therefore \left[\frac{y-x-y}{y(x+y)}\right]\frac{dy}{dx} = \frac{x+y-x}{x(x+y)}$$

$$\therefore \left[\frac{-x}{y(x+y)}\right]\frac{dy}{dx} = \frac{y}{x(x+y)}$$

$$\therefore \left(-\frac{x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-y^2}{x^2}.$$

(ii) If $\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2$, show that $\frac{dy}{dx}=-\frac{99x^2}{101y^2}$

Solution:

$$\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2$$

$$\therefore \frac{x^3-y^3}{x^3+y^3}=10^2=100$$

$$\therefore x^3-y^3=100x^3+100y^3$$

$$\therefore 101y^3=-99x^3 \qquad \therefore y^3=\frac{-99}{101}x^3$$

Differentiating both sides w.r.t. x, we get

$$3y^2\frac{dy}{dx}=\frac{-99}{101}\times 3x^2$$

$$\therefore \frac{dy}{dx}=-\frac{99x^2}{101y^2}.$$

(iii) If $\log_5\left(\frac{x^4+y^4}{x^4-y^4}\right)=2$, show that $\frac{dy}{dx}=-\frac{12x^3}{13y^3}$

Solution:

$$\log_5\left(\frac{x^4+y^4}{x^4-y^4}\right)=2$$

$$\frac{x^4+y^4}{x^4-y^4}=5^2$$

$$\frac{x^4+y^4}{x^4-y^4}=25$$

$x^4 + y^4 = 25x^4 - 25y^4$

$26y^4 = 24x^4$

Differentiating both sides w.r.t. x, we get

$$26\frac{d}{dx}y^4 = 24\frac{d}{dx}x^4$$

$$26.4y^3.\frac{dy}{dx} = 24.4.x^3$$

$$26y^3.\frac{dy}{dx} = 24.x^3$$

$$\frac{dy}{dx} = \frac{24x^3}{26y^3}$$

$$\frac{dx}{dx} = \frac{12x^3}{13y^3}$$

(iv) If $e^x + e^y = e^{x+y}$, then show that $\frac{dy}{dx} = -e^{y-x}$

Solution:

$e^x + e^y = e^{x+y}$ ……(1)

Differentiating both sides w.r.t. x, we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}.\frac{d}{dx}(x+y)$$

$$\therefore\ e^x + e^y.\frac{dy}{dx} = e^{x+y}.\left(1 + \frac{dy}{dx}\right)$$

$\therefore\ e^x + e^y \frac{dy}{dx} = e^{x+y} + e^{x+y} \frac{dy}{dx}$

$\therefore\ (e^y - e^{x+y}) \frac{dy}{dx} = e^{x+y} - e^x$

$\therefore\ \frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$

$= \frac{e^x + e^y - e^x}{e^y - e^x - e^y}$ ... [By (1)]

$= \frac{e^y}{-e^x} = -e^{y-x}$.

(v) If $\sin^{-1}\left(\frac{x^5-y^5}{x^5+y^5}\right) = \frac{\pi}{6}$, show that $\frac{dy}{dx} = \frac{x^4}{3y^4}$

Solution:

$\sin^{-1}\left(\frac{x^5-y^5}{x^5+y^5}\right) = \frac{\pi}{6}$

$\frac{x^5-y^5}{x^5+y^5} = \sin\frac{\pi}{6} = \frac{1}{2}$

$2x^5 - 2y^5 = x^5 + y^5$

$3y^5 = x^5$

Differentiating both sides w.r.t. x, we get

$3 \times 5y^4 \frac{dy}{dx} = 5x^4$

$\therefore \frac{dy}{dx} = \frac{x^4}{3y^4}$

(vi) If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution:

$x^y = e^{x-y}$

$\log x^y = \log e^{x-y}$

$y \log x = (x - y) \log e$

$y \log x = (x - y)$ ….. [$\because \log e = 1$]

$y + y \log x = x - y$

$y + y \log x = x$

$y(1 + \log x) = x$

$y = \frac{x}{1+\log x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+\log x}\right)$$

$$= \frac{(1+\log x)\cdot\frac{d}{dx}(x) - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)\cdot 1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}.$$

(vii) If $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+\ldots \infty}}}$, then show that $\frac{dy}{dx}=\frac{\sin x}{1-2y}$

Solution:

$y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x+\ldots \infty}}}$

$y^2 = \cos x + \sqrt{\cos x+\sqrt{\cos x+\ldots \infty}}$

$y^2 = \cos x + y$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=-\sin x+\frac{dy}{dx}$$

$$\therefore\ (1-2y)\frac{dy}{dx}=\sin x$$

$$\therefore\ \frac{dy}{dx}=\frac{\sin x}{1-2y}.$$

(viii) If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots \infty}}}$, then show that $\frac{dy}{dx}=\frac{1}{x(2y-1)}$

Solution:

$y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots \infty}}}$

$$\therefore\ y^2=\log x+\sqrt{\log x+\sqrt{\log x+\ldots \infty}}$$

$$\therefore\ y^2=\log x+y$$

Differentiating both sides w.r.t. x, we get

$$2y\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx}$$

$$\therefore (2y-1)\frac{dy}{dx}=\frac{1}{x} \qquad \therefore \frac{dy}{dx}=\frac{1}{x(2y-1)}.$$

(ix) If $y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$, then show that $\frac{dy}{dx} = \frac{y^2}{x(1-\log y)}$

Solution:

$y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$

$$\therefore \log y = \log\left(x^{x^{x^{\cdot^{\cdot^{\infty}}}}}\right)$$

$$= x^{x^{\cdot^{\cdot^{\infty}}}} \cdot \log x$$

$$\therefore \log y = y \log x \qquad \ldots (1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}(\log x) + (\log x)\frac{dy}{dx}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = y \times \frac{1}{x} + (\log x)\frac{dy}{dx}$$

$$\therefore \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \left(\frac{1-y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1-\log y)}. \qquad \ldots [\text{By } (1)]$$

(x) If $e^y = y^x$, then show that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$

Solution:

$e^y = y^x$

$\log e^y = \log y^x$

$y \log e = x \log y$

$y = x \log y$ …… [∵log e = 1] ……….(1)

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = x\frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x)$

$\therefore \frac{dy}{dx} = x \times \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \times 1$

$\therefore \frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$

$\therefore \left(1 - \frac{x}{y}\right)\frac{dy}{dx} = \log y$

$\therefore \left(\frac{y-x}{y}\right)\frac{dy}{dx} = \log y$

$\therefore \frac{dy}{dx} = \frac{y \log y}{y - x}$

$= \frac{y \log y}{y - \left(\frac{y}{\log y}\right)}$ … [By (1)]

$\therefore \frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$.

$e^y = y^x$

$\therefore \log e^y = \log y^x$

$\therefore y \log e = x \log y$

$\therefore y = x \log y \qquad \ldots [\because \log e = 1]$

$\therefore x = \dfrac{y}{\log y}$

Differentiating both sides w.r.t. x, we get

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y}{\log y}\right)$$

$$= \frac{(\log y) \cdot \frac{d}{dy}(y) - y \cdot \frac{d}{dy}(\log y)}{(\log y)^2}$$

$$= \frac{(\log y) \times 1 - y \times \frac{1}{y}}{(\log y)^2}$$

$$= \frac{\log y - 1}{(\log y)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{(\log y)^2}{\log y - 1}.$$

Ex 1.4

Question 1.

Find $\frac{dy}{dx}$ if

(i) $x = at^2$, $y = 2at$

Solution:

$x = at^2$, $y = 2at$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= a \times 2t = 2at$$

$$\text{and } \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t)$$

$$= 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}.$$

(ii) x = a cot θ, y = b cosec θ

Solution:

x = a cot θ, y = b cosec θ

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}(\cot\theta)=a(-\operatorname{cosec}^2\theta)$$

$$=-a\operatorname{cosec}^2\theta$$

$$\text{and } \frac{dy}{d\theta}=b\frac{d}{d\theta}(\operatorname{cosec}\theta)=b(-\operatorname{cosec}\theta\cot\theta)$$

$$=-b\operatorname{cosec}\theta\cot\theta$$

$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{-b\operatorname{cosec}\theta\cot\theta}{-a\operatorname{cosec}^2\theta}$$

$$=\frac{b}{a}\cdot\frac{\cot\theta}{\operatorname{cosec}\theta}=\frac{b}{a}\times\frac{\cos\theta}{\sin\theta}\times\sin\theta$$

$$=\left(\frac{b}{a}\right)\cos\theta.$$

(iii) x = $\sqrt{a^2+m^2}$, y = log $(a^2 + m^2)$

Solution:

x = $\sqrt{a^2+m^2}$, y = log $(a^2 + m^2)$

Differentiating x and y w.r.t. m, we get

$$\frac{dx}{dm}=\frac{d}{dm}\left(\sqrt{a^2+m^2}\right)$$

$$=\frac{1}{2\sqrt{a^2+m^2}}\cdot\frac{d}{dm}(a^2+m^2)$$

$$= \frac{1}{2\sqrt{a^2+m^2}} \times (0+2m) = \frac{m}{\sqrt{a^2+m^2}}$$

and $\frac{dy}{dm} = \frac{d}{dm}[\log(a^2+m^2)]$

$$= \frac{1}{a^2+m^2} \cdot \frac{d}{dm}(a^2+m^2)$$

$$= \frac{1}{a^2+m^2} \times (0+2m) = \frac{2m}{a^2+m^2}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dm)}{(dx/dm)} = \frac{\left(\frac{2m}{a^2+m^2}\right)}{\left(\frac{m}{\sqrt{a^2+m^2}}\right)}$$

$$= \frac{2}{\sqrt{a^2+m^2}}.$$

(iv) x = sin θ, y = tan θ

Solution:

x = sin θ, y = tan θ

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

$$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{\sec^2\theta}{\cos\theta}$$

$$= \sec^3\theta.$$

(v) x = a(1 – cos θ), y = b(θ – sin θ)
Solution:
x = a(1 – cos θ), y = b(θ – sin θ)
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(1 - \cos\theta)$$

$$= a[0 - (-\sin\theta)] = a\sin\theta$$

and $\frac{dy}{d\theta} = b\frac{d}{d\theta}(\theta - \sin\theta)$

$$= b(1 - \cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{b(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{b \times 2\sin^2(\theta/2)}{a \times 2\sin(\theta/2)\cos(\theta/2)} = \left(\frac{b}{a}\right)\tan\left(\frac{\theta}{2}\right)$$

(vi) x = $\left(t + \frac{1}{t}\right)^a$, y = $a^{t+\frac{1}{t}}$, where a > 0, a ≠ 1 and t ≠ 0
Solution:

x = $\left(t+\frac{1}{t}\right)^{a}$, y = $a^{t+\frac{1}{t}}$ ....(1)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(t+\frac{1}{t}\right)^{a}=a\left(t+\frac{1}{t}\right)^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$=a\left(t+\frac{1}{t}\right)^{a-1}\cdot\left(1-\frac{1}{t^{2}}\right)$$

and $\frac{dy}{dt}=\frac{d}{dt}\left[a^{\left(t+\frac{1}{t}\right)}\right]$

$$=a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$=a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\left(1-\frac{1}{t^{2}}\right)$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\left(1-\frac{1}{t^{2}}\right)}{a\left(t+\frac{1}{t}\right)^{a-1}\cdot\left(1-\frac{1}{t^{2}}\right)}$$

$$=\frac{a^{t+\frac{1}{t}}\cdot\log a\cdot\left(t+\frac{1}{t}\right)}{a\cdot\left(t+\frac{1}{t}\right)^{a}}$$

$$=\frac{y\cdot\log a\cdot\left(\frac{t^{2}+1}{t}\right)}{ax}\qquad\text{... [By (1)]}$$

$= \frac{y(t^2+1)\log a}{axt}$.

(vii) x = $\cos^{-1}\left(\frac{2t}{1+t^2}\right)$, y = $\sec^{-1}\left(\sqrt{1+t^2}\right)$

Solution:

x = $\cos^{-1}\left(\frac{2t}{1+t^2}\right)$, y = $\sec^{-1}\left(\sqrt{1+t^2}\right)$

Put t = tan θ Then θ = $\tan^{-1}t$

$\therefore\ x = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right), y = \sec^{-1}(\sqrt{1+\tan^2\theta})$

$\therefore\ x = \cos^{-1}(\sin 2\theta), y = \sec^{-1}(\sqrt{\sec^2\theta})$

$\therefore\ x = \cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right], y = \sec^{-1}(\sec\theta)$

$\therefore\ x = \frac{\pi}{2}-2\theta,\quad y = \theta$

$\therefore\ x = \frac{\pi}{2}-2\tan^{-1}t,\ y = \tan^{-1}t$

Differentiating x and y w.r.t. t, we get

$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{\pi}{2}\right) - 2\frac{d}{dt}(\tan^{-1}t)$

$= 0 - 2\times\frac{1}{1+t^2} = \frac{-2}{1+t^2}$

and $\frac{dy}{dt} = \frac{d}{dt}(\tan^{-1}t) = \frac{1}{1+t^2}$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{1}{1+t^2}\right)}{\left(\frac{-2}{1+t^2}\right)}$$

$$= -\frac{1}{2}.$$

(viii) $x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Solution:

$x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Put $t = \cos\theta$. Then $\theta = \cos^{-1}t$

$x = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$\therefore\ x = \cos^{-1}(\cos 3\theta),\ y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$\therefore\ x = 3\theta \quad \text{and} \quad y = \theta$$

$$\therefore\ x = 3\cos^{-1}t \quad \text{and} \quad y = \cos^{-1}t$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = 3\frac{d}{dt}(\cos^{-1}t)$$

$$= 3 \times \frac{-1}{\sqrt{1-t^2}} = \frac{-3}{\sqrt{1-t^2}}$$

and $\frac{dy}{dt}=\frac{d}{dt}(\cos^{-1}t)=\frac{-1}{\sqrt{1-t^2}}$

$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\frac{-1}{\sqrt{1-t^2}}\right)}{\left(\frac{-3}{\sqrt{1-t^2}}\right)}$$

$$=\frac{1}{3}.$$

Alternative Method :

$x=\cos^{-1}(4t^3-3t),\ t=\tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Put $t=\cos\theta$.

Then $x=\cos^{-1}(4\cos^3\theta-3\cos\theta)$,

$y=\tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$

$\therefore\ x=\cos^{-1}(\cos 3\theta),\ y=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\tan^{-1}(\tan\theta)$

$\therefore\ x=3\theta,\ y=\theta$

$\therefore\ x=3y$

$\therefore\ y=\frac{1}{3}x$

$\therefore\ \frac{dy}{dx}=\frac{1}{3}\frac{d}{dx}(x)$

$$= \frac{1}{3} \times 1 = \frac{1}{3}.$$

Question 2.

Find $\frac{dy}{dx}$, if

(i) $x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$ at $\theta = \frac{\pi}{6}$

Solution:

$x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\operatorname{cosec}\theta)^2 = 2\operatorname{cosec}\theta \cdot \frac{d}{d\theta}(\operatorname{cosec}\theta)$$

$$= 2\operatorname{cosec}\theta(-\operatorname{cosec}\theta\cot\theta)$$

$$= -2\operatorname{cosec}^2\theta\cot\theta$$

$$\text{and } \frac{dy}{d\theta} = \frac{d}{d\theta}(\cot\theta)^3 = 3\cot^2\theta \cdot \frac{d}{d\theta}(\cot\theta)$$

$$= 3\cot^2\theta \cdot (-\operatorname{cosec}^2\theta)$$

$$= -3\cot^2\theta \cdot \operatorname{cosec}^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-3\cot^2\theta \cdot \operatorname{cosec}^2\theta}{-2\operatorname{cosec}^2\theta \cdot \cot\theta}$$

$$= \frac{3}{2}\cot\theta$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } \theta = \frac{\pi}{6}} = \frac{3}{2}\cot\frac{\pi}{6} = \frac{3\sqrt{3}}{2}.$$

(ii) $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{3}$

Solution:

$x = a\cos^3\theta$, $y = a\sin^3\theta$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta)^3$$

$$= a \times 3\cos^2\theta \cdot \frac{d}{d\theta}(\cos\theta)$$

$$= 3a\cos^2\theta(-\sin\theta) = -3a\cos^2\theta\sin\theta$$

$$\text{and } \frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta)^3$$

$$= a \times 3\sin^2\theta \cdot \frac{d}{d\theta}(\sin\theta)$$

$$= 3a\sin^2\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\tan\theta$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } \theta=\frac{\pi}{3}} = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

(iii) $x = t^2 + t + 1$, $y = \sin(\frac{\pi t}{2}) + \cos(\frac{\pi t}{2})$ at $t = 1$

Solution:

$x = t^2 + t + 1$, $y = \sin(\frac{\pi t}{2}) + \cos(\frac{\pi t}{2})$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(t^2 + t + 1)$$

$$= 2t + 1 + 0 = 2t + 1$$

and $\frac{dy}{dt} = \frac{d}{dt}\left[\sin\left(\frac{\pi t}{2}\right)\right] + \frac{d}{dt}\left[\cos\left(\frac{\pi t}{2}\right)\right]$

$$= \cos\left(\frac{\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{\pi t}{2}\right) + \left[-\sin\left(\frac{\pi t}{2}\right)\right] \cdot \frac{d}{dt}\left(\frac{\pi t}{2}\right)$$

$$= \cos\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1 - \sin\left(\frac{\pi t}{2}\right) \times \frac{\pi}{2} \times 1$$

$$= \frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right)\right]$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right)\right]}{2t + 1}.$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=1} = \frac{\frac{\pi}{2}\left[\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right]}{2(1) + 1}$$

$$= \frac{\frac{\pi}{2}(0 - 1)}{3} = -\frac{\pi}{6}.$$

(iv) x = 2 cos t + cos 2t, y = 2 sin t – sin 2t at t = $\frac{\pi}{4}$

Solution:

x = 2 cos t + cos 2t, y = 2 sin t – sin 2t

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(2\cos t+\cos 2t)$$

$$=2\frac{d}{dt}(\cos t)+\frac{d}{dt}(\cos 2t)$$

$$=2(-\sin t)+(-\sin 2t)\cdot\frac{d}{dt}(2t)$$

$$=-2\sin t-\sin 2t\times 2\times 1$$

$$=-2\sin t-2\sin 2t$$

and $\frac{dy}{dt}=\frac{d}{dt}(2\sin t-\sin 2t)$

$$=2\frac{d}{dt}(\sin t)-\frac{d}{dt}(\sin 2t)$$

$$=2\cos t-\cos 2t\cdot\frac{d}{dt}(2t)$$

$$=2\cos t-\cos 2t\times 2\times 1$$

$$=2\cos t-2\cos 2t$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{2\cos t-2\cos 2t}{-2\sin t-2\sin 2t}$$

$$=\frac{\cos t-\cos 2t}{-\sin t-\sin 2t}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}}=\frac{\cos\frac{\pi}{4}-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{4}-\sin\frac{\pi}{2}}$$

$$= \frac{\frac{1}{\sqrt{2}} - 0}{-\frac{1}{\sqrt{2}} - 1} = \frac{-1}{1+\sqrt{2}}$$

$$= \frac{-1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$= \frac{-(1-\sqrt{2})}{1-2} = 1 - \sqrt{2}.$$

(v) x = t + 2 sin(πt), y = 3t – cos(πt) at t = $\frac{1}{2}$

Solution:

x = t + 2 sin(πt), y = 3t – cos(πt)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}[t + 2\sin(\pi t)]$$

$$= \frac{d}{dt}(t) + 2 \cdot \frac{d}{dt}[\sin(\pi t)]$$

$$= 1 + 2 \times \cos(\pi t) \cdot \frac{d}{dx}(\pi t)$$

$$= 1 + 2\cos(\pi t) \times \pi \times 1$$

$$= 1 + 2\pi\cos(\pi t)$$

and $\frac{dy}{dt} = \frac{d}{dt}[3t - \cos(\pi t)]$

$$= 3\frac{d}{dt}(t) - \frac{d}{dt}[\cos(\pi t)]$$

$$= 3 \times 1 - [-\sin(\pi t)] \cdot \frac{d}{dt}(\pi t)$$
$$= 3 + \sin(\pi t) \times \pi \times 1$$
$$= 3 + \pi \sin(\pi t)$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3 + \pi \sin(\pi t)}{1 + 2\pi \cos(\pi t)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t = \frac{1}{2}} = \frac{3 + \pi \sin\left(\frac{\pi}{2}\right)}{1 + 2\pi \cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{3 + \pi \times 1}{1 + 2\pi(0)} = 3 + \pi.$$

Question 3.

(i) If x = $a\sqrt{\sec\theta - \tan\theta}$, y = $a\sqrt{\sec\theta + \tan\theta}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

x = $a\sqrt{\sec\theta - \tan\theta}$, y = $a\sqrt{\sec\theta + \tan\theta}$

$$\therefore \frac{x}{a} = \sqrt{\sec\theta - \tan\theta}, \frac{y}{a} = \sqrt{\sec\theta + \tan\theta}$$

$$\therefore \sec\theta - \tan\theta = \frac{x^2}{a^2} \quad \ldots (1)$$

$$\sec\theta + \tan\theta = \frac{y^2}{a^2} \quad \ldots (2)$$

Adding (1) and (2), we get

$$2\sec\theta = \frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{x^2+y^2}{a^2}$$

$$\therefore\ \sec\theta = \frac{x^2+y^2}{2a^2}$$

Subtracting (1) from (2), we get

$$2\tan\theta = \frac{y^2}{a^2} - \frac{x^2}{a^2} = \frac{y^2-x^2}{a^2}$$

$$\therefore\ \tan\theta = \frac{y^2-x^2}{2a^2}$$

$\therefore\ \sec^2\theta - \tan^2\theta = 1$ gives,

$$\left(\frac{x^2+y^2}{2a^2}\right)^2 - \left(\frac{y^2-x^2}{2a^2}\right)^2 = 1$$

$$\therefore\ (x^2+y^2)^2 - (y^2-x^2)^2 = 4a^4$$

$$\therefore\ (x^4+2x^2y^2+y^4) - (y^4-2x^2y^2+x^4) = 4a^4$$

$$\therefore\ 4x^2y^2 = 4a^4$$

$$\therefore\ x^2y^2 = a^4$$

Differentiating both sides w.r.t. x, we get

$$x^2\cdot\frac{d}{dx}(y^2) + y^2\cdot\frac{d}{dx}(x^2) = 0$$

$$\therefore\ x^2 \times 2y\frac{dy}{dx} + y^2 \times 2x = 0$$

$$\therefore\ 2x^2y\frac{dy}{dx} = -2xy^2$$

$\therefore \frac{dy}{dx} = -\frac{y}{x}$.

(ii) If x = $e^{\sin 3t}$, y = $e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$

Solution:

x = $e^{\sin 3t}$, y = $e^{\cos 3t}$

log x = log $e^{\sin 3t}$, log y = log $e^{\cos 3t}$

log x = (sin 3t)(log e), log y = (cos 3t)(log e)

log x = sin 3t, log y = cos 3t ….. (1) [∵ log e = 1]

Differentiating both sides w.r.t. t, we get

$$\frac{1}{x} \cdot \frac{dx}{dt} = \frac{d}{dt}(\sin 3t) = \cos 3t \cdot \frac{d}{dt}(3t)$$

$$= \cos 3t \times 3 = 3 \cos 3t$$

$$\text{and } \frac{1}{y} \cdot \frac{dy}{dt} = \frac{d}{dt}(\cos 3t) = -\sin 3t \cdot \frac{d}{dx}(3t)$$

$$= -\sin 3t \times 3 = -3 \sin 3t$$

$$\therefore \frac{dx}{dt} = 3x \cos 3t \text{ and } \frac{dy}{dt} = -3y \sin 3t$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-3y \sin 3t}{3x \cos 3t}.$$

$$= \frac{-y \sin 3t}{x \cos 3t} = \frac{-y \log x}{x \log y}. \qquad \ldots \text{[By (1)]}$$

(iii) If x = $\frac{t+1}{t-1}$, y = $\frac{1-t}{t+1}$, then show that y2 – $\frac{dy}{dx}$ = 0.
Solution:
x = $\frac{t+1}{t-1}$, y = $\frac{1-t}{t+1}$

$$\therefore\ y=\frac{1}{\left(\frac{t+1}{1-t}\right)}=\frac{-1}{\left(\frac{t+1}{t-1}\right)}$$

$$\therefore\ y=-\frac{1}{x}$$

$$\therefore\ xy=-1 \quad \ldots (1)$$

Differentiating both sides w.r.t. t, we get

$$x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)=0$$

$$\therefore\ x\frac{dy}{dx}+y\times 1=0$$

$$\therefore\ -\frac{1}{y}\frac{dy}{dx}+y=0$$

$$\therefore\ -\frac{dy}{dx}+y^2=0$$

$$\therefore\ y^2-\frac{dy}{dx}=0.$$

(iv) If x = a $\cos^3 t$, y = a $\sin^3 t$, then show that $\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$
Solution:
x = a $\cos^3 t$, y = a $\sin^3 t$

x = a $\cos^3 t$, y = a $\sin^3 t$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dt}(\cos t)^3 = a\cdot 3(\cos t)^2\frac{d}{dt}(\cos t)$$

$$= 3a\cos^2 t(-\sin t) = -3a\cos^2 t\sin t$$

and $\frac{dy}{dt} = a\frac{d}{dt}(\sin t)^3 = a\cdot 3(\sin t)^2\frac{d}{dt}(\sin t)$

$$= 3a\sin^2 t\cdot\cos t$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t} \quad \ldots (1)$$

Now, $x = a\cos^3 t \quad \therefore \cos^3 t = \frac{x}{a}$

$$\therefore \cos t = \left(\frac{x}{a}\right)^{1/3}$$

$y = a\sin^3 t \quad \therefore \sin^3 t = \frac{y}{a} \quad \therefore \sin t = \left(\frac{y}{a}\right)^{1/3}$

$\therefore$ from (1), $\frac{dy}{dx} = -\frac{y^{1/3}/a^{1/3}}{x^{1/3}/a^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$.

Alternative Method :

$x = a\cos^3 t,\ y = a\sin^3 t$

$\therefore \cos^3 t = \frac{x}{a},\ \sin^3 t = \frac{y}{a}$

$\therefore \cos t = \left(\frac{x}{a}\right)^{1/3},\ \sin t = \left(\frac{y}{a}\right)^{1/3}$

$\therefore \cos^2 t + \sin^2 t = 1$ gives

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$\therefore x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = 0$$

$$\therefore \frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}.$$

(v) If x = 2 $\cos^4$(t + 3), y = 3 $\sin^4$(t + 3), show that $\frac{dy}{dx} = -\sqrt{\frac{3y}{2x}}$

Solution:

x = 2 $\cos^4$(t + 3), y = 3 $\sin^4$(t + 3)

$$\therefore \cos^4(t+3) = \frac{x}{2}, \sin^4(t+3) = \frac{y}{3}$$

$$\therefore \cos^2(t+3) = \sqrt{\frac{x}{2}}, \sin^2(t+3) = \sqrt{\frac{y}{3}}$$

$$\because \cos^2(t+3) + \sin^2(t+3) = 1$$

$$\therefore \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} = 1$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{2}}\frac{d}{dx}(\sqrt{x})+\frac{1}{\sqrt{3}}\frac{d}{dx}(\sqrt{y})=0$$

$$\therefore \frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{3}}\times\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\therefore \frac{1}{2\sqrt{3}\cdot\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{2}\cdot\sqrt{x}}$$

$$\therefore \frac{dy}{dx}=-\frac{\sqrt{3}\cdot\sqrt{y}}{\sqrt{2}\cdot\sqrt{x}}=-\sqrt{\frac{3y}{2x}}.$$

(vi) If $x = \log(1 + t^2)$, $y = t - \tan^{-1}t$, show that $\frac{dy}{dx} = \frac{\sqrt{e^x-1}}{2}$

Solution:

$x = \log(1 + t^2)$, $y = t - \tan^{-1}t$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}[\log(1+t^2)]=\frac{1}{1+t^2}\cdot\frac{d}{dt}(1+t^2)$$

$$=\frac{1}{1+t^2}\times(0+2t)=\frac{2t}{1+t^2}$$

and $\frac{dy}{dt}=\frac{d}{dt}(t)-\frac{d}{dt}(\tan^{-1}t)$

$$=1-\frac{1}{1+t^2}=\frac{1+t^2-1}{1+t^2}=\frac{t^2}{1+t^2}$$

$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\frac{t^2}{1+t^2}\right)}{\left(\frac{2t}{1+t^2}\right)}=\frac{t}{2}$$

Now, $x=\log(1+t^2)$

$\therefore 1+t^2=e^x$

$\therefore t^2=e^x-1$

$\therefore t=\sqrt{e^x-1}$

$\therefore \frac{dy}{dx}=\frac{\sqrt{e^x-1}}{2}$.

(vii) If x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}$, show that sin x + $\frac{dy}{dx}$ = 0

Solution:

x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}[\sin^{-1}(e^t)]$$

$$=\frac{1}{\sqrt{1-(e^t)^2}}\cdot\frac{d}{dt}(e^t)$$

$$=\frac{1}{\sqrt{1-e^{2t}}}\times e^t=\frac{e^t}{\sqrt{1-e^{2t}}}$$

and $\frac{dy}{dt}=\frac{d}{dt}(\sqrt{1-e^{2t}})$

$$= \frac{1}{2\sqrt{1-e^{2t}}} \cdot \frac{d}{dt}(1-e^{2t})$$

$$= \frac{1}{2\sqrt{1-e^{2t}}} \times (0 - e^{2t} \times 2)$$

$$= \frac{-e^{2t}}{\sqrt{1-e^{2t}}}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{-e^{2t}}{\sqrt{1-e^{2t}}}\right)}{\left(\frac{e^{t}}{\sqrt{1-e^{2t}}}\right)}$$

$$= -e^{t}$$

$$= -\sin x \qquad \ldots [\because x = \sin^{-1}(e^{t})]$$

$$\therefore \sin x + \frac{dy}{dx} = 0.$$

(viii) If x = $\frac{2bt}{1+t^2}$, y = $a\left(\frac{1-t^2}{1+t^2}\right)$, show that $\frac{dx}{dy} = -\frac{b^2 y}{a^2 x}$

Solution:

x = $\frac{2bt}{1+t^2}$, y = $a\left(\frac{1-t^2}{1+t^2}\right)$

Put $t = \tan\theta$.

Then $x = b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$, $y = a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\therefore x = b\sin 2\theta, \; y = a\cos 2\theta$

$$\therefore \frac{x}{b} = \sin 2\theta, \ \frac{y}{a} = \cos 2\theta$$

$$\therefore \left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = \sin^2 2\theta + \cos^2 2\theta$$

$$\therefore \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Differentiating both sides w.r.t. y, we get

$$\frac{1}{b^2} \times 2x\frac{dx}{dy} + \frac{1}{a^2} \times 2y = 0$$

$$\therefore \frac{2x}{b^2}\frac{dx}{dy} = \frac{-2y}{a^2}$$

$$\therefore \frac{dx}{dy} = -\frac{b^2 y}{a^2 x}.$$

Question 4.
(i) Differentiate x sin x w.r.t tan x.
Solution:
Let u = x sinx and v = tan x
Then we want to find $\frac{du}{dv}$
Differentiating u and v w.r.t. x, we get

$$\frac{du}{dx} = \frac{d}{dx}(x \sin x)$$

$$= x\frac{d}{dx}(\sin x) + (\sin x) \cdot \frac{d}{dx}(x)$$

$= x\cos x + (\sin x) \times 1$

$= x\cos x + \sin x$

and $\frac{dv}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$

$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{x\cos x + \sin x}{\sec^2 x}.$

(ii) Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

Let u = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and v = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then we want to find $\frac{du}{dv}$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$.

$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\tan^{-1} x$

$\therefore \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1} x) = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$

Also, $v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1} x$

$$\therefore \frac{dv}{dx}=2\frac{d}{dx}(\tan^{-1}x)=2\times\frac{1}{1+x^2}=\frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{2}{1+x^2}\right)}=1.$$

Alternative Method :

Let $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then we want to find $\frac{du}{dv}$

Put $x=\tan\theta$.

Then $u=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$

and $v=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta$

$\therefore\ u=v$

Differentiating both sides w.r.t. v, we get

$\frac{du}{dv}=1.$

(iii) Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Solution:

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and

$v = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$. Then we want to find $\frac{du}{dv}$.

Put $x = \cos\theta$. Then $\theta = \cos^{-1}x$.

$$\therefore\ u = \tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right) = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}(\cot\theta) = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}x$$

$$\therefore\ \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(\cos^{-1}x)$$

$$= 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$v = \sec^{-1}\left(\frac{1}{2x^2-1}\right) = \cos^{-1}(2x^2-1)$$

$$= \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta = 2\cos^{-1}x$$

$$\therefore\ \frac{dv}{dx} = 2\cdot\frac{d}{dx}(\cos^{-1}x) = \frac{-2}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-2} = -\frac{1}{2}.$$

(iv) Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. $\tan^{-1}x$

Solution:

Let u = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and v = $\tan^{-1}x$

Then we want to find $\frac{du}{dv}$

Put x = tan θ. Then θ = $\tan^{-1}x$.

$$\therefore\ u=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta$$

$$\therefore\ u=2\tan^{-1}x$$

$$\therefore\ \frac{du}{dx}=2\cdot\frac{d}{dx}(\tan^{-1}x)=2\times\frac{1}{1+x^2}$$

$$=\frac{2}{1+x^2}$$

Also, $v=\tan^{-1}x$

$$\therefore\ \frac{dv}{dx}=\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)}=2.$$

(v) Differentiate 3x w.r.t. $\log_x 3$.

Solution:

Let u = 3x and v = $\log_x 3$.

Then we want to find $\frac{du}{dv}$

Differentiating u and v w.r.t. x, we get

Differentiating u and v w.r.t. x, we get

$\frac{du}{dx} = \frac{d}{dx}(3^x) = 3^x \cdot \log 3$

and $\frac{dv}{dx} = \frac{d}{dx}(\log_x 3) = \frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$

$= \log 3 \cdot \frac{d}{dx}(\log x)^{-1}$

$= (\log 3)(-1)(\log x)^{-2} \cdot \frac{d}{dx}(\log x)$

$= \frac{-\log 3}{(\log x)^2} \times \frac{1}{x} = \frac{-\log 3}{x(\log x)^2}$

$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{3^x \cdot \log 3}{\left[\frac{-\log 3}{x(\log x)^2}\right]}$

$= -x(\log x)^2 \cdot 3^x.$

(vi) Differentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ w.r.t. $\sec^{-1}x$.

Solution:

Let u = $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ and v = $\sec^{-1}x$

Then we want to find $\frac{du}{dv}$.

Differentiating u and v w.r.t. x, we get

$\frac{du}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$

$$\frac{\cos x}{1+\sin x}=\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$=\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$\therefore \frac{du}{dx}=\frac{d}{dx}\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$$

$$=\frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{1}{2}\frac{d}{dx}(x)$$

$$=0-\frac{1}{2}\times 1=-\frac{1}{2}$$

and $\frac{dv}{dx}=\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$

$$\therefore \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{x\sqrt{x^2-1}}\right)}=-\frac{x\sqrt{x^2-1}}{2}.$$

(vii) Differentiate x^x w.r.t. $x^{\sin x}$.
Solution:
Let $u = x^x$ and $v = x^{\sin x}$
Then we want to find $\frac{du}{dx}$.
Take, $u = x^x$
$\log u = \log x^x = x \log x$
Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

Also, $v = x^{\sin x}$

$$\therefore \log v = \log x^{\sin x} = (\sin x)(\log x)$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[(\sin x)(\log x)]$$

$$= (\sin x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\sin x)$$

$$= (\sin x) \times \frac{1}{x} + (\log x)(\cos x)$$

$$\therefore \frac{dv}{dx} = v\left[\frac{\sin x}{x} + (\log x)(\cos x)\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + (\log x)(\cos x)\right]$$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{x^x(1+\log x)}{x^{\sin x}\left[\frac{\sin x}{x} + (\log x)(\cos x)\right]}$$

$$= \frac{x^x(1+\log x) \times x}{x^{\sin x}[\sin x + x\cos x \cdot \log x]}$$

$$= \frac{(1+\log x) \cdot x^{x+1-\sin x}}{\sin x + x\cos x \cdot \log x}.$$

(viii) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Solution:

Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and v = $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Then we want to find $\frac{du}{dv}$

u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put x = tan θ. Then θ = $\tan^{-1}x$ and

and $v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

Then we want to find $\frac{du}{dv}$.

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

Put $x=\tan\theta$. Then $\theta=\tan^{-1}x$ and

$$\frac{\sqrt{1+x^2}-1}{x}=\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$=\frac{\sec\theta-1}{\tan\theta}=\frac{\frac{1}{\cos\theta}-1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$=\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$=\tan\left(\frac{\theta}{2}\right)$$

$$\therefore\ u=\tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

$$\therefore\ \frac{du}{dx}=\frac{1}{2}\frac{d}{dx}(\tan^{-1}x)$$

$$=\frac{1}{2}\times\frac{1}{1+x^2}=\frac{1}{2(1+x^2)}$$

$$v=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$$

Put $x=\sin\theta$. Then $\theta=\sin^{-1}x$ and

$$\frac{2x\sqrt{1-x^2}}{1-2x^2}=\frac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}$$

$$=\frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}=\frac{\sin 2\theta}{\cos 2\theta}=\tan 2\theta$$

$\therefore\ v=\tan^{-1}(\tan 2\theta)=2\theta=2\sin^{-1}x$

$\therefore\ \frac{dv}{dx}=2\frac{d}{dx}(\sin^{-1}x)$

$$=2\times\frac{1}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left[\frac{1}{2(1+x^2)}\right]}{\left(\frac{2}{\sqrt{1-x^2}}\right)}$$

$$=\frac{1}{2(1+x^2)}\times\frac{\sqrt{1-x^2}}{2}=\frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

Ex 1.5

Question 1.
Find the second order derivatives of the following:
(i) $2x^5 - 4x^3 - \frac{2}{x^2} - 9$
Solution:
Let $y = 2x^5 - 4x^3 - \frac{2}{x^2} - 9$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 4x^3 - \frac{2}{x^2} - 9\right)$$
$$= 2\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^{-2}) - \frac{d}{dx}(9)$$
$$= 2 \times 5x^4 - 4 \times 3x^2 - 2(-2)x^{-3} - 0$$
$$= 10x^4 - 12x^2 + 4x^{-3}$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(10x^4 - 12x^2 + 4x^{-3})$$
$$= 10\frac{d}{dx}(x^4) - 12\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-3})$$
$$= 10 \times 4x^3 - 12 \times 2x + 4(-3)x^{-4}$$
$$= 40x^3 - 24x - \frac{12}{x^4}.$$

(ii) e^{2x} . tan x

Solution:

Let y = e^{2x} . tan x

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{2x} \cdot \tan x)$$

$$= e^{2x} \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(e^{2x})$$

$$= e^{2x} \times \sec^2 x + \tan x \times e^{2x} \cdot \frac{d}{dx}(2x)$$

$$= e^{2x} \cdot \sec^2 x + e^{2x} \cdot \tan x \times 2$$

$$= e^{2x}(\sec^2 x + 2\tan x)$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{2x}(\sec^2 x + 2\tan x)]$$

$$= e^{2x} \cdot \frac{d}{dx}(\sec^2 x + 2\tan x) + (\sec^2 x + 2\tan x)\frac{d}{dx}(e^{2x})$$

$$= e^{2x}\left[\frac{d}{dx}(\sec x)^2 + 2\frac{d}{dx}(\tan x)\right] +$$

$$(\sec^2 x + 2\tan x) \times e^{2x} \cdot \frac{d}{dx}(2x)$$

$$= e^{2x}[2\sec x \cdot \frac{d}{dx}(\sec x) + 2\sec^2 x] + (\sec^2 x + 2\tan x)e^{2x} \times 2$$

$$= e^{2x}(2\sec x \cdot \sec x \tan x + 2\sec^2 x) + 2e^{2x}(\sec^2 x + 2\tan x)$$

$$= 2e^{2x}(\sec^2 x \tan x + \sec^2 x + \sec^2 x + 2\tan x)$$

$$= 2e^{2x}[\sec^2 x(\tan x + 1) + 1 + \tan^2 x + 2\tan x]$$

$$= 2e^{2x}[\sec^2 x(\tan x + 1) + 1 + \tan^2 x + 2\tan x]$$
$$= 2e^{2x}[\sec^2 x(1 + \tan x) + (1 + \tan x)^2]$$
$$= 2e^{2x}[(1 + \tan x)(\sec^2 x + 1 + \tan x)]$$
$$= 2e^{2x}[(1 + \tan x)(1 + \tan^2 x + 1 + \tan x)]$$
$$= 2e^{2x}(1 + \tan x)(2 + \tan x + \tan^2 x).$$

(iii) e^{4x} . cos 5x

Solution:

Let y = e^{4x} . cos 5x

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{4x} \cdot \cos 5x)$$
$$= e^{4x} \cdot \frac{d}{dx}(\cos 5x) + \cos 5x \cdot \frac{d}{dx}(e^{4x})$$
$$= e^{4x} \cdot (-\sin 5x) \cdot \frac{d}{dx}(5x) + \cos 5x \times e^{4x} \cdot \frac{d}{dx}(4x)$$
$$= -e^{4x} \cdot \sin 5x \times 5 + e^{4x} \cos 5x \times 4$$
$$= e^{4x}(4\cos 5x - 5\sin 5x)$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{4x}(4\cos 5x - 5\sin 5x)]$$
$$= e^{4x}\frac{d}{dx}(4\cos 5x - 5\sin 5x) + (4\cos 5x - 5\sin 5x) \cdot \frac{d}{dx}(e^{4x})$$

$$= e^{4x}[4(-\sin 5x)\cdot\frac{d}{dx}(5x) - 5\cos 5x\cdot\frac{d}{dx}(5x)] + (4\cos 5x - 5\sin 5x)\times e^{4x}\cdot\frac{d}{dx}(4x)$$

$$= e^{4x}[-4\sin 5x\times 5 - 5\cos 5x\times 5] + (4\cos 5x - 5\sin 5x)e^{4x}\times 4$$

$$= e^{4x}(-20\sin 5x - 25\cos 5x + 16\cos 5x - 20\sin 5x)$$

$$= e^{4x}(-9\cos 5x - 40\sin 5x)$$

$$= -e^{4x}(9\cos 5x + 40\sin 5x).$$

(iv) $x^3 . \log x$

Solution:

Let $y = x^3 . \log x$

Then, $\frac{dy}{dx} = \frac{d}{dx}(x^3\cdot\log x)$

$$= x^3\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(x^3)$$

$$= x^3\times\frac{1}{x} + (\log x)\times 3x^2$$

$$= x^2 + 3x^2\log x$$

$$= x^2(1 + 3\log x)$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}[x^2(1 + 3\log x)]$

$$= x^2 \cdot \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x) \cdot \frac{d}{dx}(x^2)$$

$$= x^2\left(0 + 3 \times \frac{1}{x}\right) + (1 + 3\log x) \times 2x$$

$$= 3x + 2x + 6x\log x$$

$$= 5x + 6x\log x = x(5 + 6\log x).$$

(v) log(log x)

Solution:

Let y = log(log x)

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}[\log(\log x)]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \times \frac{1}{x} = \frac{1}{x\log x}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(x\log x)^{-1}$$

$$= (-1)(x\log x)^{-2} \cdot \frac{d}{dx}(x\log x)$$

$$= \frac{-1}{(x\log x)^2} \cdot \left[x\frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)\right]$$

$$= \frac{-1}{(x\log x)^2} \cdot \left[x \times \frac{1}{x} + (\log x) \times 1\right]$$

$$= -\frac{1+\log x}{(x\log x)^2}.$$

(vi) x^x

Solution:

$y = x^x$

$\log y = \log x^x = x \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(x\log x)$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(x)$$

$$= \frac{x}{x} + (\log x)(1) = 1 + \log x$$

$$\therefore \frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1+\log x) \quad \ldots (1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[x^x(1+\log x)]$$

$$= x^x\cdot\frac{d}{dx}(1+\log x) + (1+\log x)\cdot\frac{d}{dx}(x^x)$$

$$= x^x\left(0+\frac{1}{x}\right) + (1+\log x)\cdot x^x(1+\log x) \quad \ldots \text{[By (1)]}$$

$= x^{x-1} + x^x (1 + \log x)^2.$

Question 2.
Find $\frac{d^2y}{dx^2}$ of the following:
(i) x = a(θ – sin θ), y = a (1 – cos θ)
Solution:
x = a(θ – sin θ), y = a (1 – cos θ)
Differentiating x and y w.r.t. θ, we get
$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta)$ = a(1 – cos θ) …….(1)

$$\text{and } \frac{dy}{d\theta} = a\frac{d}{d\theta}(1 - \cos\theta)$$

$$= a[0 - (-\sin\theta)] = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\cot\left(\frac{\theta}{2}\right)\right]$$

$$= \frac{d}{d\theta}\left[\cot\left(\frac{\theta}{2}\right)\right]\cdot\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\cdot\frac{d}{d\theta}\left(\frac{\theta}{2}\right)\times\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= -\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2}\times\frac{1}{a(1-\cos\theta)} \quad \ldots [\text{By } (1)]$$

$$= -\frac{1}{2a}\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{1}{4a}\cdot\operatorname{cosec}^4\left(\frac{\theta}{2}\right).$$

(ii) $x = 2at^2$, $y = 4at$

Solution:

$x = 2at^2$, $y = 4at$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a\cdot\frac{d}{dt}(t^2)$$

$$= 2a \times 2t = 4at \quad \ldots (1)$$

and $\frac{dy}{dt} = \frac{d}{dt}(4at) = 4a\frac{d}{dt}(t)$

$$= 4a \times 1 = 4a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{4a}{4at} = \frac{1}{t}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}(t^{-1}) \times \frac{dt}{dx}$

$$= -1(t)^{-2} \times \frac{1}{\left(\frac{dx}{dt}\right)}$$

$$= -\frac{1}{t^2} \times \frac{1}{4at} \quad \ldots \text{[By (1)]}$$

$$= -\frac{1}{4at^3}.$$

(iii) $x = \sin\theta$, $y = \sin^3\theta$ at $\theta = \frac{\pi}{2}$

Solution:

$x = \sin\theta$, $y = \sin^3\theta$

Differentiating x and y w.r.t. θ, we get,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta \quad \ldots (1)$$

$$\text{and } \frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta)^3 = 3(\sin\theta)^2 \cdot \frac{d}{d\theta}(\sin\theta)$$

$$= 3\sin^2\theta \cdot \cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{3\sin^2\theta\cos\theta}{\cos\theta}$$

$$= 3\sin^2\theta$$

$$\text{and } \frac{d^2y}{dx^2} = 3\frac{d}{dx}(\sin\theta)^2$$

$$= 3\frac{d}{d\theta}(\sin\theta)^2 \times \frac{d\theta}{dx}$$

$$= 3 \times 2\sin\theta \cdot \frac{d}{d\theta}(\sin\theta) \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$= 6\sin\theta \cdot \cos\theta \times \frac{1}{\cos\theta} \qquad \ldots \text{[By (1)]}$$

$$= 6\sin\theta$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at } \theta = \frac{\pi}{2}} = 6\sin\frac{\pi}{2}$$

$$= 6 \times 1 = 6.$$

Alternative Method :

$x = \sin\theta,\ y = \sin^3\theta$

$\therefore\ y = x^3$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{d^2y}{dx^2} = 3\frac{d}{dx}(x^2) = 3 \times 2x = 6x$$

If $\theta = \frac{\pi}{2}$, then $x = \sin\frac{\pi}{2} = 1$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at } \theta = \frac{\pi}{2}} = \left(\frac{d^2y}{dx^2}\right)_{\text{at } x = 1} = 6(1) = 6.$$

(iv) $x = a\cos\theta$, $y = b\sin\theta$ at $\theta = \frac{\pi}{4}$

Solution:
x = a cos θ, y = b sin θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta)=a\frac{d}{d\theta}(\cos\theta)$$

$$=a(-\sin\theta)=-a\sin\theta \quad \ldots (1)$$

and $$\frac{dy}{d\theta}=\frac{d}{d\theta}(b\sin\theta)=b\frac{d}{dx}(\sin\theta)$$

$$=b\cos\theta$$

$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{b\cos\theta}{-a\sin\theta}$$

$$=\left(-\frac{b}{a}\right)\cot\theta$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\left(-\frac{b}{a}\right)\cot\theta\right]$$

$$=-\frac{b}{a}\cdot\frac{d}{d\theta}(\cot\theta)\times\frac{d\theta}{dx}$$

$$=\left(-\frac{b}{a}\right)(-\operatorname{cosec}^2\theta)\times\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$=\left(\frac{b}{a}\right)\operatorname{cosec}^2\theta\times\frac{1}{-a\sin\theta} \quad \ldots \text{[By (1)]}$$

$$= \left(-\frac{b}{a^2}\right) \text{cosec}^3\theta$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at } \theta=\frac{\pi}{4}} = \left(-\frac{b}{a^2}\right) \text{cosec}^3\frac{\pi}{4}$$

$$= \frac{-b}{a^2} \times (\sqrt{2})^3$$

$$= -\frac{2\sqrt{2}b}{a^2}.$$

Question 3.

(i) If x = at^2 and y = 2at, then show that $xy\frac{d^2y}{dx^2} + a = 0$

Solution:

x = at^2, y = 2at ....(1)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= a \times 2t = 2at \qquad \dots (2)$$

$$\text{and } \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t)$$

$$= 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}(t^{-1}) \cdot \frac{dt}{dx}$$

$$= (-1)t^{-2} \cdot \frac{1}{\left(\frac{dx}{dt}\right)} = \frac{-1}{t^2} \times \frac{1}{2at} \quad \ldots \text{[By (2)]}$$

$$= -\frac{1}{2at^3}.$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{(at^2)(2at)} \times a$$

$$= -\frac{a}{xy} \quad \ldots \text{[By (1)]}$$

$$\therefore xy\frac{d^2y}{dx^2} = -a$$

$$\therefore xy\frac{d^2y}{dx^2} + a = 0.$$

(ii) If y = $e^{m \tan^{-1} x}$, show that $\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-m)\frac{dy}{dx} = 0$

Solution:

y = $e^{m \tan^{-1} x}$ ……..(1)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{m\tan^{-1}x}\right)$$

$$= e^{m\tan^{-1}x} \cdot \frac{d}{dx}\left(m\tan^{-1}x\right)$$

$$= e^{m\tan^{-1}x} \times m \times \frac{1}{1+x^2}$$

$$\therefore (1+x^2)\frac{dy}{dx} = my \quad \ldots \text{[By (1)]}$$

Differentiating again w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2)=m\frac{dy}{dx}$$

$$\therefore\ (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}(0+2x)=m\frac{dy}{dx}$$

$$\therefore\ (1+x^2)\frac{d^2y}{dx^2}+2x\cdot\frac{dy}{dx}=m\frac{dy}{dx}.$$

$$\therefore\ (1+x^2)\frac{d^2y}{dx^2}+(2x-m)\frac{dy}{dx}=0.$$

(iii) If x = cos t, y = e^{mt}, show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$

Solution:

x = cos t, y = e^{mt}

$$\therefore\ t=\cos^{-1}x \text{ and } y=e^{m\cos^{-1}x} \quad \dots (1)$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(e^{m\cos^{-1}x})$$

$$=e^{m\cos^{-1}x}\cdot\frac{d}{dx}(m\cos^{-1}x)$$

$$=e^{m\cos^{-1}x}\times m\times\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore\ \sqrt{1-x^2}\cdot\frac{dy}{dx}=-my \quad \dots \text{[By (1)]}$$

$$\therefore\ (1-x^2)\left(\frac{dy}{dx}\right)^2=m^2y^2$$

Differentiating again w.r.t. x, we get

$$(1-x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(1-x^2)=m^2\cdot\frac{d}{dx}(y^2)$$

$$\therefore\ (1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(0-2x)=m^2\times 2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=m^2y$$

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0.$$

(iv) If y = x + tan x, show that $\cos^2 x\cdot\frac{d^2y}{dx^2}-2y+2x=0$

Solution:

y = x + tan x

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(x+\tan x)$$

$$=1+\sec^2 x$$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}(1+\sec^2 x)$

$$=\frac{d}{dx}(1)+\frac{d}{dx}(\sec x)^2$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(\sec x)^2$$

$$= 0 + 2\sec x \cdot \frac{d}{dx}(\sec x)$$

$$= 2\sec x \cdot \sec x \tan x$$

$$= 2\sec^2 x \tan x$$

$$\therefore \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x$$

$$= \cos^2 x(2\sec^2 x \tan x) - 2(x + \tan x) + 2x$$

$$= \cos^2 x \times \frac{2}{\cos^2 x} \times \tan x - 2x - 2\tan x + 2x$$

$$= 2\tan x - 2\tan x.$$

$$\therefore \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0.$$

(v) If $y = e^{ax} . \sin(bx)$, show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

Solution:

$y = e^{ax} . \sin(bx)$ ....(1)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax} \cdot \sin(bx)]$$

$$= e^{ax} \cdot \frac{d}{dx}[\sin(bx)] + \sin(bx) \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot \cos(bx) \cdot \frac{d}{dx}(bx) + \sin(bx) \times e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= e^{ax} \cdot \cos(bx) \times b + e^{ax} \cdot \sin(bx) \times a$$

$$\therefore \; y_1 = e^{ax}[b\cos(bx) + a\sin(bx)] \qquad \dots (2)$$

Differentiating again w.r.t. x, we get

$$\frac{dy_1}{dx} = \frac{d}{dx}[e^{ax}\{b\cos(bx) + a\sin(bx)\}]$$

$$= e^{ax} \cdot \frac{d}{dx}[b\cos(bx) + a\sin(bx)] + [b\cos(bx) + a\sin(bx)] \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot [b\{-\sin(bx)\} \cdot \frac{d}{dx}(bx) + a\cos(bx) \cdot \frac{d}{dx}(bx)] + [b\cos(bx) + a\sin(bx)] \times e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= e^{ax}[-b\sin(bx) \times b + a\cos(bx) \times b] + [b\cos(bx) + a\sin(bx)]e^{ax} \times a$$

$$= e^{ax}[-b^2\sin(bx) + ab\cos(bx) + ab\cos(bx) + a^2\sin(bx)]$$

$$\therefore \; y_2 = e^{ax}[-b^2\sin(bx) + 2ab\cos(bx) + a^2\sin(bx)] \dots (3)$$

$$\therefore \; y_2 - 2ay_1 + (a^2 + b^2)y$$

$$= e^{ax}[-b^2\sin(bx) + 2ab\cos(bx) + a^2\sin(bx)] - 2a \cdot e^{ax}[b\cos(bx) + a\sin(bx)] + (a^2 + b^2)e^{ax}\sin(bx) \quad \dots \text{[By (1), (2) and (3)]}$$

$$= e^{ax}[-b^2\sin bx + 2ab\cos(bx) + a^2\sin(bx) - 2ab\cos(bx) - 2a^2\sin(bx) + a^2\sin(bx) + b^2\sin(bx)]$$

$= e^{ax} \times 0$

$\therefore\ y_2 - 2ay_1 + (a^2 + b^2)y = 0.$

(vi) If $\sec^{-1}\left(\frac{7x^3-5y^3}{7x^3+5y^3}\right) = m$, show that $\frac{d^2y}{dx^2} = 0$

Solution:

Solution : $\sec^{-1}\left(\frac{7x^3 - 5y^3}{7x^3 + 5y^3}\right) = m$

$\therefore\ \frac{7x^3 - 5y^3}{7x^3 + 5y^3} = \sec m = k$ ... (Say)

$\therefore\ 7x^3 - 5y^3 = 7kx^3 + 5ky^3$

$\therefore\ (5k + 5)y^3 = (7 - 7k)x^2$

$\therefore\ \frac{y^3}{x^3} = \frac{7 - 7k}{5k + 5}$

$\therefore\ \frac{y}{x} = \left(\frac{7 - 7k}{5k + 5}\right)^{\frac{1}{3}} = p$, where p is a constant.

$\therefore\ \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}(p)$

$\therefore\ \frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2} = 0$

$\therefore\ x\frac{dy}{dx} - y \times 1 = 0$

$\therefore\ x\frac{dy}{dx}=y$

$\therefore\ \frac{dy}{dx}=\frac{y}{x}$ ... (1)

$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}\right)$

$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}\right)$

$=\frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}$

$=\frac{x\left(\frac{y}{x}\right)-y\times 1}{x^2}$ ... [By (1)]

$=\frac{y-y}{x^2}=\frac{0}{x^2}=0$

Note : $\frac{dy}{dx}=\frac{y}{x}$, where $\frac{y}{x}=p$,

$\therefore\ \frac{dy}{dx}=p$, where p is a constant.

$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}(p)=0.$

(vii) If $2y=\sqrt{x+1}+\sqrt{x-1}$, show that $4(x^2-1)y_2+4xy_1-y=0$.

Solution:

2y = $\sqrt{x+1}+\sqrt{x-1}$ ...... (1)

Differentiating both sides w.r.t. x, we get

$$\therefore\ 2\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x+1})+\frac{d}{dx}(\sqrt{x-1})$$

$$=\frac{1}{2\sqrt{x+1}}(1+0)+\frac{1}{2\sqrt{x-1}}(1-0)$$

$$\therefore\ 2\frac{dy}{dx}=\frac{1}{2\sqrt{x+1}}+\frac{1}{2\sqrt{x-1}}$$

$$=\frac{\sqrt{x-1}+\sqrt{x+1}}{2\sqrt{x+1}\cdot\sqrt{x-1}}$$

$$=\frac{2y}{2\sqrt{x^2-1}} \quad \text{... [By (1)]}$$

$$\therefore\ 2\sqrt{x^2-1}\frac{dy}{dx}=y$$

$$\therefore\ 4(x^2-1)\cdot\left(\frac{dy}{dx}\right)^2=y^2$$

Differentiating both sides w.r.t. x, we get

$$4(x^2-1)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}[4(x^2-1)]=2y\frac{dy}{dx}$$

$$\therefore\ 4(x^2-1)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot 4(2x)=2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ on both sides, we get

$$4(x^2-1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}=y$$

$$\therefore\ 4(x^2-1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-y=0$$

$$\therefore\ 4(x^2-1)y_2+4xy_1-y=0.$$

(viii) If y = $\log\left(x+\sqrt{x^2+a^2}\right)^m$, show that $\left(x^2+a^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

Solution:

y = $\log\left(x+\sqrt{x^2+a^2}\right)^m$ = $m\log\left(x+\sqrt{x^2+a^2}\right)$

$$\therefore\ \frac{dy}{dx}=m\frac{d}{dx}[\log(x+\sqrt{x^2+a^2})]$$

$$=m\times\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x^2+a^2)\right]$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\times(2x+0)\right]$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$\therefore\ \frac{dy}{dx}=\frac{m}{\sqrt{x^2+a^2}}$$

$$\therefore\ \sqrt{x^2+a^2}\frac{dy}{dx}=m$$

$$\therefore\ (x^2+a^2)\left(\frac{dy}{dx}\right)^2 = m^2$$

Differentiating both sides w.r.t. x, we get

$$(x^2+a^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(x^2+a^2)=\frac{d}{dx}(m^2)$$

$$\therefore\ (x^2+a^2)\times 2\frac{dy}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2\times(2x+0)=0$$

$$\therefore\ (x^2+a^2)\cdot 2\frac{dy}{dx}\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=0$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$

(ix) If y = sin(m $\cos^{-1}$x), then show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

Solution:

y = sin(m $\cos^{-1}$x)

$\sin^{-1}$y = m $\cos^{-1}$x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=m\times\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\cdot\frac{dy}{dx} = -m\sqrt{1-y^2}$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 - m^2y^2$$

Differentiating both sides w.r.t. x, we get

$$(1-x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(1-x^2) = 0 - m^2\cdot\frac{d}{dx}(y^2)$$

$$\therefore (1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

(x) If y = log(log 2x), show that $xy_2 + y_1(1 + xy_1) = 0$.

Solution:

y = log(log 2x)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[\log(\log 2x)]$$

$$= \frac{1}{\log 2x} \cdot \frac{d}{dx}(\log 2x)$$

$$= \frac{1}{\log 2x} \times \frac{1}{2x} \cdot \frac{d}{dx}(2x)$$

$$= \frac{1}{\log 2x} \times \frac{1}{2x} \times 2$$

$$\therefore \frac{dy}{dx} = \frac{1}{x \log 2x}$$

$$\therefore (\log 2x) \cdot \frac{dy}{dx} = \frac{1}{x} \quad \ldots (1)$$

Differentiating both sides w.r.t. x, we get

$$(\log 2x) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\log 2x) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \cdot \frac{d}{dx}(2x) = -\frac{1}{x^2}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \times 2 = -\frac{1}{x^2}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} = -\frac{1}{x} \cdot \frac{1}{x}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \left[(\log 2x) \cdot \frac{dy}{dx}\right] \cdot \frac{dy}{dx} = -\frac{1}{x}\left[(\log 2x) \cdot \frac{dy}{dx}\right]$$

... [By (1)]

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=-\frac{1}{x}\frac{dy}{dx}$$

$$\therefore\ x\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=-\frac{dy}{dx}$$

$$\therefore\ x\frac{d^2y}{dx^2}+\frac{dy}{dx}+x\left(\frac{dy}{dx}\right)^2=0$$

$$\therefore\ x\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(1+x\frac{dy}{dx}\right)=0$$

$$\therefore\ xy_2+y_1(1+xy_1)=0.$$

(xi) If $x^2 + 6xy + y^2 = 10$, show that $\frac{d^2y}{dx^2}=\frac{80}{(3x+y)^3}$

Solution:

$x^2 + 6xy + y^2 = 10$ …… (1)

Differentiating both sides w.r.t. x, we get

$$2x+6\left[x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)\right]+2y\frac{dy}{dx}=0$$

$$\therefore\ 2x+6x\frac{dy}{dx}+6y\times1+2y\frac{dy}{dx}=0$$

$$\therefore\ (6x+2y)\frac{dy}{dx}=-2x-6y$$

$$\therefore\ \frac{dy}{dx}=\frac{-2(x+3y)}{2(3x+y)}=-\left(\frac{x+3y}{3x+y}\right) \qquad \ldots (2)$$

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{x+3y}{3x+y}\right)$$

$$= -\left[\frac{(3x+y)\cdot\frac{d}{dx}(x+3y)-(x+3y)\cdot\frac{d}{dx}(3x+y)}{(3x+y)^2}\right]$$

$$= -\left[\frac{(3x+y)\left(1+3\frac{dy}{dx}\right)-(x+3y)\left(3+\frac{dy}{dx}\right)}{(3x+y)^2}\right]$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left\{1-\frac{3(x+3y)}{3x+y}\right\}+\right.$$

$$\left.(x+3y)\left(3-\frac{x+3y}{3x+y}\right)\right] \quad \text{... [By (2)]}$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left(\frac{3x+y-3x-9y}{3x+y}\right)+\right.$$

$$\left.(x+3y)\left(\frac{9x+3y-x-3y}{3x+y}\right)\right]$$

$$= \frac{1}{(3x+y)^2}\left[8y+\frac{(x+3y)(8x)}{3x+y}\right]$$

$$= \frac{1}{(3x+y)^2}\left[\frac{8y(3x+y)+(x+3y)8x}{3x+y}\right]$$

$$= \frac{24xy+8y^2+8x^2+24xy}{(3x+y)^3}$$

$$= \frac{8x^2+48xy+8y^2}{(3x+y)^3} = \frac{8(x^2+6xy+y^2)}{(3x+y)^3}$$

$$= \frac{8(10)}{(3x+y)^3} \quad \ldots \text{[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}.$$

(xii) If x = a sin t – b cos t, y = a cos t + b sin t, Show that $\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}$

Solution:

x = a sin t – b cos t, y = a cos t + b sin t

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dx}(\sin t) - b\frac{d}{dt}(\cos t)$$

$$= a\cos t - b(-\sin t) = a\cos t + b\sin t$$

$$\text{and } \frac{dy}{dt} = a\frac{d}{dx}(\cos t) - b\frac{d}{dt}(\sin t)$$

$$= a(-\sin t) + b\cos t = -a\sin t + b\cos t$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-a\sin t + b\cos t}{a\cos t + b\sin t}$$

$$= -\left(\frac{a\sin t - b\cos t}{a\cos t + b\sin t}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \quad \ldots (1)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$= -\left[\frac{y \times 1 - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \text{... [By (1)]}$$

$$= -\left[\frac{y^2 + x^2}{y^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}.$$

Question 4.
Find the nth derivative of the following:
(i) $(ax + b)^m$
Solution:
Let $y = (ax + b)^m$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(ax+b)^m$$

$$= m(ax+b)^{m-1} \cdot \frac{d}{dx}(ax+b)$$

$$= m(ax+b)^{m-1} \times (a \times 1 + 0)$$

$$= am(ax+b)^{m-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[am(ax+b)^{m-1}]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$= -\left[\frac{y \times 1 - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \text{... [By (1)]}$$

$$= -\left[\frac{y^2 + x^2}{y^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}.$$

Question 4.
Find the nth derivative of the following:
(i) $(ax + b)^m$
Solution:
Let $y = (ax + b)^m$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(ax + b)^m$$

$$= m(ax + b)^{m-1} \cdot \frac{d}{dx}(ax + b)$$

$$= m(ax + b)^{m-1} \times (a \times 1 + 0)$$

$$= am(ax + b)^{m-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[am(ax + b)^{m-1}]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[am(ax+b)^{m-1}]$$

$$= am\frac{d}{dx}(ax+b)^{m-1}$$

$$= am(m-1)(ax+b)^{m-2} \cdot \frac{d}{dx}(ax+b)$$

$$= am(m-1)(ax+b)^{m-2} \times (a \times 1 + 0)$$

$$= a^2m(m-1)(ax+b)^{m-2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}[a^2m(m-1)(ax+b)^{m-2}]$$

$$= a^2m(m-1)\frac{d}{dx}(ax+b)^{m-2}$$

$$= a^2m(m-1)(m-2)(ax+b)^{m-3}\frac{d}{dx}(ax+b)$$

$$= a^2m(m-1)(m-2)(ax+b)^{m-3} \times (a \times 1 + 0)$$

$$= a^3m(m-1)(m-2)(ax+b)^{m-3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = a^n m(m-1)(m-2) \ldots (m-n+1)(ax+b)^{m-n}$$

Case (i) : If $m > 0$, $m > n$, then

$$\frac{d^ny}{dx^n} = \frac{a^n \cdot m(m-1)(m-2)\ldots(m-n+1)(m-n)\ldots 3 \cdot 2 \cdot 1}{(m-n)(m-n-1)\ldots 3 \cdot 2 \cdot 1} \times (ax+b)^{m-n}$$

$\therefore \frac{d^n y}{dx^n} = \frac{a^n \cdot m!\,(ax+b)^{m-n}}{(m-n)!}$, if $m > 0$, $m > n$.

Case (ii) : If $m > 0$ and $m < n$, then its m^{th} order derivative is a constant and every derivatives after m^{th} order are zero.

$\therefore \frac{d^n y}{dx^n} = 0$, if $m > 0$, $m < n$.

Case (iii) : If $m > 0$, $m = n$, then

$$\frac{d^n y}{dx^n} = a^n \cdot n(n-1)(n-2)\ldots(n-n+1)(ax+b)^{n-n}$$

$$= a^n \cdot n(n-1)(n-2)\ldots 1 \cdot (ax+b)^0$$

$\therefore \frac{d^n y}{dx^n} = a^n \cdot n!$, if $m > 0$, $m = n$.

(ii) $\frac{1}{x}$

Solution:

Let y = $\frac{1}{x}$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$= \frac{(-1)^1 1!}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x^2}\right) = -1\frac{d}{dx}(x^{-2})$$

$$= (-1)(-2)x^{-3} = \frac{(-1)^2 \cdot 1 \cdot 2}{x^3}$$

$$= \frac{(-1)^2 2!}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^2 \cdot 2!}{x^3}\right] = (-1)^2 \cdot 2! \frac{d}{dx}(x^{-3})$$

$$= (-1)^2 \cdot 2! \cdot (-3) x^{-4}$$

$$= \frac{(-1)^3 \times 3 \cdot 2!}{x^4} = \frac{(-1)^3 \cdot 3!}{x^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = \frac{(-1)^n \cdot n!}{x^{n+1}}.$$

(iii) e^{ax+b}

Solution:

Let $y = e^{ax+b}$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{ax+b}) = e^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= e^{ax+b} \times (a \times 1 + 0) = ae^{ax+b}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(ae^{ax+b}) = a \cdot \frac{d}{dx}(e^{ax+b})$$

$$= ae^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= ae^{ax+b} \times (a \times 1 + 0) = a^2 \cdot e^{ax+b}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}[a^2 e^{ax+b}] = a^2 \frac{d}{dx}(e^{ax+b})$$

$$= a^2 e^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= a^2 e^{ax+b} \times (a \times 1 + 0) = a^3 \cdot e^{ax+b}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = a^n \cdot e^{ax+b}.$$

(iv) a^{px+q}

Solution:

Let y = a^{px+q}

(v) log(ax + b)

Solution:

Let y = log(ax + b)

Then $\frac{dy}{dx} = \frac{d}{dx}[\log(ax+b)]$

$$= \frac{1}{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= \frac{1}{ax+b} \times (a \times 1 + 0) = \frac{a}{ax+b}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\right) = a\frac{d}{dx}(ax+b)^{-1}$$

$$= a(-1)(ax+b)^{-2} \cdot \frac{d}{dx}(ax+b)$$

$$= \frac{(-1)a}{(ax+b)^2} \times (a \times 1 + 0)$$

$$= \frac{(-1)a^2}{(ax+b)^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^1 a^2}{(ax+b)^2}\right] = (-1)^1 a^2 \cdot \frac{d}{dx}(ax+b)^{-2}$$

$$= (-1)^1 a^2 \cdot (-2)(ax+b)^{-3} \cdot \frac{d}{dx}(ax+b)$$

$$= \frac{(-1)^2 \cdot 1 \cdot 2 \cdot a^2}{(ax+b)^3} \times (a \times 1 + 0)$$

$$= \frac{(-1)^2 \cdot 2!\, a^3}{(ax+b)^3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)!\, a^n}{(ax+b)^n}.$$

(vi) cos x

Solution:

Let y = cos x

Then $\frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$

$$= \cos\left(\frac{\pi}{2} + x\right)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-\sin x)=-\cos x$$

$$=\cos(\pi+x)=\cos\left(\frac{2\pi}{2}+x\right)$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}(-\cos x)=-\frac{d}{dx}(\cos x)$$

$$=-(-\sin x)=\sin x$$

$$=\cos\left(\frac{3\pi}{2}+x\right)$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n}=\cos\left(\frac{n\pi}{2}+x\right).$$

(vii) sin(ax + b)
Solution:
Let y = sin(ax + b)

Then $\frac{dy}{dx}=\frac{d}{dx}[\sin(ax+b)]$

$$=\cos(ax+b)\cdot\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b)\times(a\times 1+0)$$

$$=a\sin\left[\frac{\pi}{2}+(ax+b)\right]$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[a\cos(ax+b)]$$

$$=a\frac{d}{dx}[\cos(ax+b)]$$

$$=a[-\sin(ax+b)]\cdot\frac{d}{dx}(ax+b)$$

$$=a[-\sin(ax+b)]\times(a\times 1+0)$$

$$=a^2\cdot\sin[\pi+(ax+b)]$$

$$=a^2\cdot\sin\left[\frac{2\pi}{2}+(ax+b)\right]$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}[-a^2\sin(ax+b)]$$

$$=-a^2\frac{d}{dx}[\sin(ax+b)]$$

$$=-a^2\cdot\cos(ax+b)\cdot\frac{d}{dx}(ax+b)$$

$$=-a^2\cdot\cos(ax+b)\times(a\times 1+0)$$

$$=a^3\cdot\sin\left[\frac{3\pi}{2}+(ax+b)\right]$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n}=a^n\cdot\sin\left[\frac{n\pi}{2}+(ax+b)\right].$$

(viii) cos(3 – 2x)

Solution:

Let y = cos (3 – 2x)

Then $\frac{dy}{dx} = \frac{d}{dx}[\cos(3 - 2x)]$

= $\cos(3 - 2x) . \frac{d}{dx}(3 - 2x)$

= cos (3 – 2x) x (a x 1 + 0)

= $a \cos\left[\frac{\pi}{2} + (3 - 2x)\right]$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[a \cos(3 - 2x)]$

= $a \frac{d}{dx}[\cos(3 - 2x)]$

= $a[-\cos(3 - 2x)] . \frac{d}{dx}(3 - 2x)$

= a[– cos(3 – 2x)] x (a x 1 + 0)

= a^2 . cos[π + (3 – 2x)]

= $a^2 . \cos\left[\frac{2\pi}{2} + (3 - 2x)\right]$

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[-a^2 \cos(ax + b)\right]$

$= -a^2 \frac{d}{dx}[\cos(3 - 2x)]$

$= -a^2 . \cos(3 - 2x) . \frac{d}{dx}(3 - 2x)$

= – a^2. cos(3 – 2x) x (a x 1 + 0)

$= a^3 . \cos\left[\frac{3\pi}{2} + (3 - 2x)\right]$

In general, the n^{th} order derivative is given by

$\frac{d^n y}{dx^n} = (-2)^n \cos\left[\frac{n\pi}{2} + (3 - 2x)\right]$.

(ix) log(2x + 3)

Solution:

Let y = log (2x + 3)

Then $\frac{dy}{dx} = \frac{d}{dx}[\log(2x + 3)]$

$= \frac{1}{2x + 3} . \frac{d}{dx}(2x + 3)$

$= \frac{1}{2x + 3} \times (a \times 1 + 0)$

$= \frac{a}{2x + 3}$

$$\frac{d^2y}{dx^2} = \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{a}{2x+3}\right)$$

$= a\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)^{-1}$

$= a(-1)(2x+3)^{-2}.\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)$

$= \frac{(-1)a}{(2x+3)^2} \times (a \times 1 + 0)$

$= \frac{(-1)a}{(2x+3)^2}$

$$\frac{d^3y}{dx^3} = \frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{(-1)^1a^2}{(2x+3)^2}\right]$$

$= (-1)^1a^2.\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)^{-2}$

$= (-1)^1a^2.(-2)(2x+3)^{-3}.\frac{\mathrm{d}}{\mathrm{dx}}(2x+3)$

$= \frac{(-1)^2.1.2.a^2}{(2x+3)^3} \times (a \times 1 + 0)$

$= \frac{(-1)\text{-}2.2!a^3}{(2x+3)^3}$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^2} = \frac{(-1)^{n-1}.(n-1)!2^n}{(2x+3)^n}.$$

(x) $\frac{1}{3x-5}$

Solution:

Let y = $\frac{1}{3x-5}$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(3x-5)$$

$$= -1(3x-5)^{-2} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{-1}{(3x-5)^2} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^1 \cdot 3}{(3x-5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(-1)^1 \cdot 3}{(3x-5)^2}\right]$$

$$= (-1)^1 \cdot 3 \frac{d}{dx}(3x-5)^{-2}$$

$$= (-1)^1 \cdot 3 \cdot (-2)(3x-5)^{-3} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^2 \cdot 3 \cdot 2}{(3x-5)^3} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}\right]$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot \frac{d}{dx}(3x-5)^{-3}$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot (-3)(3x-5)^{-4} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^3 \times 3 \cdot 2! \times 3^2}{(3x-5)^4} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^3 \cdot 3! \cdot 3^3}{(3x-5)^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = \frac{(-1)^n \cdot n! \cdot 3^n}{(3x-5)^{n+1}}.$$

(xi) $y = e^{ax} . \cos(bx + c)$

Solution:

$y = e^{ax} . \cos(bx + c)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax} \cdot \cos(bx+c)]$$

$$= e^{ax} \cdot \frac{d}{dx}[\cos(bx+c)] + \cos(bx+c) \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot [-\sin(bx+c)] \cdot \frac{d}{dx}(bx+c) +$$

$$\cos(bx+c) \cdot e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= -e^{ax} \sin(bx+c) \times (b \times 1 + 0) +$$

$$e^{ax} \cos(bx+c) \times a \times 1$$

$$= e^{ax} [a\cos(bx+c) - b\sin(bx+c)]$$

$$= e^{ax} \cdot \sqrt{a^2+b^2} \left[\frac{a}{\sqrt{a^2+b^2}} \cos(bx+c) - \right.$$

$$\left. \frac{b}{\sqrt{a^2+b^2}} \sin(bx+c) \right]$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$ and $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$

Then $\tan\alpha = \frac{b}{a}$ $\quad \therefore \ \alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore \ \frac{dy}{dx} = e^{ax} \cdot \sqrt{a^2+b^2} [\cos\alpha \cdot \cos(bx+c) -$$

$$\sin\alpha \cdot \sin(bx+c)]$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{1}{2}} \cdot \cos(bx+c+\alpha)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^{ax} \cdot (a^2+b^2)^{\frac{1}{2}} \cdot \cos(bx+c+\alpha)]$$

$$= (a^2+b^2)^{\frac{1}{2}} \cdot \frac{d}{dx}[e^{ax} \cdot \cos(bx+c+\alpha)]$$

$$= (a^2+b^2)^{\frac{1}{2}} [e^{ax} \cdot \frac{d}{dx}\{\cos(bx+c+\alpha)\} + \cos(bx+c+\alpha) \cdot \frac{d}{dx}(e^{ax})]$$

$$= (a^2+b^2)^{\frac{1}{2}} [e^{ax} \cdot \{-\sin(bx+c+\alpha)\} \cdot \frac{d}{dx}(bx+c+\alpha) + \cos(bx+c+\alpha) \cdot e^{ax} \cdot \frac{d}{dx}(ax)]$$

$$= (a^2+b^2)^{\frac{1}{2}} [-e^{ax} \sin(bx+c+\alpha) \times (b \times 1 + 0 + 0) + \cos(bx+c+\alpha) \cdot e^{ax} \times a \times 1]$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{1}{2}} [a\cos(bx+c+\alpha) - b\sin(bx+c+\alpha)]$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{1}{2}} \cdot \sqrt{a^2+b^2} \left[\frac{a}{\sqrt{a^2+b^2}}\cos(bx+c+\alpha) - \frac{b}{\sqrt{a^2+b^2}}\sin(bx+c+\alpha)\right]$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{2}{2}} [\cos\alpha \cdot \cos(bx+c+\alpha) - \sin\alpha \cdot \sin(bx+c+\alpha)]$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{2}{2}} \cdot \cos(bx+c+\alpha+\alpha)$$

$$= e^{ax} \cdot (a^2+b^2)^{\frac{2}{2}} \cdot \cos(bx+c+2\alpha)$$

Similarly,

$$\frac{d^3y}{dx^3} = e^{ax} \cdot (a^2+b^2)^{\frac{3}{2}} \cdot \cos(bx+c+3\alpha)$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = e^{ax} \cdot (a^2 + b^2)^{\frac{n}{2}} \cdot \cos(bx + c + n\alpha),$$

where $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore \frac{d^n y}{dx^n} = e^{ax} \cdot (a^2 + b^2)^{\frac{n}{2}} \cdot \cos\left[bx + c + n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

(xii) $y = e^{8x} . \cos(6x + 7)$

Solution:

$y = e^{8x} . \cos(6x + 7)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax}. \cos(6x + 7)]$$

$$= e^{ax}. \frac{d}{dx}[\cos(6x + 7)] + \cos(6x + 7). \frac{d}{dx}(e^{ax})$$

$$= e^{ax}. [-\sin(6x + 7)]. \frac{d}{dx}(6x + 7) + \cos(6x + 7). e^{ax}. \frac{d}{dx}(ax)$$

$= -e^{ax} \sin(6x + 7) \times (b \times 1 + 0) + e^{ax}\cos(6x + 7) \times a \times 1$

$= e^{ax} [a \cos(6x + 7) - b \sin(6x + 7)]$

$$= e^{ax}. \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{a^2 + b^2}}\cos(6x + 7) - \frac{b}{\sqrt{a^2 + b^2}}\sin(6x + 7)\right]$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos x$ and $\frac{b}{\sqrt{a^2+b^2}} = \sin x$

Then tan ∞ = $\frac{b}{a}$

∴ ∞ = $\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore \frac{dy}{dx} = e^{ax}.\sqrt{a^2+b^2}[\cos\infty.\cos(bx+c) - \sin\infty.\sin(bx+c)]$

$= e^{ax}.(a^2+b^2)^{\frac{1}{2}}.\cos(6x+7+x)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{ax}.(a^2+b^2)^{\frac{1}{2}}.\cos(6x+7+\infty)\right]$

$= (a^2+b^2)^{\frac{1}{2}}.\frac{d}{dx}[e^{ax}.\cos(6x+7+\infty)]$

$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}.\frac{d}{dx}\{\cos(6x+7+\infty)\} + \cos(6x+7+\infty).\frac{d}{dx}(e^{ax})\right]$

$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}.\{-\sin(6x+7+\infty)\}.\frac{d}{dx}(6x+7+\infty) + \cos(6x+7+\infty).e^{ax}.\frac{d}{dx}(ax)\right]$

$= (a^2+b^2)^{\frac{1}{2}}[-e^{ax}\sin(6x+7+\infty)\times(b\times1+0+0) + \cos(6x+7+\infty).e^{ax}\times a\times1]$

$= e^{ax}.(a^2+b^2)^{\frac{1}{2}}[a\cos(6x+7+\infty) - b\sin(6x+7+\infty)]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(6x+7+\infty)=\frac{b}{\sqrt{a^2+b^2}}\sin(6x+7+\infty)\right]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}[\cos\infty.\cos(6x+7+\infty)-\sin\infty.\sin(6x+7+\infty)$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+\infty+\infty)$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+2\infty)$

Similarly.

$\frac{d^3y}{dx^3}=e^{ax}.\left(a^2+b^2\right)^{\frac{3}{2}}.\cos(6x+7+3\infty)$

In general, the nth order derivative is given by

$\frac{d^ny}{dx^n}=e^{ax}.\left(a^2+b^2\right)^{\frac{n}{2}}.\cos(6x+7+n\infty),$

Where ∞ = $\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore\ \frac{d^ny}{dx^n}=e^{8x}.(10)^n.\cos\left[6x+7+n\tan^{-1}\left(\frac{3}{4}\right)\right]$

Maharashtra Board Solutions Class 12 Arts & Science Maths (Part 2)

About About Maharashtra State Board (MSBSHSE)

The Maharashtra State Board of Secondary and Higher Secondary Education or MSBSHSE (Marathi: महाराष्ट्र राज्य माध्यमिक आणि उच्च माध्यमिक शिक्षण मंडळ), is an **autonomous and statutory body established in 1965**. The board was amended in the year 1977 under the provisions of the Maharashtra Act No. 41 of 1965.

The Maharashtra State Board of Secondary & Higher Secondary Education (MSBSHSE), Pune is an independent body of the Maharashtra Government. There are more than 1.4 million students that appear in the examination every year. The Maha State Board conducts the board examination twice a year. This board conducts the examination for SSC and HSC.

The Maharashtra government established the Maharashtra State Bureau of Textbook Production and Curriculum Research, also commonly referred to as Ebalbharati, in 1967 to take up the responsibility of providing quality textbooks to students from all classes studying under the Maharashtra State Board. MSBHSE prepares and updates the curriculum to provide holistic development for students. It is designed to tackle the difficulty in understanding the concepts with simple language with simple illustrations. Every year around 10 lakh students are enrolled in schools that are affiliated with the Maharashtra State Board.

FAQs

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How many state boards are there in Maharashtra?

The Maharashtra State Board of Secondary & Higher Secondary Education, conducts the HSC and SSC Examinations in the state of Maharashtra through its nine Divisional Boards located at Pune, Mumbai, Aurangabad, Nasik, Kolhapur, Amravati, Latur, Nagpur and Ratnagiri.

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