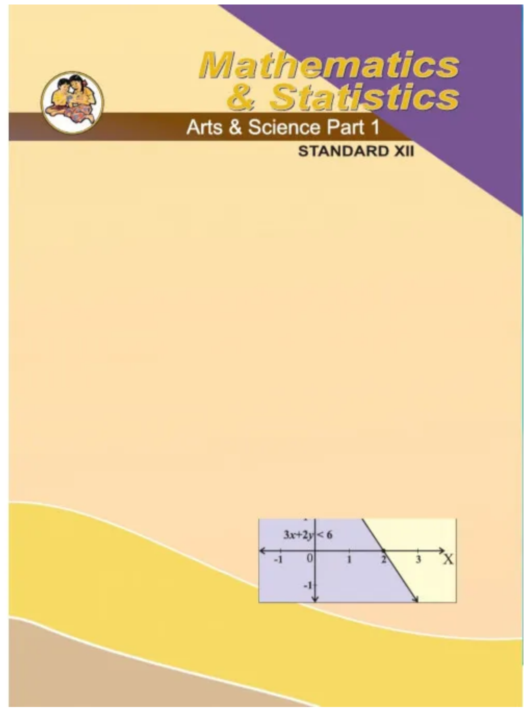
Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 1): Chapter 5- Vectors

Class 12 - Chapter 5 Vectors



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Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 1): Chapter 5- Vectors

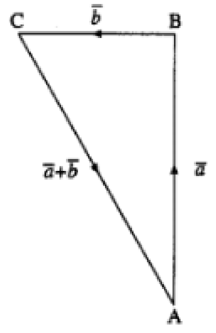
Class 12: Maths Chapter 5 solutions. Complete Class 12 Maths Chapter 5 Notes.

Maharashtra Board Solutions Class 12-Arts & Science Maths (Part 1): Chapter 5- Vectors

Maharashtra Board 12th Maths Chapter 5, Class 12 Maths Chapter 5 solutions

Ex 5.1

Question 1.
The vector $\bar{a}$ is directed due north and $|\bar{a}|$ = 24. The vector $\bar{b}$ is directed due west and $|\bar{b}|$ = 7.
Find $|\bar{a}+\bar{b}|$.
Solution:



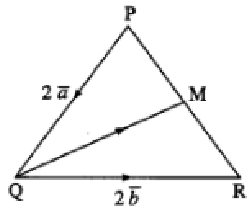
Let $\overline{AB}$ = $\bar{a}$, $\overline{BC}$ = $\bar{b}$
Then $\overline{AC}$ = $\overline{AB}$ + $\overline{BC}$ = a + b
Given : $|\bar{a}|$ = $|\overline{AB}|$ = l(AB) = 24 and
$|\bar{b}|$ = $|\overline{BC}|$ = l(BC) = 7
∴ ∠ABC = 90°
∴ $[l(AC)]^2 = [l(AB)]^2 + [l(BC)]^2$
$= (24)^2 + (7)^2 = 625$
∴ l(AC) = 25 ∴ $|\overline{AC}|$ = 25
∴ $|\bar{a}+\bar{b}|$ = $|\overline{AC}|$ = 25.

Question 2.

In the triangle PQR, $\overline{PQ}$ = 2$\bar{a}$ and $\overline{QR}$ = 2$\bar{b}$. The mid-point of PR is M. Find following vectors in terms of $\bar{a}$ and $\bar{b}$.

(i) $\overline{PR}$

Solution:



Given : $\overline{PQ}$ = 2$\bar{a}$, $\overline{QR}$ = 2$\bar{b}$

(i) $\overline{PR}$ = $\overline{PQ}$ + $\overline{QR}$

= 2$\bar{a}$ + 2$\bar{a}$.

(ii) $\overline{PM}$

Solution:

∵ M is the midpoint of PR

∴ $\overline{PM} = \frac{1}{2}\overline{PR} = \frac{1}{2}[2\bar{a} + 2\bar{b}]$

= $\bar{a}$ + $\bar{b}$.

(iii) $\overline{QM}$

Solution:

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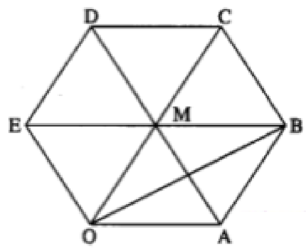
$\overline{RM} = \frac{1}{2}(\overline{RP}) = -\frac{1}{2}\overline{PR} = -\frac{1}{2}(2\bar{a} + 2\bar{b})$
$= -\bar{a} - \bar{b}$
$\therefore \overline{QM} = \overline{QR} + \overline{RM}$
$= 2\bar{b} - \bar{a} - \bar{b}$
$= \bar{b} - \bar{a}.$

Question 3.
OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively, referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.
Solution:



Given : $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{a}$ Let AD, BE, OC meet at M.
Then M bisects AD, BE, OC.
$\overline{AB} = \overline{AO} + \overline{OB} = -\overline{OA} + \overline{OB} = -\bar{a} + \bar{b} = \bar{b} - \bar{a}$
$\because$ OABM is a parallelogram
$\therefore \overline{OM} = \overline{AB} = \bar{b} - \bar{a}$
$\overline{OC} = 2\overline{OM} = 2(\bar{b} - \bar{a}) = 2\bar{b} - 2\bar{a}$
$\overline{OD} = \overline{OC} + \overline{CD}$
$= \overline{OC} - \overline{DC}$

$= \overline{OC} - \overline{OA}$ ... [∵ OA = DC and OA ∥ DC]

$= 2\bar{b} - 2\bar{a} - \bar{a}$

$= 2\bar{b} - 3\bar{a}$

$\overline{OE} = \overline{OM} + \overline{ME}$

$= (\bar{b} - \bar{a}) - \overline{EM}$

$= \bar{b} - \bar{a} - \bar{a}$ ... [∵ EM = OA and EM ∥ OA]

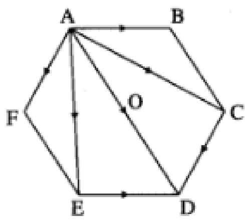
$= \bar{b} - 2\bar{a}$

Hence, the position vectors of C, D and E are $2\bar{b} - 2\bar{a}$, $2\bar{b} - 3\bar{a}$ and $\bar{b} - 2\bar{a}$ respectively.

Question 4.

If ABCDEF is a regular hexagon, show that $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 6\overline{AO}$, where O is the center of the hexagon.

Solution:



ABCDEF is a regular hexagon.

∴ $\overline{AB} = \overline{ED}$ and $\overline{AF} = \overline{CD}$

∴ by the triangle law of addition of vectors,

$\overline{AC} + \overline{AF} = \overline{AC} + \overline{CD} = \overline{AD}$

$$\overline{AC} + \overline{AF} = \overline{AC} + \overline{CD} = \overline{AD}$$
$$\overline{AE} + \overline{AB} = \overline{AE} + \overline{ED} = \overline{AD}$$
$$\therefore \text{ LHS} = \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$$
$$= \overline{AD} + (\overline{AC} + \overline{AF}) + (\overline{AE} + \overline{AB})$$
$$= \overline{AD} + \overline{AD} + \overline{AD}$$
$$= 3\overline{AD} = 3(2\overline{AO}) \quad \text{... [O is midpoint of AD]}$$
$$= 6\overline{AO}.$$

Question 5.

Check whether the vectors $2\hat{i} + 2\hat{j} + 3\hat{k}$, + $-3\hat{i} + 3\hat{j} + 2\hat{k}$, + $3\hat{i} + 4\hat{k}$ form a triangle or not.

Solution:

Let, if possible, the three vectors form a triangle ABC

with $\overline{AB} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\overline{BC} = 3\hat{i} + 3\hat{j} + 2\hat{k}$, $\overline{AC} = 3\hat{i} + 4\hat{k}$

Now, $\overline{AB} + \overline{BC}$

$= (2\hat{i} + 2\hat{j} + 3\hat{k}) + (-3\hat{i} + 3\hat{j} + 2\hat{k})$

$= -\hat{i} + 5\hat{j} + 5\hat{k} \neq 3\hat{i} + 4\hat{k} = \overline{AC}$

Hence, the three vectors do not form a triangle.

Question 6.

In the figure 5.34 express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$. Find a vector in the direction of $\bar{a}$ = $\hat{i} - 2\hat{j}$ that has magnitude 7 units.

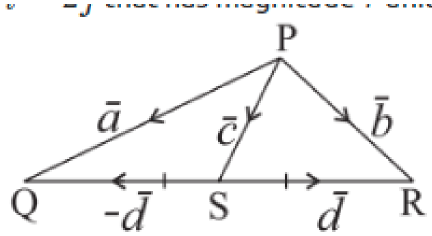


Fig 5.34

Solution:

$\overline{PQ} = \overline{PS} + \overline{SQ}$

$\therefore \bar{a} = \bar{c} - \bar{d}$ ... (1)

$\overline{PR} = \overline{PS} + \overline{SR}$

$\therefore \bar{b} = \bar{c} + \bar{d}$ ... (2)

Adding equations (1) and (2), we get

$\bar{a} + \bar{b} = (\bar{c} - \bar{d}) + (\bar{c} + \bar{d}) = 2\bar{c}$

$$\therefore \ \bar{c} = \frac{1}{2}(\bar{a} + \bar{b}) = \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b}$$

Substituting for $\bar{c}$ in (2), we get

$$\bar{d} = \bar{b} - \bar{c} = \bar{b} - \frac{1}{2}(\bar{a} + \bar{b})$$

$$= \frac{2\bar{b} - (\bar{a} + \bar{b})}{2}$$

$$= \frac{1}{2}(\bar{b} - \bar{a}) = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$$

Hence, $\bar{c} = \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b}$ and $\bar{d} = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$.

Question 7.
Find the distance from (4, -2, 6) to each of the following :
(a) The XY-plane
Solution:
Let the point A be (4, -2, 6).
Then,
The distance of A from XY-plane = |z| = 6

(b) The YZ-plane
Solution:
The distance of A from YZ-plane = |x| = 4

(c) The XZ-plane
Solution:
The distance of A from ZX-plane = |y| = 2

(d) The X-axis
Solution:
The distance of A from X-axis
= $\sqrt{y^2+z^2}$ = $\sqrt{(-2)^2+6^2}$ = $\sqrt{40}$ = $2\sqrt{10}$

(e) The Y-axis
Solution:
The distance of A from Y-axis

= $\sqrt{z^2 + x^2}$ = $\sqrt{6^2 + 4^2}$ = $\sqrt{52}$ = $2\sqrt{13}$

(f) The Z-axis
Solution:
The distance of A from Z-axis
= $\sqrt{x^2 + y^2}$ = $\sqrt{4^2 + (-2)^2}$ = $\sqrt{20}$ = $2\sqrt{5}$

Question 8.
Find the coordinates of the point which is located :
(a) Three units behind the YZ-plane, four units to the right of the XZ-plane and five units above the XY-plane.
Solution:
Let the coordinates of the point be (x, y, z).
Since the point is located 3 units behind the YZ- j plane, 4 units to the right of XZ-plane and 5 units , above the XY-plane,
x = -3, y = 4 and z = 5
Hence, coordinates of the required point are (-3, 4, 5)

(b) In the YZ-plane, one unit to the right of the XZ-plane and six units above the XY-plane.
Solution:
Let the coordinates of the point be (x, y, z).
Since the point is located in the YZ plane, x = 0. Also, the point is one unit to the right of XZ-plane and six units above the XY-plane.
∴ y = 1, z = 6.

Hence, coordinates of the required point are (0, 1, 6).

Question 9.
Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).
Solution:
Let A = (1, 1, 0), B = (1, 0, 1), C = (0, 1, 1)

$$l(AB) = \sqrt{(1-1)^2 + (1-0)^2 + (0-1)^2} = \sqrt{0+1+1} = \sqrt{2}$$
$$l(BC) = \sqrt{(1-0)^2 + (0-1)^2 + (1-1)^2} = \sqrt{1+1+0} = \sqrt{2}$$
$$l(CA) = \sqrt{(0-1)^2 + (1-1)^2 + (1-0)^2} = \sqrt{1+0+1} = \sqrt{2}$$
$\therefore$ $l(AB) = l(BC) = l(CA)$

$\therefore$ the triangle is equilateral

$$\therefore \text{ its area} = \frac{\sqrt{3}}{4}(\text{side})^2$$
$$= \frac{\sqrt{3}}{4}(\sqrt{2})^2$$
$$= \frac{\sqrt{3}}{2} \text{ sq units.}$$

Question 10.
If $\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ and initial point A ≡ (1, 5, ,0). Find the terminal point B.
Solution:
Let $\bar{a}$ and $\bar{b}$ be the position vectors of A and B.

Given : A = (1, 5, 0) $\therefore \bar{a} = \hat{i} + 5\hat{j}$
Now, $\overline{AB} = 2\hat{i} - 4\hat{j} + 7\hat{k}$
$\therefore \bar{b} - \bar{a} = 2\hat{i} - 4\hat{j} + 7\hat{k}$
$\therefore \bar{b} = (2\hat{i} - 4\hat{j} + 7\hat{k}) + \bar{a}$
$= (2\hat{i} - 4\hat{j} + 7\hat{k}) + (\hat{i} + 5\hat{j})$
$= 3\hat{i} + \hat{j} + 7\hat{k}$
Hence, the terminal point B = (3, 1, 7).

Question 11.
Show that the following points are collinear :
(i) A (3, 2, -4), B (9, 8, -10), C (-2, -3, 1).
Solution:
Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of the points.
A = (3, 2, -4), B = (9, 8, -10) and C = (-2, -3, 1) respectively.

Then, $\bar{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$, $\bar{b} = 9\hat{i} + 8\hat{j} - 10\hat{k}$,
$\bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$

$\overline{AB} = \bar{b} - \bar{a}$
$= (9\hat{i} + 8\hat{j} - 10\hat{k}) - (3\hat{i} + 2\hat{j} - 4\hat{k})$
$= 6\hat{i} + 6\hat{j} - 6\hat{k}$ ... (1)

and $\overline{BC} = \bar{c} - \bar{b}$
$= (-2\hat{i} - 3\hat{j} + \hat{k}) - (9\hat{i} + 8\hat{j} - 10\hat{k})$
$= -11\hat{i} - 11\hat{j} + 11\hat{k}$
$= -11(\hat{i} + \hat{j} - \hat{k}) = -\frac{11}{6}(6\hat{i} + 6\hat{j} - 6\hat{k})$

$$= -\frac{11}{6}\overline{AB} \quad \text{... [By (1)]}$$

$\therefore \overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$
$\therefore$ they are parallel to each other.
But they have the point B in common.
$\therefore \overline{BC}$ and $\overline{AB}$ are collinear vectors.
Hence, the points A, B and C are collinear.

(ii) P (4, 5, 2), Q (3, 2, 4), R (5, 8, 0).
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of the points.
P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0) respectively.

Then $\bar{a} = 4\hat{i} + 5\hat{j} + 2\hat{k}$, $\bar{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$, $\bar{c} = 5\hat{i} + 8\hat{j} + 0\hat{k}$

$\overline{AB} = \bar{b} - \bar{a}$

$$= \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + 2\hat{k}\right)$$

$= -\hat{i} - 3\hat{j} + 2\hat{k}$ i.e.

$$= -\left(\hat{i} + 3\hat{j} - 2\hat{k}\right) \quad \text{.....(1)}$$

and $\overline{BC} = \bar{c} - \bar{b}$

$$= \left(5\hat{i} + 8\hat{j} + 0\hat{k}\right) - \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$$

$$= 2\hat{i} + 6\hat{j} - 4\hat{k}$$

$$= 2\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$$

= 2.$\overline{AB}$ ...[By (1)]
$\therefore$ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$
$\therefore$ they are parallel to each other.
But they have the point B in common.
$\therefore$ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.
Hence, the points A, B and C are collinear.

Question 12.
If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then find the value of q.
Solution:
The vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear
$\therefore$ the coefficients of $\hat{i}, \hat{j}, \hat{k}$ are proportional

$$\therefore \frac{2}{4} = \frac{-q}{-5} = \frac{3}{6}$$

$$\therefore \frac{q}{5} = \frac{1}{2}$$

$$\therefore q = \frac{5}{2}.$$

Question 13.
Are the four points A(1, -1, 1), B(-1, 1, 1), C(1, 1, 1) and D(2, -3, 4) coplanar? Justify your answer.
Solution:
The position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ of the points A, B, C, D are

$$\bar{a} = \hat{i} - \hat{j} + \hat{k},\ \bar{b} = -\hat{i} + \hat{j} + \hat{k},\ \bar{c} = \hat{i} + \hat{j} + \hat{k},\ \bar{d} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$\therefore\ \overline{AB} = \bar{b} - \bar{a} = (-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} - \hat{j} + \hat{k})$
$= -2\hat{i} + 2\hat{j}$

$\overline{AC} = \bar{c} - \bar{a} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} - \hat{j} + \hat{k}) = 2\hat{j}$

and $\overline{AD} = \bar{d} - \bar{a} = (2\hat{i} - 3\hat{j} + 4\hat{k}) - (\hat{i} - \hat{j} + \hat{k})$
$= \hat{i} - 2\hat{j} + 3\hat{k}$

If A, B, C, D are coplanar, then there exist scalars x, y such that

$\overline{AB} = x \cdot \overline{AC} + y \cdot \overline{AD}$

$\therefore\ -2\hat{i} + 2\hat{j} = x(2\hat{j}) + y(\hat{i} - 2\hat{j} + 3\hat{k})$

$\therefore\ -2\hat{i} + 2\hat{j} = y\hat{i} + (2x - 2y)\hat{j} + 3y\hat{k}$

By equality of vectors,
y = -2 ….(1)
2x – 2y = 2 … (2)
3y = 0 … (3)
From (1), y = -2
From (3), y = 0 This is not possible.
Hence, the points A, B, C, D are not coplanar.

Question 14.
Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

Solution:

Let $\bar{a} = 2\hat{i} + \hat{j} - 4\hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$,

$\bar{c} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\bar{p} = -\hat{i} - 3\hat{j} + 4\hat{k}$.

Suppose $\bar{p} = x\bar{a} + y\bar{b} + z\bar{c}$.

Then, $-\hat{i} - 3\hat{j} + 4\hat{k} = x(2\hat{i} + \hat{j} - 4\hat{k}) + y(2\hat{i} - \hat{j} + 3\hat{k}) + z(3\hat{i} + \hat{j} - 2\hat{k})$

$\therefore \quad -\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$

By equality of vectors,

2x + 2y + 3 = -1

x – y + z = -3

-4x + 3y – 2z = 4

We have to solve these equations by using Cramer's Rule

$$D = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ -4 & 3 & -2 \end{vmatrix}$$

= 2(2 – 3) – 2(-2 + 4) + 3(3 – 4)

= -2 – 4 – 3 = -9 ≠ 0

$$D_x = \begin{vmatrix} -1 & 2 & 3 \\ -3 & -1 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= -1(2-3) - 2(6-4) + 3(-9+4)$$

$$= 1 - 4 - 15 = -18$$

$$D_y = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 1 \\ -4 & 4 & -2 \end{vmatrix}$$

$$= 2(6-4) + 1(-2+4) + 3(4-12)$$

$$= 4 + 2 - 24 = -18$$

$$D_z = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -3 \\ -4 & 3 & 4 \end{vmatrix}$$

= 2(-4 + 9) – 2(4 – 12) – 1(3 – 4)

= 10 + 16 + 1 = 27

$$\therefore\ x = \frac{D_x}{D} = \frac{-18}{-9} = 2$$

$$\therefore\ y = \frac{D_y}{D} = \frac{-18}{-9} = 2$$

$$\therefore\ z = \frac{D_z}{D} = \frac{27}{-9} = -3$$

$$\therefore\ \bar{p} = 2\bar{a} + 2\bar{b} - 3\bar{c}.$$

Ex 5.2

Question 1.
Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3 : 2
(i) internally
Solution:
It is given that the points P and Q have position vectors $\bar{p}$ = $2\hat{i} - \hat{j} + 3\hat{k}$ and $\bar{p}$ = $-5\hat{i} + 2\hat{j} - 5\hat{k}$ respectively.
(i) If R($\bar{r}$) divides the line segment PQ internally in the ratio 3 : 2, by section formula for internal division,

$$\bar{r} = \frac{3\bar{q}+2\bar{p}}{3+2} = \frac{3(-5\hat{i}+2\hat{j}-5\hat{k})+2(2\hat{i}-\hat{j}+3\hat{k})}{5}$$

$$= \frac{-11\hat{i}+4\hat{j}-9\hat{k}}{5} = \frac{1}{5}(-11\hat{i}+4\hat{j}-9\hat{k})$$

$$\therefore \text{ coordinates of R} = \left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$$

Hence, the position vector of R is $\frac{1}{5}(-11\hat{i}+4\hat{j}-9\hat{k})$

and the coordinates of R are $\left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$.

(ii) externally.
Solution:
If R($\bar{r}$) divides the line segment joining P and Q externally in the ratio 3 : 2, by section formula for external division,

$$\bar{r}=\frac{3\bar{q}-2\bar{p}}{3-2}=\frac{3(-5\hat{i}+2\hat{j}-5\hat{k})-2(2\hat{i}-\hat{j}+3\hat{k})}{3-2}$$
$$=-19\hat{i}+8\hat{j}-21\hat{k}$$

∴ coordinates of R = (-19, 8, -21).
Hence, the position vector of R is $-19\hat{i}+8\hat{j}-21\hat{k}$ and coordinates of R are (-19, 8, -21).

Question 2.
Find the position vector of midpoint M joining the points L (7, -6, 12) and N (5, 4, -2).
Solution:
The position vectors $\bar{l}$ and $\bar{n}$ of the points L(7, -6, 12) and N (5, 4, -2) are given by

$$\bar{l}=7\hat{i}-6\hat{j}+12\hat{k},\ \bar{n}=5\hat{i}+4\hat{j}-2\hat{k}$$

If M($\bar{m}$) is the midpoint of LN, by midpoint formula,

$$\bar{m}=\frac{\bar{l}+\bar{n}}{2}=\frac{(7\hat{i}-6\hat{j}+12\hat{k})+(5\hat{i}+4\hat{j}-2\hat{k})}{2}$$
$$=\frac{1}{2}(12\hat{i}-2\hat{j}+10\hat{k})=6\hat{i}-\hat{j}+5\hat{k}$$

∴ coordinates of M = (6, -1, 5).
Hence, position vector of M is $6\hat{i}-\hat{j}+5\hat{k}$ and the coordinates of M are (6, -1, 5).

Question 3.
If the points A(3, 0, p), B (-1, q, 3) and C(-3, 3, 0) are collinear, then find
(i) The ratio in which the point C divides the line segment AB.
Solution:
Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be the position vectors of A, B and C respectively.
Then $\bar{a} = 3\hat{i} + 0 \cdot \hat{j} + p\hat{k}$, $\bar{b} = -\hat{i} + q\hat{j} + 3\hat{k}$ and $\bar{c} = -3\hat{i} + 3\hat{j} + 0\hat{k}$.
As the points A, B, C are collinear, suppose the point C divides line segment AB in the ratio λ : 1.
∴ by the section formula,

$$\bar{c} = \frac{\lambda \cdot \bar{b} + 1 \cdot \bar{a}}{\lambda + 1}$$

$$\therefore \ -3\hat{i} + 3\hat{j} + 0\cdot\hat{k} = \frac{\lambda(-\hat{i} + q\hat{j} + 3\hat{k}) + (3\hat{i} + 0\cdot\hat{j} + p\hat{k})}{\lambda + 1}$$

$$\therefore \ (\lambda + 1)(-3\hat{i} + 3\hat{j} + 0\cdot\hat{k}) = (-\lambda\hat{i} + \lambda q\hat{j} + 3\lambda\hat{k}) + (3\hat{i} + 0\cdot\hat{j} + p\hat{k})$$

$$\therefore \ -3(\lambda + 1)\hat{i} + 3(\lambda + 1)\hat{j} + 0\cdot\hat{k} = (-\lambda + 3)\hat{i} + \lambda q\hat{j} + (3\lambda + p)\hat{k}$$

By equality of vectors, we have,
$-3(\lambda + 1) = -\lambda + 3$ … (1)
$3(\lambda + 1) = \lambda q$ … (2)
$0 = 3\lambda + p$ … (3)
From equation (1), $-3\lambda - 3 = -\lambda + 3$
∴ $-2\lambda = 6$ ∴ $\lambda = -3$
∴ C divides segment AB externally in the ratio 3 : 1.

(ii) The values of p and q.
Solution:
Putting λ = -3 in equation (2), we get
3(-3 + 1) = -3q
∴ -6 = -3q ∴ q = 2
Also, putting λ = -3 in equation (3), we get
0 = -9 + p ∴ p = 9
Hence p = 9 and q = 2.

Question 4.
The position vector of points A and B are $6\bar{a} + 2\bar{b}$ and $\bar{a} - 3\bar{b}$. If the point C divides AB in the ratio 3 : 2 then show that the position vector of C is $3\bar{a} - \bar{b}$.
Solution:
Let $\bar{c}$ be the position vector of C.
Since C divides AB in the ratio 3 : 2,

$$\bar{c} = \frac{3(\bar{a} - 3\bar{b}) + 2(6\bar{a} + 2\bar{b})}{3 + 2}$$

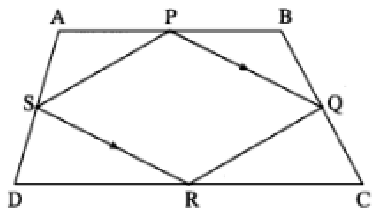
$$= \frac{3\bar{a} - 9\bar{b} + 12\bar{a} + 4\bar{b}}{5}$$

$$= \frac{1}{5}(15\bar{a} - 5\bar{b}) = 3\bar{a} - \bar{b}$$

Hence, the position vector of C is $3\bar{a} - \bar{b}$.

Question 5.
Prove that the line segments joining mid-point of adjacent sides of a quadrilateral form a parallelogram.
Solution:
Let ABCD be a quadrilateral and P, Q, R, S be the midpoints of the sides AB, BC, CD and DA respectively.
Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{p}$, $\bar{q}$, $\bar{r}$ and s be the position vectors of the points A, B, C, D, P, Q, R and S respectively.
Since P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively,

$$\bar{p}=\frac{\bar{a}+\bar{b}}{2}, \bar{q}=\frac{\bar{b}+\bar{c}}{2}, \bar{r}=\frac{\bar{c}+\bar{d}}{2} \text{ and } \bar{s}=\frac{\bar{d}+\bar{a}}{2}$$



$$\therefore\ \overline{PQ}=\bar{q}-\bar{p}$$

$$=\left(\frac{\bar{b}+\bar{c}}{2}\right)-\left(\frac{\bar{a}+\bar{b}}{2}\right)$$

$$=\frac{1}{2}(\bar{b}+\bar{c}-\bar{a}-\bar{b})=\frac{1}{2}(\bar{c}-\bar{a})$$

$$\overline{SR}=\bar{r}-\bar{s}$$

$$=\left(\frac{\bar{c}+\bar{d}}{2}\right)-\left(\frac{\bar{d}+\bar{a}}{2}\right)$$

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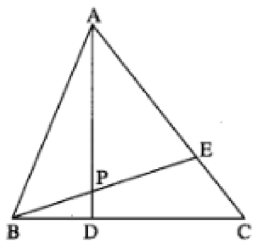
$$= \frac{1}{2}(\bar{c} + \bar{d} - \bar{d} - \bar{a}) = \frac{1}{2}(\bar{c} - \bar{a})$$

$\therefore\ \overline{PQ} = \overline{SR} \quad \therefore\ \overline{PQ} \parallel \overline{SR}$

Similarly, $\overline{QR} \parallel \overline{PS}$

∴ □PQRS is a parallelogram.

Question 6.
D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.
Solution:



Let AD and BE intersect at P.
Let A, B, C, D, E, P have position vectos $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{p}$ respectively.
D and E divide segments BC and CA internally in the ratio 2 : 3.
By the section formula for internal division,

$$\bar{d} = \frac{2\bar{c} + 3\bar{b}}{2 + 3}$$

$$\therefore\ 5\bar{d} = 2\bar{c} + 3\bar{b} \quad \text{... (1)}$$

and $\bar{e} = \frac{2\bar{a} + 3\bar{c}}{2 + 3}$

$\therefore\ 5\bar{e} = 2\bar{a} + 3\bar{c}$ ... (2)

From (1), $5\bar{d} - 3\bar{b} = 2\bar{c}$ $\quad\therefore\ 15\bar{d} - 9\bar{b} = 6\bar{c}$

From (2), $5\bar{e} - 2\bar{a} = 3\bar{c}$ $\quad\therefore\ 10\bar{e} - 4\bar{a} = 6\bar{c}$

Equating both values of $6\bar{c}$, we get

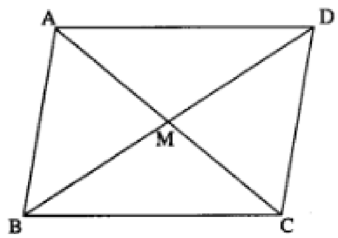
$15\bar{d} - 9\bar{b} = 10\bar{e} - 4\bar{a}$

$\therefore\ 15\bar{d} + 4\bar{a} = 10\bar{e} + 9\bar{b}$

$\therefore\ \frac{15\bar{d} + 4\bar{a}}{15 + 4} = \frac{10\bar{e} + 9\bar{b}}{10 + 9}$

LHS is the position vector of the point which divides segment AD internally in the ratio 15 : 4.
RHS is the position vector of the point which divides segment BE internally in the ratio 10 : 9.
But P is the point of intersection of AD and BE.
∴ P divides AD internally in the ratio 15 : 4 and P divides BE internally in the ratio 10 : 9.
Hence, the position vector of the point of interaction of
AD and BE is $\bar{p} = \frac{15\bar{d} + 4\bar{a}}{19} = \frac{10\bar{e} + 9\bar{b}}{19}$ and it divides AD internally in the ratio 15 : 4 and BE internally in the ratio 10 : 9.

Question 7.
Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the parallelogram ABCD. Then AB = DC and side AB || side DC.



$\therefore \overline{AB} = \overline{DC}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{a} + \bar{c} = \bar{b} + \bar{d}$

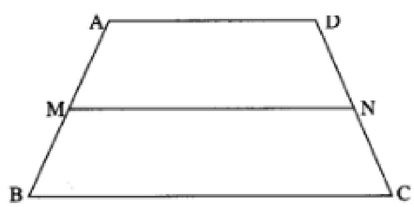
$\therefore \dfrac{\bar{a}+\bar{c}}{2} = \dfrac{\bar{b}+\bar{d}}{2}$ ... (1)

The position vectors of the midpoints of the diagonals AC and BD are $(\bar{a} + \bar{c})/2$ and $(\bar{b} + \bar{d})/2$. By (1), they are equal.
∴ the midpoints of the diagonals AC and BD are the same.
This shows that the diagonals AC and BD bisect each other.

(ii) Conversely, suppose that the diagonals AC and BD of □ ABCD bisect each other,
i. e. they have the same midpoint.
∴ the position vectors of these midpoints are equal.
$\therefore \frac{\bar{a}+\bar{c}}{2} = \frac{\bar{b}+\bar{d}}{2} \therefore \bar{a} + \bar{c} = \bar{b} + \bar{d}$
$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d} \therefore \overline{AB} = \overline{DC}$
$\therefore \overline{AB} \parallel \overline{DC}$ and $|\overline{AB}| = |\overline{DC}|$
∴ side AB || side DC and AB = DC.

∴ □ ABCD is a parallelogram.

Question 8.
Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half the sum of parallel sides.
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the trapezium ABCD, with side AD || side BC.
Then the vectors $\overline{AD}$ and $\overline{BC}$ are parallel.
∴ there exists a scalar k,
such that $\overline{AD} = k \cdot \overline{BC}$
∴ $\overline{AD} + \overline{BC} = k \cdot \overline{BC} + \overline{BC}$
= (k + 1)BC ...(1)



Let $\bar{m}$ and $\bar{n}$ be the position vectors of the midpoints M and N of the non-parallel sides AB and DC respectively.
Then seg MN is the median of the trapezium.
By the midpoint formula,

$\bar{m} = \frac{\bar{a}+\bar{b}}{2}$ and $\bar{n} = \frac{\bar{d}+\bar{c}}{2}$

$\therefore \overline{MN} = \bar{n} - \bar{m}$

$$= \left(\frac{\bar{d}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{b}}{2}\right)$$

$$= \frac{1}{2}(\bar{d}+\bar{c}-\bar{a}-\bar{b})$$

$$= \frac{1}{2}[(\bar{d}-\bar{a})+(\bar{c}-\bar{b})]$$

$$= \frac{\overline{AD}+\overline{BC}}{2} \quad \ldots (2)$$

$$= \frac{(k+1)\overline{BC}}{2} \quad \ldots \text{[By (1)]}$$

Thus $\overline{MN}$ is a scalar multiple of $\overline{BC}$

$\therefore$ $\overline{MN}$ and $\overline{BC}$ are parallel vectors

$\therefore$ $\overline{MN} \parallel \overline{BC}$ where $\overline{BC} \parallel \overline{AD}$

$\therefore$ the median MN is parallel to the parallel sides AD and BC of the trapezium.

Now $\overline{AD}$ and $\overline{BC}$ are collinear

$\therefore |\overline{AD}+\overline{BC}| = |\overline{AD}| + |\overline{BC}| = AD + BC$

$\therefore$ from (2), we have

$\therefore \overline{MN} = \frac{\overline{AD}+\overline{BC}}{2}$

$\therefore$ MN $= \frac{1}{2}$(AD + BC).

Question 9.
If two of the vertices of the triangle are A(3, 1, 4) and B(-4, 5, -3) and the centroid of a triangle is G(-1, 2, 1), then find the coordinates of the third vertex C of the triangle.
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{g}$ be the position vectors of A, B, C and G respectively.
Then $\bar{a} = 3\hat{i} + \hat{j} + 4\hat{k}$, $\bar{b} = -4\hat{i} + 5\hat{j} - 3\hat{k}$ and $\bar{g} = -\hat{i} + 2\hat{j} + \hat{k}$.
Since G is the centroid of the ΔABC, by the centroid formula,

$$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3} \quad \therefore\ 3\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore\ 3(-\hat{i} + 2\hat{j} + \hat{k}) = (3\hat{i} + \hat{j} + 4\hat{k}) + (-4\hat{i} + 5\hat{j} - 3\hat{k}) + \bar{c}$$

$$\therefore\ -3\hat{i} + 6\hat{j} + 3\hat{k} = (-\hat{i} + 6\hat{j} + \hat{k}) + \bar{c}$$

$$\therefore\ \bar{c} = (-3\hat{i} + 6\hat{j} + 3\hat{k}) - (-\hat{i} + 6\hat{j} + \hat{k})$$

$$\therefore\ \bar{c} = -2\hat{i} + 0\cdot\hat{j} + 2\hat{k}$$

$\therefore$ the coordinates of third vertex C are (-2, 0, 2).

Question 10.
InΔOAB, E is the mid-point of OB and D is the point on AB such that AD : DB = 2 : 1.
If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.
Solution:

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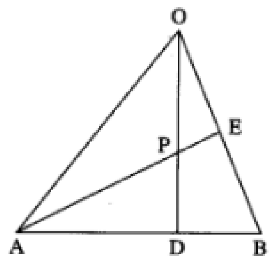


Let A, B, D, E, P have position vectors $\bar{a}$, $\bar{b}$, $\bar{d}$, $\bar{e}$, $\bar{p}$ respectively w.r.t. O.
∵ AD : DB = 2 : 1.
∴ D divides AB internally in the ratio 2 : 1.
Using section formula for internal division, we get

$$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$$

$$\therefore\ 3\bar{d} = 2\bar{b} + \bar{a} \quad \ldots (1)$$

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\overline{OB} = \frac{\bar{b}}{2}$

$$\therefore\ \bar{b} = 2\bar{e} \quad \ldots (2)$$

∴ from (1),

$$3\bar{d} = 2(2\bar{e}) + \bar{a} \quad \ldots \text{[By (2)]}$$

$$= 4\bar{e} + \bar{a}$$

$$\therefore\ \frac{3\bar{d} + 2\cdot\bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.
RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.
But OD and AE intersect at P

∴ P divides OD internally in the ratio 3 : 2.
Hence, OP : PD = 3 : 2.

Question 11.
If the centroid of a tetrahedron OABC is (1, 2, -1) where A = (a, 2, 3), B = (1, b, 2), C = (2, 1, c) respectively, find the distance of P (a, b, c) from the origin.
Solution:
Let G = (1, 2, -1) be the centroid of the tetrahedron OABC.
Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{g}$ be the position vectors of the points A, B, C, G respectively w.r.t. O.
Then $\bar{a} = a\hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = \hat{i} + b\hat{j} + 2\hat{k}$,
$\bar{c} = 2\hat{i} + \hat{j} + c\hat{k}$, $\bar{g} = \hat{i} + 2\hat{j} - \hat{k}$
By formula of centroid of a tetrahedron,

$$\bar{g} = \frac{\bar{0} + \bar{a} + \bar{b} + \bar{c}}{4}$$

$$\therefore\ 4\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore\ 4(\hat{i} + 2\hat{j} - \hat{k}) = (a\hat{i} + 2\hat{j} + 3\hat{k}) + (\hat{i} + b\hat{j} + 2\hat{k}) + (2\hat{i} + \hat{j} + c\hat{k})$$

$$\therefore\ 4\hat{i} + 8\hat{j} - 4\hat{k} = (a + 1 + 2)\hat{i} + (2 + b + 1)\hat{j} + (3 + 2 + c)\hat{k}$$

$$\therefore\ 4\hat{i} + 8\hat{j} - 4\hat{k} = (a + 3)\hat{i} + (b + 3)\hat{j} + (c + 5)\hat{k}$$

By equality of vectors
a + 3 = 4, b + 3 = 8, c + 5= -4
∴ a = 1, b = 5, c = -9
∴ P = (a, b, c) = (1, 5, -9)
Distance of P from origion = $\sqrt{1^2 + 5^2 + (-9)^2}$
= $\sqrt{1 + 25 + 81}$
= $\sqrt{107}$

Question 12.
Find the centroid of tetrahedron with vertices K(5, -7, 0), L(1, 5, 3), M(4, -6, 3), N(6, -4, 2) ?
Solution:
Let $\bar{p}$, $\bar{l}$, $\bar{m}$, $\bar{n}$ be the position vectors of the points K, L, M, N respectively w.r.t. the origin O.

Then $\bar{p} = 5\hat{i} - 7\hat{j} + 0 \cdot \hat{k}$, $\bar{l} = \hat{i} + 5\hat{j} + 3\hat{k}$,

$\bar{m} = 4\hat{i} - 6\hat{j} + 3\hat{k}$, $\bar{n} = 6\hat{i} - 4\hat{j} + 2\hat{k}$.

Let G($\bar{g}$) be the centroid of the tetrahedron.

Then by centroid formula

$$\bar{g} = \frac{\bar{p} + \bar{l} + \bar{m} + \bar{n}}{4}$$

$$= \frac{1}{4}[(5\hat{i} - 7\hat{j} + 0 \cdot \hat{k}) + (\hat{i} + 5\hat{j} + 3\hat{k}) + (4\hat{i} - 6\hat{j} + 3\hat{k}) + (6\hat{i} - 4\hat{j} + 2\hat{k})]$$

$$= \frac{1}{4}(16\hat{i} - 12\hat{j} + 8\hat{k})$$

$$= 4\hat{i} - 3\hat{j} + 2\hat{k}$$

Hence, the centroid of the tetrahedron is G = (4, -3, 2).

Ex 5.3

Question 1.
Find two unit vectors each of which is perpendicular to both $\bar{u}$ and $\bar{v}$, where $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$
Solution:
Let $\bar{u} = 2\hat{i} + \hat{j} - 2\hat{k}$
$\bar{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

$$\text{Then } \bar{u} \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2+4)\hat{i} + (-4+2)\hat{j} + (4-1)\hat{k}$$

$$= 2\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore \left|\bar{u} \times \bar{v}\right| = \sqrt{(2)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{4+4+9}$$

$$= \sqrt{17}$$

$$= \pm \frac{\bar{u} \times \bar{v}}{\left|\bar{u} \times \bar{v}\right|} = \pm \frac{2\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{17}}$$

$$= \pm \left(\frac{2}{\sqrt{17}}\hat{i} - \frac{2}{\sqrt{17}}\hat{j} + \frac{3}{\sqrt{17}}\hat{k} \right)$$

Question 2.

If $\bar{a}$ and $\bar{b}$ are two vectors perpendicular to each other, prove that $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$

Solution:

$\bar{a}$ and $\bar{b}$ are perpendicular to each other.

$$\therefore \ \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a} = 0 \quad \ldots (1)$$

$$\begin{aligned} \text{LHS} &= (\bar{a}+\bar{b})^2 \\ &= (\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b}) \\ &= \bar{a}\cdot(\bar{a}+\bar{b}) + \bar{b}\cdot(\bar{a}+\bar{b}) \\ &= \bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b} \\ &= \bar{a}\cdot\bar{a} + 0 + 0 + \bar{b}\cdot\bar{b} \quad \ldots [\text{By (1)}] \\ &= |\bar{a}|^2 + |\bar{b}|^2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (\bar{a}-\bar{b})^2 \\ &= (\bar{a}-\bar{b})\cdot(\bar{a}-\bar{b}) \\ &= \bar{a}\cdot(\bar{a}-\bar{b}) + \bar{b}\cdot(\bar{a}-\bar{b}) \\ &= \bar{a}\cdot\bar{a} - \bar{a}\cdot\bar{b} - \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b} \\ &= \bar{a}\cdot\bar{a} + \bar{b}\cdot\bar{b} \quad \ldots [\text{By (1)}] \\ &= |\bar{a}|^2 + |\bar{b}|^2 \end{aligned}$$

$\therefore$ LHS = RHS

Hence, $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$.

Question 3.

Find the values of c so that for all real x the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.

Solution:

Let $\bar{a} = xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $\bar{b} = x\hat{i} + 2\hat{j} + 2cx\hat{k}$

Consider $\bar{a}\cdot\bar{b} = (xc\hat{i} - 6\hat{j} + 3\hat{k}) \cdot (x\hat{i} + 2\hat{j} + 2cx\hat{k})$

= (xc)(x) + (-6)(2) + (3)(2cx)

= $cx^2 - 12 + 6cx$

= $cx^2 + 6cx - 12$

If the angle between $\bar{a}$ and $\bar{b}$ is obtuse, $\bar{a}\cdot\bar{b} < 0$.

$\therefore cx^2 + 6cx - 12 < 0$

$\therefore cx^2 + 6cx < 12$

$$\therefore c(x^2 + 6x) < 12$$

$$\therefore c < \frac{12}{x^2 + 6x}$$

$$\therefore c < \frac{12}{(x^2 + 6x + 9) - 9} = \frac{12}{(x+3)^2 - 9}$$

$$\therefore c < \min\left\{\frac{12}{(x+3)^2 - 9}\right\}$$

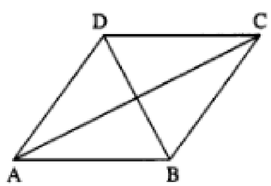
Now, $\frac{12}{(x+3)^2 - 9}$ is minimum if $(x+3)^2 - 9$ is maximum

i.e. $(x+3)^2 - 9 = \infty - 9 = \infty$

$$\therefore c < \min\left\{\frac{12}{\infty}\right\} = 0$$

∴ c < 0.
Hence, the angle between a and b is obtuse if c < 0.

Question 4.
Show that the sum of the length of projections of $\hat{p}\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where p = 2, q = 3 and r = 4, is 9.
Solution:
Let $\bar{a} = \hat{p}\hat{i} + q\hat{j} + r\hat{k}$
Projection of $\bar{a}$ on X-axis
$$= \frac{\bar{a}\cdot\hat{i}}{|\hat{i}|} = \frac{(p\hat{i} + q\hat{i} + r\hat{k})\cdot\hat{i}}{1} = p = 2$$
Similarly, projections of $\bar{a}$ on Y- and Z-axes are 3 and 4 respectively.
∴ sum of these projections = 2 + 3 + 4 = 9.

Question 5.
Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.
Solution:


Let ABCD be a quadrilateral in which

$|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{DA}|$ ... (1)

and AB || DC and AD || BC

$\therefore\ \overline{AB} = \overline{DC}$ and $\overline{AD} = \overline{BC}$ ... (2)

Now, $\overline{AC} = \overline{AB} + \overline{BC}$

and $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC}$ ... [By (2)]

$= \overline{BC} - \overline{AB}$

$\therefore\ \overline{AC} \cdot \overline{BD} = (\overline{AB} + \overline{BC}) \cdot (\overline{BC} - \overline{AB})$

$= \overline{AB} \cdot (\overline{BC} - \overline{AB}) + \overline{BC} \cdot (\overline{BC} - \overline{AB})$

$= \overline{AB} \cdot \overline{BC} - \overline{AB} \cdot \overline{AB} + \overline{BC} \cdot \overline{BC} - \overline{BC} \cdot \overline{AB}$

$= |\overline{BC}|^2 - |\overline{AB}|^2$ ... $[\because\ \overline{AB} \cdot \overline{BC} = \overline{BC} \cdot \overline{AB}]$

$= \mathbf{0}$ ... [By (1)]

$\because\ \overline{AC}, \overline{BD}$ are non-zero vectors

$\therefore\ \overline{AC}$ is perpendicular to $\overline{BD}$

Hence, the diagonals are perpendicular.

Question 6.

Determine whether $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neither.

(i) $\bar{a} = -9\hat{i} + 6\hat{j} + 15\hat{k}$, $\bar{b} = 6\hat{i} - 4\hat{j} - 10\hat{k}$

Solution:

$\bar{a} = -9\hat{i} + 6\hat{j} + 15\hat{k} = -3(3\hat{i} - 2\hat{j} - 5\hat{k})$

$= -\frac{3}{2}(6\hat{i} - 4\hat{j} - 19\hat{k})$

$\therefore \bar{a} = -\frac{3}{2}\bar{b}$

i.e. $\bar{a}$ is a non-zero scalar multiple of $\bar{b}$

Hence, $\bar{a}$ is parallel to $\bar{b}$.

(ii) $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

Solution:

$\bar{a} \cdot \bar{b} = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (5\hat{i} - 2\hat{j} + 4\hat{k})$

= (2)(5) + (3)(-2) + (-1)(4)

= 10 – 6 – 4 = 0

Since, $\bar{a}$, $\bar{b}$ are non-zero vectors and $\bar{a} \cdot \bar{b} = 0$, $\bar{a}$ is orthogonal to $\bar{b}$.

(iii) $\bar{a} = -\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}$, $\bar{b} = 5\hat{i} + 4\hat{j} + 3\hat{k}$.

Solution:

$$\bar{a}.\bar{b} = \left(-\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}\right).\left(5\hat{i} + 4\hat{j} + 3\hat{k}\right)$$

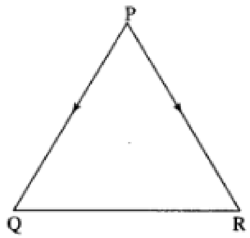
$$= \left(-\frac{3}{5}\right)(5) + \left(\frac{1}{2}\right)(4) + \left(\frac{1}{3}\right)(3)$$

= -3 + 2 + 1

= 0
Since, $\bar{a}$, $\bar{b}$ are non-zero vectors and $\bar{a} \cdot \bar{b}$ = 0
$\bar{a}$ is orthogonal to $\bar{b}$.

(iv) $\bar{a}$ = $4\hat{i} - \hat{j} + 6\hat{k}$, $\bar{a}$ = $5\hat{i} - 2\hat{j} + 4\hat{k}$
Solution:
$\bar{a} \cdot \bar{b}$ = $(4\hat{i} - \hat{j} + 6\hat{k}) \cdot (5\hat{i} - 2\hat{j} + 4\hat{k})$
= (4)(5) + (-1)(-2) + (6)(4)
= 20 + 2 + 24
= 46 ≠ 0
$\therefore$ $\bar{a}$ is not orthogonal to $\bar{b}$.
It is clear that $\bar{a}$ is not a scalar multiple of $\bar{b}$.
$\therefore$ $\bar{a}$ is not parallel to $\bar{b}$.
Hence, $\bar{a}$ is neither parallel nor orthogonal to $\bar{b}$.

Question 7.
Find the angle P of the triangle whose vertices are P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1).
Solution:
The position vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$ of the points P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1) are



$\bar{p} = -\hat{j} - 2\hat{k}, \bar{q} = 3\hat{i} + \hat{j} + 4\hat{k},$

$\bar{r} = 5\hat{i} + 7\hat{j} + \hat{k}$

$\therefore \overline{PQ} = \bar{q} - \bar{p} = (3\hat{i} + \hat{j} + 4\hat{k}) - (-\hat{j} - 2\hat{k}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$

and $\overline{PR} = \bar{r} - \bar{p} = (5\hat{i} + 7\hat{j} + \hat{k}) - (-\hat{j} - 2\hat{k}) = 5\hat{i} + 8\hat{j} + 3\hat{k}$

$\therefore \overline{PQ} \cdot \overline{PR} = (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (5\hat{i} + 8\hat{j} + 3\hat{k})$

$= (3)(5) + (2)(8) + (6)(3)$

$= 15 + 16 + 18 = 49$

$|\overline{PQ}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

$|\overline{PR}| = \sqrt{5^2 + 8^2 + 3^2} = \sqrt{25 + 64 + 9} = \sqrt{98} = 7\sqrt{2}$

Using the formula for angle between two vectors,

$$\cos P = \frac{\overline{PQ} \cdot \overline{PR}}{|\overline{PQ}||\overline{PR}|} = \frac{49}{7 \times 7\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45°$$

$\therefore P = 45°$

Question 8.

Question 8.
If $\hat{p}$, $\hat{q}$, and $\hat{r}$ are unit vectors, find

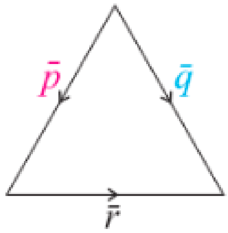
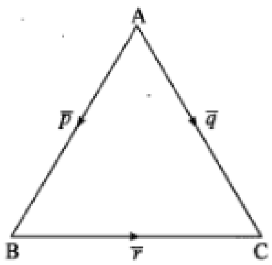


Fig.5.50

(i) $\hat{p} \cdot \hat{q}$
Solution:



Let the triangle be denoted by ABC,
where $\overline{AB} = \bar{p}$, $\overline{AC} = \bar{q}$ and $\overline{BC} = \bar{r}$
$\because \bar{p}, \bar{r}, \bar{r}$ are unit vectors.
$\therefore$ l(AB) = l(BC) = l(CA) = 1
$\therefore$ the triangle is equilateral
$\therefore \angle A = \angle B = \angle C = 60°$
(i) Using the formula for angle between two vectors,

$$\cos A = \frac{\bar{p} \cdot \bar{q}}{|\bar{p}||\bar{q}|}$$

$\therefore \cos 60° = \frac{p \cdot q}{1 \times 1}$

$\therefore \frac{1}{2} = \bar{p} \cdot \bar{q}$

$\therefore \bar{p} \cdot \bar{q} = \frac{1}{2}$.

(ii) $\hat{p} \cdot \hat{r}$

Solution:

Using $\cos B = \frac{\bar{p} \cdot \bar{r}}{|\bar{p}||\bar{r}|}$, we get $\bar{p} \cdot \bar{r} = \frac{1}{2}$

$\cos 60° = \frac{\bar{p} \cdot \bar{r}}{1 \times 1}$ $\quad \therefore \frac{1}{2} = \bar{p} \cdot \bar{r}$

$\therefore \bar{p} \cdot \bar{r} = \frac{1}{2}$.

Question 9.

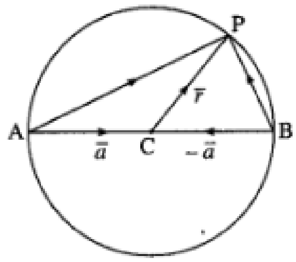
Prove by vector method that the angle subtended on semicircle is a right angle.

Solution:

Let seg AB be a diameter of a circle with centre C and P be any point on the circle other than A and B. Then ∠APB is an angle subtended on a semicircle.

Let $\overline{AC} = \overline{CB} = \bar{a}$ and $\overline{CP} = \bar{r}$.

Then $|\bar{a}| = |\bar{r}|$ ...(1)



$\overline{AP} = \overline{AC} + \overline{CP} = \bar{a} + \bar{r} = \bar{r} + \bar{a}$

$\overline{BP} = \overline{BC} + \overline{CP} = -\overline{CB} + \overline{CP} = -\bar{a} + \bar{r}$

$\therefore\ \overline{AP} \cdot \overline{BP} = (\bar{r} + \bar{a}) \cdot (\bar{r} - \bar{a})$

$= \bar{r} \cdot \bar{r} - \bar{r} \cdot \bar{a} + \bar{a} \cdot \bar{r} - \bar{a} \cdot \bar{a}$

$= |\bar{r}|^2 - |\bar{a}|^2 = 0 \quad \ldots (\because\ \bar{r} \cdot \bar{a} = \bar{a} \cdot \bar{r})$

$\therefore\ \overline{AP} \perp \overline{BP} \quad \therefore\ \angle APB$ is a right angle.

Hence, the angle subtended on a semicircle is the right angle.

Question 10.
If a vector has direction angles 45°and 60° find the third direction angle.
Solution:
Let $\alpha = 45°$, $\beta = 60°$
We have to find γ.
$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\therefore \cos^2 45° + \cos^2 60° + \cos^2\gamma = 1$

$\therefore\ \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right) + \cos^2\gamma = 1$

$$\therefore \cos^2 \gamma = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \cos \gamma = \pm \frac{1}{2}$$

$$\therefore \cos \gamma = \frac{1}{2} \text{ or } \cos \gamma = -\frac{1}{2}$$

$$\therefore \cos \gamma = \cos \frac{\pi}{3} \text{ or } \cos \gamma = -\cos \frac{\pi}{3}$$

$$= \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3}$$

$$\therefore \gamma = \frac{\pi}{3} \text{ or } \gamma = \frac{2\pi}{3}$$

Hence, the third direction angle is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Question 11.
If a line makes angles 90°, 135°, 45° with the X, Y and Z axes respectively, then find its direction cosines.
Solution:
Let l, m, n be the direction cosines of the line.
Then l = cos α, m = cos β, n = cos γ
Here, α = 90°, β = 135° and γ = 45°
∴ l = cos 90° = 0
m = cos 135° = cos (180° – 45°) = -cos 45° = $-\frac{1}{\sqrt{2}}$ and n = cos 45° = $\frac{1}{\sqrt{2}}$

∴ the direction cosines of the line are 0, $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Question 12.
If a line has the direction ratios, 4, -12, 18 then find its direction cosines.
Solution:
The direction ratios of the line are a = 4, b = -12, c = 18.
Let l, m, n be the direction cosines of the line.

$$\text{Then } l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{4}{\sqrt{4^2+(-12)^2+(18)^2}}$$

$$= \frac{4}{\sqrt{16+144+324}} = \frac{4}{22} = \frac{2}{11}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-12}{\sqrt{4^2+(-12)^2+(18)^2}}$$

$$= \frac{-12}{\sqrt{16+144+324}} = \frac{-12}{22} = \frac{-6}{11}$$

$$\text{and } n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{18}{\sqrt{4^2+(-12)^2+(18)^2}}$$

$$= \frac{18}{\sqrt{16+144+324}} = \frac{18}{22} = \frac{9}{11}$$

Hence, the direction cosines of the line are $\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}$.

Question 13.
The direction ratios of $\overline{AB}$ are -2, 2, 1. If A = (4, 1, 5) and l(AB) = 6 units, find B.
Solution:
The direction ratio of $\overline{AB}$ are -2, 2, 1.
∴ the direction cosines of $\overline{AB}$ are

$$l=\frac{-2}{\sqrt{(-2)^2+2^2+1^2}}=\frac{-2}{3},$$

$$m=\frac{2}{\sqrt{(-2)^2+2^2+1^2}}=\frac{2}{3},$$

$$n=\frac{1}{\sqrt{(-2)^2+2^2+1^2}}=\frac{1}{3}.$$

$$\text{i.e. } l=\frac{-2}{3},\ m=\frac{2}{3},\ n=\frac{1}{3}$$

The coordinates of the points which are at a distance of d units from the point (x_1, y_1, z_1) are given by $(x_1 \pm ld, y_1 \pm md, z_1 \pm nd)$
Here $x_1 = 4$, $y_1 = 1$, $z_1 = 5$, d = 6, $l = -\frac{2}{3}$, $m = \frac{2}{3}$, $n = \frac{1}{3}$
∴ the coordinates of the requited points are
$\left(4 \pm \left(-\frac{2}{3}\right)6, 1 \pm \frac{2}{3}(6), 5 \pm \frac{1}{3}(6)\right)$
i.e. (4 – 4, 1 + 4, 5 + 2) and (4 + 4, 1 – 4, 5 – 2)
i.e. (0, 5, 7) and (8, -3, 3).

Question 14.
Find the angle between the lines whose direction cosines l, m, n satisfy the equations 5l + m + 3n = 0 and 5mn – 2nl + 6lm = 0.
Solution:

Given, $5l + m + 3n = 0$ …(1)
and $5mn – 2nl + 6lm = 0$ …(2)
From (1), $m = -(51 + 3n)$
Putting the value of m in equation (2), we get,
$-5(5l + 3n)n – 2nl – 6l(5l + 3n) = 0$
$\therefore -25ln – 15n^2 – 2nl – 30l^2 – 18ln = 0$
$\therefore – 30l^2 – 45ln – 15n^2 = 0$
$\therefore 2l^2 + 3ln + n^2 = 0$
$\therefore 2l^2 + 2ln + ln + n^2 = 0$
$\therefore 2l(l + n) + n(l + n) = 0$
$\therefore (l + n)(2l + n) = 0$
$\therefore l + n = 0$ or $2l + n = 0$
$l = -n$ or $n = -2l$
Now, $m = -(5l + 3n)$, therefore, if $l = -n$,
$m = -(-5n + 3n) = 2n$
$\therefore -l = \frac{m}{2} = n$
$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{1}$
$\therefore$ the direction ratios of the first line are
$a_1 = -1, b_1 = 2, c_1 = 1$
If $n = -2l$, $m = -(5l – 6l) – l$
$\therefore l = m = \frac{n}{-2}$
$\therefore \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$
$\therefore$ the direction ratios of the second line are
$a_2 = -1, b_2 = 1, c_2 = -2$
Let θ be the angle between the lines.

Then $\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

$$= \left|\frac{(-1)(1) + 2(1) + 1(-2)}{\sqrt{(-1)^2 + 2^2 + 1^2}\cdot\sqrt{1^2 + 1^2 + (-2)^2}}\right|$$

$$= \left|\frac{-1 + 2 - 2}{\sqrt{6}\cdot\sqrt{6}}\right| = \left|\frac{-1}{6}\right| = \frac{1}{6}$$

$\therefore\ \theta = \cos^{-1}\left(\frac{1}{6}\right)$.

Ex 5.4

Question 1.
If $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$ find $(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})$
Solution:

$$\text{Given : } \bar{a} = 2\hat{i} + 3\hat{j} - \hat{k},\ \bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\therefore\ \bar{a} + \bar{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 4\hat{j} + 2\hat{k}) = 3\hat{i} - \hat{j} + \hat{k}$$

$$\text{and } \bar{a} - \bar{b} = (2\hat{i} + 3\hat{j} - \hat{k}) - (\hat{i} - 4\hat{j} + 2\hat{k}) = \hat{i} + 7\hat{j} - 3\hat{k}$$

$$\therefore\ (\bar{a} + \bar{b}) \times (\bar{a} - \bar{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 7 & -3 \end{vmatrix}$$

$$= (3 - 7)\hat{i} - (-9 - 1)\hat{j} + (21 + 1)\hat{k}$$

$$= -4\hat{i} + 10\hat{j} + 22\hat{k}.$$

Question 2.
Find a unit vector perpendicular to the vectors $\hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j}$.
Solution:
Let $\bar{a} = \hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} + \hat{j}$

$$\text{Then } \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= (0 - 2)\hat{i} - (0 - 2)\hat{j} + (0 - 1)\hat{k}$$

$$= -2\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore\ |\bar{a} \times \bar{b}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

$$= \pm \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \pm \left(\frac{-2\hat{i} + 2\hat{j} - \hat{k}}{3} \right)$$

$$= \pm \left(-\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \right)$$

Question 3.
If $\bar{a} \cdot \bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.
Solution:
Let θ be the angle between $\bar{a}$ and $\bar{b}$

$$\because\ \bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore\ |\bar{a} \times \bar{b}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = 3$$

$$\therefore\ |\bar{a}||\bar{b}| \sin\theta = 3 \quad \ldots (1)$$

$\because a \cdot b = \sqrt{3}$

$\therefore |\bar{a}||\bar{b}| \cos\theta = \sqrt{3}$ ... (2)

Dividing (1) by (2), we get

$\frac{|\bar{a}||\bar{b}| \sin\theta}{|\bar{a}||\bar{b}| \cos\theta} = \frac{3}{\sqrt{3}} \quad \therefore \tan\theta = \sqrt{3} = \tan 60^\circ$

$\therefore \theta = 60^\circ$.

Question 4.

If $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.

Solution:

Given : $\bar{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (2+3)\hat{j} + (-4-1)\hat{k}$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore |\bar{a} \times \bar{b}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2}$$

$$= \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$$

$\therefore$ unit vectors perpendicular to both the vectors $\bar{a}$ and $\bar{b}$.

$$= \frac{\pm(\bar{a} \times \bar{b})}{|\bar{a} \times \bar{b}|} = \frac{\pm(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$$

$$= \frac{\pm(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$$

∴ required vectors of magnitude 5 units

$= \pm\frac{5}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

Question 5.

Find

(i) $\bar{u}\cdot\bar{v}$ if $|\bar{u}| = 2$, $|\vec{v}| = 5$, $|\bar{u} \times \bar{v}| = 8$

Solution:

Let θ be the angle between $\bar{u}$ and $\bar{v}$.

Then $|\bar{u} \times \bar{v}| = 8$ gives

$|\bar{u}||\bar{v}| \sin\theta = 8$

∴ 2 × 5 × sin θ = 8

$$\therefore \sin\theta = \frac{4}{5}$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} \qquad \ldots [\because 0 \leqslant \theta \leqslant \pi]$$

$$= \pm\sqrt{1-\left(\frac{4}{5}\right)^2} = \pm\sqrt{1-\frac{16}{25}}$$

$$= \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Now, $\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}| \cos\theta$

$$\therefore \bar{u}\cdot\bar{v} = 2 \times 5 \times \left(\pm\frac{3}{5}\right) = \pm 6.$$

(ii) $|\bar{u} \times \bar{v}|$ if $|\bar{u}| = 10$, $|\vec{v}| = 2$, $\bar{u} \cdot \bar{v} = 12$

Solution:

Let θ be the angle between $\bar{u}$ and $\bar{v}$.

Then $\bar{u} \cdot \bar{v} = 12$ gives

$|\bar{u}||\bar{v}|\cos\theta = 12$

$\therefore 10 \times 2 \times \cos\theta = 12$

$$\therefore \cos\theta = \frac{3}{5}, \text{ where } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now, $|\bar{u} \times \bar{v}| = |\bar{u}||\bar{v}| \sin\theta$

$$\therefore \bar{u} \times \bar{v} = 10 \times 2 \times \frac{4}{5} = 16$$

Question 6.

Prove that $2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b})$

Solution:

$$\text{LHS} = 2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b})$$

$$= 4[(\bar{a} - \bar{b}) \times (\bar{a} + \bar{b})]$$

$$= 4[\bar{a} \times (\bar{a} + \bar{b}) - \bar{b} \times (\bar{a} + \bar{b})]$$

$$= 4[\bar{a} \times (\bar{a} + \bar{b}) - \bar{b} \times (\bar{a} + \bar{b})]$$

$$= 4(\bar{a} \times \bar{a} + \bar{a} \times \bar{b} - \bar{b} \times \bar{a} - \bar{b} \times \bar{b})$$

$$= 8(\bar{a} \times \bar{b})$$

$$\ldots \; [\because \; \bar{a} \times \bar{a} = \bar{b} \times \bar{b} = \bar{0} \text{ and } -(\bar{b} \times \bar{a}) = \bar{a} \times \bar{b}]$$

$$= \text{RHS}$$

$$\therefore \; 2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b}).$$

Question 7.

If $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, and $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$, verify that $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

Solution:

Given : $\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\bar{c} = \hat{i} - \hat{j} + 2\hat{k}$

$$\therefore \; \bar{b} + \bar{c} = (4\hat{i} - 3\hat{j} + \hat{k}) + (\hat{i} - \hat{j} + 2\hat{k})$$

$$= 5\hat{i} - 4\hat{j} + 3\hat{k}$$

$$\text{and } \bar{a} \times (\bar{b} + \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$$

$$= (-6 + 12)\hat{i} - (3 - 15)\hat{j} + (-4 + 10)\hat{k}$$

$$= 6\hat{i} + 12\hat{j} + 6\hat{k} \qquad \ldots (1)$$

$$\text{Also, } \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$$

$$= (-2 + 9)\hat{i} - (1 - 12)\hat{j} + (-3 + 8)\hat{k}$$

$$= 7\hat{i} + 11\hat{j} + 5\hat{k}$$

and $\bar{a} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$

$$= (-4+3)\hat{i} - (2-3)\hat{j} + (-1+2)\hat{k}$$

$$= -\hat{i} + \hat{j} + \hat{k}$$

$\therefore\ \bar{a} \times \bar{b} + \bar{a} \times \bar{c} = (7\hat{i} + 11\hat{j} + 5\hat{k}) + (-\hat{i} + \hat{j} + \hat{k})$

$$= 6\hat{i} + 12\hat{j} + 6\hat{k} \quad \ldots (2)$$

From (1) and (2), we get

$\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$.

Question 8.

Find the area of the parallelogram whose adjacent sides are the vectors $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$.

Solution:

Given : $\bar{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\bar{b} = \hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore\ \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -3 & -3 \end{vmatrix}$

$$= (6+3)\hat{i} - (-6-1)\hat{j} + (-6+2)\hat{k}$$

$$= 9\hat{i} + 7\hat{j} - 4\hat{k}$$

$|\bar{a} \times \bar{b}| = \sqrt{9^2 + 7^2 + (-4)^2} = \sqrt{81 + 49 + 16} = \sqrt{146}$

Area of the parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b}$ is $|\bar{a} \times \bar{b}| = \sqrt{146}$ sq units.

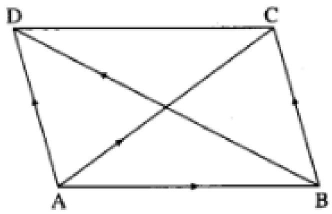
Question 9.

Show that vector area of a quadrilateral ABCD is $\frac{1}{2}$ ($\overline{AC}$ × $\overline{BD}$), where AC and BD are its diagonals.

Solution:

Let ABCD be a parallelogram.

Then $\overline{AC} = \overline{AB} + \overline{BC}$



and $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC}$ ... [$\because \overline{BC} = \overline{AD}$]

$= \overline{BC} - \overline{AB}$

$\therefore \overline{AC} \times \overline{BD} = (\overline{AB} + \overline{BC}) \times (\overline{BC} - \overline{AB})$

$= \overline{AB} \times (\overline{BC} - \overline{AB}) + \overline{BC} \times (\overline{BC} - \overline{AB})$

$= \overline{AB} \times \overline{BC} - \overline{AB} \times \overline{AB} + \overline{BC} \times \overline{BC} - \overline{BC} \times \overline{AB}$

$= \overline{AB} \times \overline{BC} + \overline{AB} \times \overline{BC}$

... [$\overline{AB} \times \overline{AB} = \overline{BC} \times \overline{BC} = \bar{0}$ and $-\overline{BC} \times \overline{AB} = \overline{AB} \times \overline{BC}$]

$\therefore \overline{AC} \times \overline{BD} = 2(\overline{AB} \times \overline{BC})$

$= 2$ (vector area of parallelogram ABCD)

$\therefore$ vector area of parallelogram ABCD $= \frac{1}{2}(\overline{AC} \times \overline{BD})$.

Question 10.
Find the area of parallelogram whose diagonals are determined by the vectors $\bar{a} = 3i - \hat{j} - 2\hat{k}$, and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$
Solution:
Given: $\bar{a} = 3i - \hat{j} - 2\hat{k}$, $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 3 & -3 \end{vmatrix}$$

$$= (3+6)\hat{i} - (-9-2)\hat{j} + (9-1)\hat{k}$$

$$= 9\hat{i} + 11\hat{j} + 8\hat{k}$$

and $|\bar{a} \times \bar{b}| = \sqrt{9^2 + 11^2 + 8^2} = \sqrt{81 + 121 + 64} = \sqrt{266}$

Area of the parallelogram having diagonals $\bar{a}$ and $\bar{b}$

$$= \frac{1}{2}|\bar{a} \times \bar{b}| = \frac{1}{2}\sqrt{266} \text{ sq units.}$$

Question 11.
If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$.
Solution:
$\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are four distinct vectors

$\therefore \bar{a} \neq \bar{b} \neq \bar{c} \neq \bar{d}$.

$\therefore \bar{a} - \bar{d} \neq \bar{0}$ and $\bar{b} - \bar{c} \neq \bar{0}$ ... (1)

Now, $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ ... (2)

and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$ ... (3)

Subtracting (3) from (2), we get

$\bar{a} \times \bar{b} - \bar{a} \times \bar{c} = \bar{c} \times \bar{d} - \bar{b} \times \bar{d}$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) = (\bar{c} - \bar{b}) \times \bar{d} = -(\bar{b} - \bar{c}) \times \bar{d}$

$= \bar{d} \times (\bar{b} - \bar{c})$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) - \bar{d} \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore (\bar{a} - \bar{d}) \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore \bar{a} - \bar{d}$ and $\bar{b} - \bar{c}$ are parallel to each other.

... [By (1)]

Question 12.

If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and, $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$

Solution:

Given $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{c} = \hat{j} - \hat{k}$

Let $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \cdot \bar{b} = 3$ gives

$(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\therefore (1)(x) + (1)(y) + (1)(z) = 3$

Also, $x + y + z = 3$ ... (1)

Also, $\bar{c} = \bar{a} \times \bar{b}$

$$\therefore \hat{j} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= (z-y)\hat{i} - (z-x)\hat{j} + (y-x)\hat{k}$$
$$= (z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k}$$

By equality of vectors,

z – y = 0 ….(2)

x – z = 1 ……(3)

y – x = -1 ……(4)

From (2), y = z.

From (3), x = 1 + z

Substituting these values of x and y in (1), we get

1 + z + z + z = 3 ∴ z = $\frac{2}{3}$

∴ y = z = $\frac{2}{3}$

∴ x = 1 + z =1 + $\frac{2}{3} = \frac{5}{3}$

$$\therefore\ \bar{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

i.e. $\bar{b} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

Question 13.

Find $\bar{a}$, if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$.

Solution:

Let $\bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \times \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{i}$

$$= x(\hat{i} \times \hat{i}) + y(\hat{j} \times \hat{i}) + z(\hat{k} \times \hat{i})$$
$$= z\hat{j} - y\hat{k} \quad \ldots \left[\because\ \hat{i} \times \hat{i} = \bar{0}, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{i} = \hat{j}\right]$$

It is given that

$\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$

$\therefore\ z\hat{j} - y\hat{k} + 2(x\hat{i} + y\hat{j} + z\hat{k}) - 5\hat{j} = 0$

$\therefore\ z\hat{j} - y\hat{k} + 2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 5\hat{j} = \bar{0}$

$\therefore\ 2x\hat{i} + (2y + z - 5)\hat{j} + (2z - y)\hat{k} = \bar{0}$

By equality of vectors
2x = 0 i.e. x = 0
2y + z – 5 = 0 … (1)
2z – y = 0 … (2)
From (2), y = 2z
Substituting y = 2z in (1), we get
4z + z = 5 ∴ z = 1
∴ y = 2z = 2(1) = 2
∴ x = 0, y = 2, z = 1
$\therefore \bar{a} = 2\hat{j} + \hat{k}$

Question 14.
If $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ and $\bar{a} \cdot \bar{b} < 0$, then find the angle between $\bar{a}$ and $\bar{b}$
Solution:
Let θ be the angle between $\bar{a}$ and $\bar{b}$.
Then $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ gives

$|ab \cos\theta| = |ab \sin\theta|$

$\therefore\ -ab\cos\theta = ab\sin\theta \quad \ldots [\because \bar{a}\cdot\bar{b} < 0]$

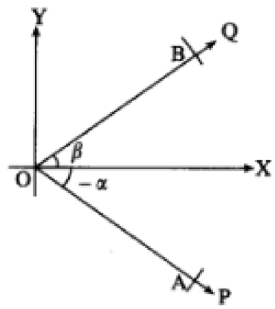
$\therefore\ -1 = \tan\theta$

$\therefore\ \tan\theta = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$

$\therefore \tan\theta = \tan\frac{3\pi}{4} \qquad \therefore \theta = \frac{3\pi}{4}$

Hence, the angle between $\bar{a}$ and $\bar{b}$ is $\frac{3\pi}{4}$.

Question 15.
Prove by vector method that sin (α + β) = sinα·cosβ+cosα·sinβ.
Solution:



Let ∠XOP and ∠XOQ be in standard position and m∠XOP = -α, m∠XOQ = β.
Take a point A on ray OP and a point B on ray OQ such that
OA = OB = 1.
Since cos (-α) = cos α
and sin (-α) = -sin α,
A is (cos (-α), sin (-α)),
i.e. (cos α, – sin α)
B is (cos β, sin β)

$\therefore \overline{OA} = (\cos\alpha)\bar{i} - (\sin\alpha)\cdot\bar{j} + 0\cdot\bar{k}$

$\overline{OB} = (\cos\beta)\cdot\bar{i} + (\sin\beta)\cdot\bar{j} + 0\cdot\bar{k}.$

$$\therefore \overline{OA} \times \overline{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$= (\cos\alpha\sin\beta + \sin\alpha\cos\beta)\bar{k} \quad \dots (1)$$

The angle between $\overline{OA}$ and $\overline{OB}$ is $\alpha + \beta$.

Also $\overline{OA}$, $\overline{OB}$ lie in the XY-plane.

∴ the unit vector perpendicular to $\overline{OA}$ and $\overline{OB}$ is $\bar{k}$.

∴ $\overline{OA} \times \overline{OB}$ = [OA·OB sin (α + β)]$\bar{k}$

= sin(α + β)·$\bar{k}$ …(2)

∴ from (1) and (2),

sin (α + β) = sin α cos β + cos α sin β.

Question 16.

Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are

(i) -2, 1, -1 and -3, -4, 1

Solution:

Let a, b, c be the direction ratios of the vector which is perpendicular to the two lines whose direction ratios are -2, 1, -1 and -3, -4, 1

∴ -2a + b – c = 0 and -3a – 4b + c = 0

$$\therefore \frac{a}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{b}{\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$$

$$\therefore \frac{a}{1-4}=\frac{b}{3+2}=\frac{c}{8+3}$$

$$\therefore \frac{a}{-3}=\frac{b}{5}=\frac{c}{11}$$

$\therefore$ the required direction ratios are -3, 5, 11

Alternative Method:

Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are -2, 1, -1 and -3, -4, 1 respectively.

Then $\bar{a} = -2\hat{i} + \hat{j} - \hat{k}$ and $\bar{b} = -3\hat{i} - 4\hat{j} + \hat{k}$

The vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

$$= (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

Hence, the required direction ratios are -3, 5, 11.

(ii) 1, 3, 2 and -1, 1, 2

Solution:

Question 17.

Prove that two vectors whose direction cosines are given by relations al + bm + cn = 0 and fmn + gnl + hlm = 0 are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

Solution:

Given, al + bm + cn = 0 …(1)
and fmn + gnl + hlm = 0 …..(2)
From (1), n = $-\left(\frac{al+bm}{c}\right)$ …..(3)
Substituting this value of n in equation (2), we get
(fm + gl)·[latex]-\left(\frac{a l+b m}{c}\right)[/latex] + hlm = 0
∴ -(aflm + bfm^2 + agl^2 + bglm) + chlm = 0
∴ agl^2 + (af + bg – ch)lm + bfm^2 = 0 … (4)
Note that both l and m cannot be zero, because if l = m = 0, then from (3), we get
n = 0, which is not possible as l^2 + m^2 + n^2 = 1.
Let us take m # 0.
Dividing equation (4) by m^2, we get
ag$\left(\frac{l}{m}\right)^2$ + (af + bg – ch)$\left(\frac{l}{m}\right)$ + bf = 0 … (5)
This is quadratic equation in $\left(\frac{l}{m}\right)$.
If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines given by the equation (1) and (2), then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of the equation (5).
From the quadratic equation (5), we get

$$\text{product of roots} = \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$$

$$\therefore \frac{l_1 l_2}{m_1 m_2} = \frac{(f/a)}{(g/b)}$$

$$\therefore \frac{l_1 l_2}{(f/a)} = \frac{m_1 m_2}{(g/b)}$$

Similarly, we can show that,

$$\frac{l_1 l_2}{(f/a)} = \frac{n_1 n_2}{(h/c)}$$

$$\therefore \frac{l_1 l_2}{(f/a)} = \frac{m_1 m_2}{(g/b)} = \frac{n_1 n_2}{(h/c)} = \lambda \qquad \text{... (Say)}$$

$$\therefore l_1 l_2 = \lambda\left(\frac{f}{a}\right), m_1 m_2 = \lambda\left(\frac{g}{b}\right), n_1 n_2 = \lambda\left(\frac{h}{c}\right)$$

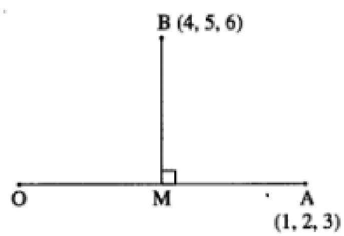
Now, the lines are perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

i.e. if $\lambda\left(\frac{f}{a}\right) + \lambda\left(\frac{g}{b}\right) + \lambda\left(\frac{h}{c}\right) = 0$

i.e. if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$.

Question 18.
If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and point A.
Solution:



Let M be the foot of the perpendicular drawn from B to the line joining O and A.
Let M = (x, y, z)
OM has direction ratios x – 0, y – 0, z – 0 = x, y, z

OA has direction ratios 1 – 0, 2 – 0, 3 – 0 = 1, 2, 3
But O, M, A are collinear.
∴ $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ = k ...(Let)
∴ x = k, y = 2k, z = 3k
∴ M = (k, 2k, 3k)
∵ BM has direction ratios
k – 4, 2k – 5, 3k – 6
BM is perpendicular to OA
∴ (l)(k – 4) + 2(2k – 5) + 3(3k – 6)
∴ = k – 4 + 4k – 10 + 9k – 18 = 0
∴ 14k = 32
∴ k = $\frac{16}{7}$
∴ M = (k, 2k, 3k) = ($\frac{16}{7}$, $\frac{32}{7}$, $\frac{48}{7}$)

Ex 5.5

Question 1.

Find $\bar{a}\cdot(\bar{b} \times \bar{c})$, if $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

$$\bar{a}\cdot(\bar{b} \times \bar{c}) = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & -1 \\ -5 & 2 & 3 \end{vmatrix}$$

= 3(9 + 2) + 1 (6 – 5) + 4(4 + 15)

= 33 + 1 + 76

= 110.

Question 2.

If the vectors $3\hat{i} + 5\hat{k}, 4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are to co-terminus edges of the parallelo piped, then find the volume of the parallelopiped.

Solution:

Let $\bar{a} = 3\hat{i} + 5\hat{k}, \bar{b} = 4\hat{i} + 2\hat{j} - 3\hat{k}, \bar{c} = 3\hat{i} + \hat{j} + 4\hat{k}$

$$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = = \begin{vmatrix} 3 & 0 & 5 \\ 4 & 2 & -3 \\ 3 & 1 & 4 \end{vmatrix}$$

= 3(8 + 3) – 0(16 + 9) + 5(4 – 6)

= 33 – 0 – 10 = 23

∴ volume of the parallelopiped = $[\bar{a}\bar{b}\bar{c}]$

= 23 cubic units.

Question 3.

If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}, \hat{i} + 2\hat{k}$ and $\hat{i} - p\hat{j}$ are coplanar, then find the value of p.

Solution:

Let $\bar{a} = -3\hat{i} + 4\hat{j} - 2\hat{k}, \bar{b} = \hat{i} + 2\hat{k}, \bar{c} = \hat{i} - p\hat{j}$

Then, $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$$\therefore \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$$

∴ -3(0 + 2p) – 4(0 – 2) – 2(-p – 0) = 0

∴ -6p + 8 + 2p = 0

P = 2.

Question 4.
Prove that :
(i) [latex]\bar{a} \bar{b}+\bar{c} \bar{a}+\bar{b}+\bar{c}[/latex] = 0
Solution:

$$[\bar{a} \quad \bar{b}+\bar{c} \quad \bar{a}+\bar{b}+\bar{c}]$$
$$=\bar{a}\cdot[(\bar{b}+\bar{c})\times(\bar{a}+\bar{b}+\bar{c})]$$
$$=\bar{a}\cdot(\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{c}\times\bar{b}+\bar{c}\times\bar{c})$$
$$=\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{b}\times\bar{b})+\bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{b})+\bar{a}\cdot(\bar{c}\times\bar{c})$$
$$=0+0+\bar{a}\cdot(\bar{b}\times\bar{c})+0-\bar{a}\cdot(\bar{b}\times\bar{c})+0$$
$$=0.$$

(ii) $(\bar{a}-2\bar{b}-\bar{c})\cdot[(\bar{a}-\bar{b})\times\bar{a}-\bar{b}-\bar{c}] = 3[\bar{a}-\bar{b}-\bar{c}]$
Question is modified.
$(\bar{a}-2\bar{b}-\bar{c})\ [(\bar{a}-\bar{b})\times\bar{a}-\bar{b}-\bar{c}] = 3[\bar{a}\ \bar{b}\ \bar{c}]$
Solution:

$$(\bar{a}+2\bar{b}-\bar{c})\cdot[(\bar{a}-\bar{b})\times(\bar{a}-\bar{b}-\bar{c})]$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{a}\times\bar{a}-\bar{a}\times\bar{b}-\bar{a}\times\bar{c}-\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c})$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{0}-\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}+\bar{0}+\bar{b}\times\bar{c})$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{c}\times\bar{a}+\bar{b}\times\bar{c})$$
$$=\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{b}\times\bar{c})+2\bar{b}\cdot(\bar{c}\times\bar{a})+2\bar{b}\cdot(\bar{b}\times\bar{c})-$$

$\bar{c}\cdot(\bar{c}\times\bar{a}) - \bar{c}\cdot(\bar{b}\times\bar{c})$

$= 0 + \bar{a}\cdot(\bar{b}\times\bar{c}) + 2\bar{b}\cdot(\bar{c}\times\bar{a}) + 2\times 0 - 0 - 0$

$= [\bar{a}\ \bar{b}\ \bar{c}] + 2[\bar{b}\ \bar{c}\ \bar{a}]$

$= [\bar{a}\ \bar{b}\ \bar{c}] + 2[\bar{a}\ \bar{b}\ \bar{c}] = 3[\bar{a}\ \bar{b}\ \bar{c}].$

Question 5.
If $\bar{c} = 3\bar{a} - 2\bar{b}$ prove that $[\bar{a}\ \bar{b}\ \bar{c}] = 0$
Solution:
We use the results : $\bar{b}\times\bar{b} = 0$ and if in a scalar triple product, two vectors are equal, then the scalar triple product is zero.

$$
\begin{aligned}
[\bar{a}\ \ \bar{b}\ \ \bar{c}] &= \bar{a}\cdot(\bar{b}\times\bar{c}) \\
&= \bar{a}\cdot[\bar{b}\times(3\bar{a} - 2\bar{b})] \\
&= \bar{a}\cdot(3\bar{b}\times\bar{a} - 2\bar{b}\times\bar{b}) \\
&= \bar{a}\cdot(3\bar{b}\times\bar{a} - \bar{0}) \\
&= 3\bar{a}\cdot(\bar{b}\times\bar{a}) = 3\times 0 = 0.
\end{aligned}
$$

Alternative Method :

$\bar{c} = 3\bar{a} - 2\bar{b}$

$\therefore$ $\bar{c}$ is a linear combination of $\bar{a}$ and $\bar{b}$

$\therefore$ $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore$ $[\bar{a}\ \bar{b}\ \bar{c}] = 0.$

Question 6.
If u = $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{k}$, $\bar{v} = 3\hat{\mathbf{i}} + \hat{k}$ and $\bar{w} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ are given vectors, then find
(i) $[\bar{u} + \bar{w}]\cdot[(\bar{w} \times \bar{r}) \times (\bar{r} \times \bar{w})]$
Question is modified.
If $\bar{u} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{k}$, $\bar{r} = 3\hat{\mathbf{i}} + \hat{k}$ and $\bar{w} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ are given vectors, then find $[\bar{u} + \bar{w}]\cdot[(\bar{u} \times \bar{r}) \times (\bar{r} \times \bar{w})]$
Solution:

$$\bar{u} + \bar{w} = (\hat{i} - 2\hat{j} + \hat{k}) + (\hat{j} - \hat{k})$$
$$= \hat{i} - \hat{j}$$
$$\bar{u} \times \bar{r} = = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$
$$= (-2-0)\hat{i} - (1-3)\hat{j} + (0+6)\hat{k}$$
$$= -2\hat{i} + 2\hat{j} + 6\hat{k}$$
$$\bar{r} \times \bar{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$
$$= (0-1)\hat{i} - (-3-0)\hat{j} + (3-0)\hat{k}$$
$$= -\hat{i} + 3\hat{j} + 3\hat{k}$$
$$\text{Now, } (\bar{u} + \bar{w})\cdot[(\bar{u} \times \bar{r}) \times (\bar{r} \times \bar{w})] = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 2 & 6 \\ -1 & 3 & 3 \end{vmatrix}$$

= 1(6 – 18) + 1 (-6 + 6) + 0
= -12 + 0 + 0 = -12.

Question 7.
Find the volume of a tetrahedron whose vertices are A(-1, 2, 3) B(3, -2, 1), C (2, 1, 3) and D(-1, -2, 4).
Solution:
The position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ of the points A, B, C and D w.r.t. the origin are $\bar{a}$ = $-\hat{i}+2\hat{j}+3\hat{k}$, $\bar{b}=3\hat{i}-2\hat{j}+\hat{k}$, $\bar{c}=2\hat{i}+\hat{j}+3\hat{k}$ and
$\bar{d}=-\hat{i}-2\hat{j}+4\hat{k}$.

$$\therefore\ \overline{AB}=\bar{b}-\bar{a}=(3\hat{i}-2\hat{j}+\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$$
$$=4\hat{i}-4\hat{j}-2\hat{k}$$
$$\overline{AC}=\bar{c}-\bar{a}=(2\hat{i}+\hat{j}+3\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$$
$$=3\hat{i}-\hat{j}$$
and $\overline{AD}=\bar{d}-\bar{a}=(-\hat{i}-2\hat{j}+4\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$
$$=-4\hat{j}+\hat{k}$$
$$\therefore\ [\overline{AB}\ \overline{AC}\ \overline{AD}]=\begin{vmatrix}4 & -4 & -2\\ 3 & -1 & 0\\ 0 & -4 & 1\end{vmatrix}$$
$$=4(-1+0)+4(3-0)-2(-12+0)$$
$$=-4+12+24=32$$
$$\therefore\ \text{volume of the tetrahedron}=\frac{1}{6}\left|[\overline{AB}\ \overline{AC}\ \overline{AD}]\right|$$
$$=\frac{1}{6}(32)=\frac{16}{3}\text{ cu units.}$$

Question 8.
If $\bar{a} = \hat{i} + 2\hat{j} + 3$, $\bar{b} = 3\hat{i} + 2\hat{j}$ and $\bar{c} = ,2\hat{i} + \hat{j} + 3$ then verify that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$
Solution:

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= (6-0)\hat{i} - (9-0)\hat{j} + (3-4)\hat{k}$$

$$= 6\hat{i} - 9\hat{j} - \hat{k}$$

$$\therefore\ \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & -9 & -1 \end{vmatrix}$$

$$= (-2+27)\hat{i} - (-1-18)\hat{j} + (-9-12)\hat{k}$$

$$= 25\hat{i} + 19\hat{j} - 21\hat{k} \quad \dots (1)$$

$$\bar{a} \cdot \bar{c} = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (2\hat{i} + \hat{j} + 3\hat{k})$$

$$= (1)(2) + (2)(1) + (3)(3)$$

$$= 2 + 2 + 9 = 13$$

$$\therefore\ (\bar{a} \cdot \bar{c})\bar{b} = 13(3\hat{i} + 2\hat{j}) = 39\hat{i} + 26\hat{j}$$

Also, $\bar{a} \cdot \bar{b} = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 2\hat{j})$

$$= (1)(3) + (2)(2) + (3)(0)$$

$$= 3 + 4 + 0 = 7$$

$$\therefore\ (\bar{a} \cdot \bar{b})\bar{c} = 7(2\hat{i} + \hat{j} + 3\hat{k}) = 14\hat{i} + 7\hat{j} + 21\hat{k}$$

$$\therefore\ (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$$

$= (39\hat{i} + 26\hat{j}) - (14\hat{i} + 7\hat{j} + 21\hat{k})$

$= 25\hat{i} + 19\hat{j} - 21\hat{k}$ ... (2)

From (1) and (2), we get

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

Question 9.

If, $\bar{a} = \hat{i} - 2\hat{j}$, $\bar{b} = \hat{i} + 2\hat{j}$ and $\bar{c} = 2\hat{i} + \hat{j} - 2$ then find

(i) $\bar{a} \times (\bar{b} \times \bar{c})$

Solution:

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (-4 - 0)\hat{i} - (-2 - 0)\hat{j} + (1 - 4)\hat{k}$$

$$= -4\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\therefore \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ -4 & 2 & -3 \end{vmatrix}$$

$$= (6 - 0)\hat{i} - (-3 - 0)\hat{j} + (2 - 8)\hat{k}$$

$$= 6\hat{i} + 3\hat{j} - 6\hat{k}$$

(ii) $(\bar{a} \times \bar{b}) \times \bar{c}$ Are the results same? Justify.

Solution:

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + (2-(-2))\hat{k}$$

$$= 4\hat{k}$$

$$\therefore \left(\bar{a} \times \bar{b}\right) \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (0-4)\hat{i} - (0-8)\hat{j} + (0-0)\hat{k}$$

$$= -4\hat{i} + 8\hat{j}$$

$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

Question 10.
Show that $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$
Solution:

$$\text{LHS} = \bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b})$$

$$= (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} + (\bar{b}\cdot\bar{a})\bar{c} - (\bar{b}\cdot\bar{c})\bar{a} + (\bar{c}\cdot\bar{b})\bar{a} - (\bar{c}\cdot\bar{a})\bar{b}$$

$$= (\bar{c}\cdot\bar{a})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} + (\bar{a}\cdot\bar{b})\bar{c} - (\bar{b}\cdot\bar{c})\bar{a} + (\bar{b}\cdot\bar{c})\bar{a} - (\bar{c}\cdot\bar{a})\bar{b}$$

$$\dots \quad [\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}]$$

$$= \bar{0} = \text{RHS.}$$

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- Chapter 2- Matrices
- Chapter 3- Trigonometric Functions
- Chapter 4- Pair of Straight Lines
- Chapter 5- Vectors
- Chapter 6- Line and Plane
- Chapter 7- Linear Programming

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