

● Differential Equation

An equation containing an independent variable and a dependent variable and the derivatives of the dependent variable with respect to the independent variable is called an ordinary differential equation.

$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$ is called an ordinary differential equation.

For instance, (1) $\frac{dy}{dx} + y = x$

$$(2) \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} = \sin x - 5$$

$$(3) \quad e^{\frac{dy}{dx}} = 3 \text{ etc. are differential equations.}$$

● Order and Degree of a Differential Equation

The order of the highest order derivative of the dependent variable with respect to the independent variable occurring in a given differential equation is called the **order of the differential equation**.

When a differential equation is in polynomial form in derivatives, the highest power of the highest order derivative occurring in the differential equation is called the degree of the differential equation. Obviously to obtain the degree of a differential equation, we should make the equation free from radicals and fractional powers. The degree of a differential equation is a positive integer.

● Formation of a Differential Equation

A functional relation with n arbitrary constants is $f(x, y, c_1, c_2, \dots, c_n) = 0$, where $c_1, c_2, \dots, c_n$, are arbitrary constants. Differentiating it n times; we get $(n+1)$ equations with the given relation. By eliminating arbitrary constants; we get a differential equation.

The order of the differential equation = The number of arbitrary constants.

● Solution of a Differential Equation

If the differential equation $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$ is obtained from functional relation

$f(x, y, c_1, c_2, \dots, c_n) = 0$, we say $f(x, y, c_1, c_2, \dots, c_n) = 0$ is the general solution of given differential equation.

- If we can find definite values of the arbitrary constants occurring in the general solution of the differential equation under some conditions on the given variable x , y and derivatives

$\frac{dy}{dx}, \frac{d^2y}{dx^2}$...etc., then the solution of the differential equation with definite values of arbitrary constants is called the particular solution and the given conditions are called initial conditions or boundary conditions.

- A solution other than the general solution of a differential equation which cannot be obtained as a particular solution from the general solution is called a singular solution.
- A first order and first degree differential equation is represented by $\frac{dy}{dx} = F(x, y)$ or $f(x, y) dx + g(x, y) dy = 0$.

● Variable Separable Method

If the general form of first order and first degree differential equation is $p(x) dx + q(y) dy = 0$, then it is said to be in variable separable form. $p(x)$ is a function of x only and $q(y)$ is a function of y only.

Now, $\int p(x) dx + \int q(y) dy = c$ (c is an arbitrary constant) is the general solution.

Homogeneous Differential Equation :

If a two variable function $f(x, y)$ can be written as $x^n \phi\left(\frac{y}{x}\right)$, then the function $f(x, y)$ is called a **homogeneous function of degree n** . $\frac{dy}{dx} = f(x, y) = x^n \phi\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

- Assuming $\frac{y}{x} = v$, the equation can be converted into variable separable form. In this way we can find the solution of the equation.
- **Linear Differential Equation**

If $P(x)$ and $Q(x)$ are functions of variable x , then the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is called a linear differential equation.

- Let $\frac{dy}{dx} + P(x)y = Q(x)$ be a given linear differential equation. To find the solution of this differential equation, multiply by integrating factor (I.F.) $e^{\int P(x) dx}$ on both the sides. In this way, we can make it easily integrable.

Multiple Choice Questions

- (1) The degree and the order of $y = x \frac{d^2 y}{dx^2} + 3 \sqrt{1 + \left(\frac{d^3 y}{dx^3}\right)^3}$ are and respectively.
- (A) 2, 2 (B) 3, 3 (C) 2, 1 (D) 3, not defined

Sol. : Remove the radicals from the differential equation.

$$\left(y - x \frac{d^2 y}{dx^2}\right)^2 = 9 \left(1 + \left(\frac{d^3 y}{dx^3}\right)^3\right)$$

$$y^2 - 2yx \frac{d^2 y}{dx^2} + x^2 \left(\frac{d^2 y}{dx^2}\right)^2 = 9 + 9 \left(\frac{d^3 y}{dx^3}\right)^3$$

$\therefore$ The order and the degree of the differential equation are 3 and 3 respectively. **Ans. : (B)**

- (2) The order and the degree of differential equation $5 \frac{d^2 y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$ are and respectively.
- (A) 1, 6 (B) 1, 2 (C) 2, 2 (D) 2, not defined

Sol. : $\left(5 \frac{d^2 y}{dx^2}\right)^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3$

$$\therefore 25 \left(\frac{d^2 y}{dx^2}\right)^2 = 1 + 3 \left(\frac{dy}{dx}\right)^2 + 3 \left(\frac{dy}{dx}\right)^4 + \left(\frac{dy}{dx}\right)^6$$

$\therefore$ Hence the order is 2 and the degree is 2.

Ans. : (C)

- (3) The order and the degree of $2x = e^{\frac{dy}{dx}}$ are and

(A) 1, 1

(B) 1, not defined

(C) 2, 2

(D) not defined, 1

Sol. : $2x = e^{\frac{dy}{dx}}$

$$\therefore \log 2x = \frac{dy}{dx}$$

$\therefore$ The differential equation has order 1 and degree 1.

Ans. : (A)

- (4) The order and the degree of the differential equation $\frac{d^2 y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$ are and respectively .

(A) 4 and 2

(B) 1 and 2

(C) 1 and 4

(D) 2 and 4

Sol. : $\left(\frac{d^2 y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^2$

$\therefore$ The order is 2 and the degree is 4.

Ans. : (D)

- (5) The degree of the differential equation $e^x + \sin\left(\frac{dy}{dx}\right) = 3$ is

(A) 0

(B) 1

(C) 2

(D) None of these

Sol. : Here we can not write the differential equation as a polynomial in derivatives.

So, the degree of the differential equation is not defined.

Ans. : (D)

- (6) The order and the degree of $\sqrt[3]{\frac{d^2 y}{dx^2}} = \sqrt{\frac{dy}{dx}}$ are and respectively .

(A) 1, 2

(B) 2, 1

(C) 1, 1

(D) 2, 2

Sol. : $\left(\frac{d^2 y}{dx^2}\right)^{\frac{1}{3}} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$

$$\therefore \left(\frac{d^2 y}{dx^2}\right)^{\frac{1}{3} \times 6} = \left(\frac{dy}{dx}\right)^{\frac{1}{2} \times 6}$$

$$\therefore \left(\frac{d^2 y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3$$

$\therefore$ It is clear that the order is 2 and the degree is also 2.

Ans. : (D)

(7) The degree of $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(c \frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$ is

- (A) 1 (B) 2 (C) 6 (D) not defined

Sol. : $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \left(c \frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$

$$\therefore \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2} \times 6} = \left(c \frac{d^2y}{dx^2}\right)^{\frac{1}{3} \times 6}$$

$$\therefore \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = \left(c \frac{d^2y}{dx^2}\right)^2$$

$\therefore$ The degree is 2.

Ans. : (B)

(8) The differential equation of the population of rabbit at time t is $\frac{dp}{dt} = \frac{1}{2} p(t) - 200$.

If $p(0) = 100$, then $p(t) = \dots$

- (A) $400 + 300e^{\frac{t}{2}}$, if $p > 400$ (B) $300 - 200e^{\frac{-t}{2}}$
(C) $600 - 500e^{\frac{t}{2}}$ (D) $400 - 300e^{\frac{-t}{2}}$

Sol. : $\frac{dp}{dt} = \frac{p - 400}{2}$

$$\frac{dp}{p - 400} = \frac{1}{2} dt$$

$$\int \frac{dp}{p - 400} = \int \frac{1}{2} dt$$

$$\therefore \log |p - 400| = \frac{1}{2} t + c$$

If $t = 0$, then $p = 100$

$$\therefore \log 300 = c$$

$$\log |p - 400| = \frac{1}{2} t + \log 300$$

$$\therefore |p - 400| = 300 e^{\frac{1}{2} t} . \quad \text{So, } p - 400 = 300 e^{\frac{1}{2} t}, \text{ if } p > 400$$

Ans. : (A)

$$p = 400 - 300 e^{\frac{1}{2} t}, \text{ if } p < 400$$

(9) $y = c_1 e^x - c_2 e^{-x}$ is the general solution of a differential equation. The order of the differential equation is, where c_1 and c_2 are arbitrary constants.

- (A) 0 (B) 1 (C) 2 (D) not defined

Sol. : $y_1 = c_1 e^x + c_2 e^{-x}$

$$\therefore y_2 = c_1 e^x - c_2 e^{-x}$$

$$\therefore y_2 = y$$

$\therefore$ The order is 2.

Note : The solution of the differential equation has two arbitrary constants. So the differential equation has order 2. **Ans. : (C)**

(10) Suppose a function f satisfies $f(x) \geq 0, \forall x \in [0, 1]$

If $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt, 0 \leq x \leq 1$ and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Sol. : $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt, 0 \leq x \leq 1$

Differentiate w.r.t. $t, \sqrt{1 - (f'(x))^2} = f(x)$

$$\therefore 1 - \left(\frac{dy}{dx}\right)^2 = y^2 \quad (y = f(x))$$

$$\therefore \frac{dy}{dx} = \pm \sqrt{1 - y^2}$$

$$\therefore \frac{dy}{\sqrt{1 - y^2}} = \pm dx$$

Integrating it, we get $\sin^{-1} y = c \pm x$

$f(0) = 0$. So $c = 0$.

$$\therefore \sin^{-1} y = \pm x$$

$$\therefore y = f(x) = \pm \sin x$$

Also, $f(x) \geq 0, \forall x \in [0, 1]$

$$\therefore f(x) = \sin x$$

Also $\sin x < x, \forall x \in [0, 1]$

$$\therefore f(x) < x, \forall x \in [0, 1]$$

$$\therefore f\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } f\left(\frac{1}{3}\right) < \frac{1}{3}$$

Ans. : (C)

(11) If the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ has the general solution $y = \frac{x}{\log|cx|}$, then $\phi\left(\frac{x}{y}\right) = \dots\dots$

(A) $\frac{-x^2}{y^2}$

(B) $\frac{y^2}{x^2}$

(C) $\frac{x^2}{y^2}$

(D) $\frac{-y^2}{x^2}$

Sol. : Let $y = vx$. So, $x \frac{dv}{dx} + v = \frac{dy}{dx}$

From the differential equation,

$$x \frac{dv}{dx} + v = v + \phi\left(\frac{1}{v}\right) \Rightarrow \frac{dv}{\phi\left(\frac{1}{v}\right)} = \frac{dx}{x}$$

Integrate, $\int \frac{dv}{\phi\left(\frac{1}{v}\right)} = \log |cx|$, where c is an arbitrary constant. (1)

$y = \frac{x}{\log |cx|}$ is the general solution.

$$\therefore \log |cx| = \frac{x}{y} = \frac{1}{v}$$

From (1), $\int \frac{dv}{\phi\left(\frac{1}{v}\right)} = \frac{1}{v}$

Differentiate on both the sides, $\frac{1}{\phi\left(\frac{1}{v}\right)} = -\frac{1}{v^2}$

$$\therefore \phi\left(\frac{1}{v}\right) = -v^2 \quad \text{So, } \phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2} \quad \text{Ans. : (D)}$$

(12) Suppose function f is differentiable on $(0, \infty)$. $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ and $f(1) = 1$. Then

$$f(x) = \dots + c$$

(A) $\frac{1}{3x^3}$ (B) $3x + \frac{3}{2x^2}$ (C) $\frac{1}{3x} + \frac{2x^2}{3}$ (D) $\frac{1}{x} + 2x^2$

Sol. : $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$.

$$\therefore \lim_{t \rightarrow x} \frac{2t f(x) - x^2 f'(t)}{1} = 1$$

$$\therefore 2x f(x) - x^2 f'(x) = 1$$

$$\therefore x^2 \frac{dy}{dx} - 2xy = -1 \quad (y = f(x))$$

$\therefore \frac{dy}{dx} - \frac{2}{x} y = \frac{-1}{x^2}$ is a linear differential equation.

$$\text{Integrating factor} = e^{-\int \frac{2}{x} dx} = e^{-2 \log x} = \frac{1}{x^2} \quad (|x|^2 = x^2)$$

$$\therefore \text{The general solution is } y\left(\frac{1}{x^2}\right) = \int \left(\frac{-1}{x^2}\right)\left(\frac{1}{x^2}\right) dx + c$$

$$y\left(\frac{1}{x^2}\right) = \frac{1}{3x^3} + c$$

$$\text{Also, } f(1) = 1 \text{ so, } 1 = \frac{1}{3} + c \Rightarrow c = \frac{2}{3}$$

$$\therefore y \frac{1}{x^2} = \frac{1}{3x^3} + \frac{2}{3} \quad \text{So, } y = \frac{1}{3x} + \frac{2x^2}{3} \quad \text{Ans. : (C)}$$

- (13) If the differential equation $(p - x^2) \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx} = q$ has the solution $y = (\sin^{-1} x)^2 + A \cos^{-1} x + B$ (where A, B are arbitrary constants), then $p + q = \dots$

- (A) 1 (B) 2 (C) 3 (D) 4

$$\text{Sol. : } y = (\sin^{-1} x)^2 + A \cos^{-1} x + B$$

$$\text{Differentiating on both the sides, } \frac{dy}{dx} = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} - \frac{A}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2} y' = 2 \sin^{-1} x - A$$

$$\text{Again differentiating, } \frac{1}{2\sqrt{1-x^2}} (-2x) y' + \sqrt{1-x^2} y'' = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore -xy' + (1-x^2) y'' = 2$$

$$\therefore (1-x^2) y'' - xy' = 2$$

$$\therefore p = 1, q = 2. \text{ So, } p + q = 3$$

Ans. : (C)

- (14) A function f is differentiable on $\mathbb{R}$ and $f(1) = 1$. The y -intercept of the tangent at point $P(x, y)$ of curve $y = f(x)$ is equal to the cube of the x coordinate, then $f(-3) = \dots$

- (A) 9 (B) -9 (C) 10 (D) 1

Sol. : The equation of the tangent to the curve at $P(x, y)$ is

$$Y - y = f'(x)(X - x)$$

$$y\text{-intercept} = y - x f'(x)$$

$$\text{We are given that } y - x f'(x) = x^3$$

$$y - x \frac{dy}{dx} = x^3$$

$$\therefore \frac{dy}{dx} - \frac{1}{x} y = -x^2 \text{ is a linear differential equation.}$$

$$\text{I.F.} = e^{\int \frac{-1}{x} dx} = e^{-\log|x|} = \frac{1}{x}$$

$$\therefore \text{Solution : } y \frac{1}{x} = -\int x dx = -\frac{1}{2} x^2 + c$$

$$\therefore f(x) = -\frac{1}{2} x^3 + cx$$

$$\text{Also, } f(1) = 1$$

$$\therefore 1 = -\frac{1}{2} (1) + c. \text{ So } c = \frac{3}{2}$$

$$\therefore f(x) = \frac{-1}{2}x^3 + \frac{3}{2}x$$

$$\therefore f(-3) = 9$$

Ans. : (A)

- (15) The normal at point $P(x, y)$ of the curve $y = f(x)$ intersects X-axis at point Q. If it is passing through another point $(0, k)$ on the curve and PQ is constant k , then

$$(A) \quad (1+k)^2 x^2 + y^2 = k^2$$

$$(B) \quad x^2 + (1+x^2)y^2 = k^2$$

$$(C) \quad x^2 + y^2 = k^2$$

$$(D) \quad x^2 + y^2 = 1$$

Sol. : The equation of the normal at point $P(x, y)$ of the curve $y = f(x)$ is

$$Y - y = -\frac{dx}{dy}(X - x)$$

It intersects X-axis at point Q. So, $Q\left(x + y \frac{dy}{dx}, 0\right)$.

$$PQ^2 = \left(y \frac{dy}{dx}\right)^2 + y^2 = k^2$$

$$\therefore \left(y \frac{dy}{dx}\right)^2 = k^2 - y^2$$

$$\therefore y \frac{dy}{dx} = \pm \sqrt{k^2 - y^2}$$

$$\therefore \frac{y dy}{\sqrt{k^2 - y^2}} = \pm dx$$

$$\text{Integrating, } \int \frac{y dy}{\sqrt{k^2 - y^2}} = \int \pm dx. \quad \text{So, } -\sqrt{k^2 - y^2} = \pm x + c$$

The curve is passing through $(0, k)$

$$\therefore -\sqrt{k^2 - k^2} = \pm(0) + c$$

$$\therefore c = 0$$

$$\therefore -\sqrt{k^2 - y^2} = \pm x$$

$$\therefore k^2 - y^2 = x^2$$

$$\therefore x^2 + y^2 = k^2$$

Ans. : (C)

- (16) Suppose f is a periodic function defined on $\left[0, \frac{\pi}{2}\right]$.

$$\int_0^x (f'(t) - \sin 2t) dt = \int_x^0 f(t) \tan t dt \quad \text{and } f(0) = 1, \text{ then }$$

$$(A) \quad \text{a maximum value of } f \text{ will be } f(0) = 1 \quad (B) \quad \text{a minimum value of } f \text{ will be } f(0) > 1$$

$$(C) \quad f\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} - 1$$

$$(D) \quad f\left(\frac{\pi}{3}\right) = 1$$

Sol. : Differentiate the equation on both the sides.

$$f'(x) - \sin 2x = -f(x) \tan x$$

Let, $y = f(x)$. So, $y' + y \tan x = \sin 2x$ is the linear differential equation obtained.

$$\text{I.F. } e^{\int \tan x dx} = \sec x$$

$$\begin{aligned} \text{The solution of the differential equation is } y(x) &= \cos x \left[\int 2 \sin x dx + c \right] \\ &= c \cos x - 2 \cos^2 x \end{aligned}$$

$$\text{Here, } y(0) = 1. \text{ So, } c = 3 \quad [1 = c - 2]$$

$$\begin{aligned} \therefore f(x) &= y = 3 \cos x - 2 \cos^2 x \\ &= -2 \left(\cos x - \frac{3}{4} \right)^2 + \frac{9}{8} \end{aligned}$$

Since $x \in \left[0, \frac{\pi}{2}\right]$, $0 \leq \cos x \leq 1$. When $\cos x = \frac{3}{4}$, then f is maximum $= \frac{9}{8}$

The maximum value of $f(x) = f(0) = 1$ is not true.

$$f\left(\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} - 1 \text{ and } f\left(\frac{\pi}{3}\right) = 1$$

Ans. : (C, D)

(17) The differential equation $y^5 x + y - x \frac{dy}{dx} = 0$ has the general solution

$$(A) \quad \frac{x^4}{4} + \frac{1}{5} \left(\frac{x}{y} \right)^5 = c$$

$$(B) \quad \frac{x^5}{5} + \frac{1}{4} \left(\frac{x}{y} \right)^4 = c$$

$$(C) \quad \left(\frac{x}{y} \right)^5 + \frac{x^4}{4} = c$$

$$(D) \quad (xy)^4 + \frac{x^5}{5} = c$$

Sol. : Here the differential equation is $y^5 x + y - x \frac{dy}{dx} = 0$. So, $y^5 x dx + y dx - x dy = 0$

$$\text{Multiply by } \frac{x^3}{y^5}, \quad x^4 dx + \frac{x^3}{y^3} \left(\frac{y dx - x dy}{y^2} \right) = 0$$

$$x^4 dx + u^3 du = 0 \quad (\text{Taking } u = \frac{x}{y})$$

$$\text{Integrating, } \frac{x^5}{5} + \frac{u^4}{4} = c$$

$$\therefore \frac{x^5}{5} + \frac{1}{4} \left(\frac{x}{y} \right)^4 = c$$

Ans. : (B)

(18) $\frac{dy}{dx} = \frac{y}{x} + 2 \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ has the solution

$$(A) \quad x^2 \phi\left(\frac{y}{x}\right) = k$$

$$(B) \quad y^2 \phi\left(\frac{y}{x}\right) = k$$

$$(C) \quad \phi\left(\frac{y}{x}\right) = kx^2$$

$$(D) \quad \phi\left(\frac{y}{x}\right) = ky^2$$

Sol. : Let $\frac{y}{x} = u$. So $\frac{dy}{dx} = u + x \frac{du}{dx}$

$\therefore$ The differential equation becomes $u + x \frac{du}{dx} = u + 2 \frac{\phi(u)}{\phi'(u)}$

$\therefore x \frac{du}{dx} = 2 \frac{\phi(u)}{\phi'(u)}$

$\therefore \frac{\phi'(u)}{\phi(u)} du = 2 \frac{dx}{x}$

Integrating, $\log |\phi(u)| = \log x^2 + \log k$

$\therefore |\phi(u)| = kx^2$

$\therefore \phi\left(\frac{y}{x}\right) = kx^2$, k is arbitrary constant

Ans. : (C)

- (19) The tangent at point (x, y) on the curve $y = f(x)$ makes an angle of measure $\tan^{-1}(2x + 3y)$ with positive direction of X-axis. If the curve is passing through the point $(1, 2)$, then its equation is

(A) $6x + 9y + 2 = 26 e^{3(x-1)}$

(B) $6x + 9y - 2 = 26 e^{3(x-1)}$

(C) $6x - 9y + 2 = 26 e^{3(x-1)}$

(D) $6x - 9y - 2 = 26 e^{3(x-1)}$

Sol. : $\frac{dy}{dx} = \tan \left[\tan^{-1}(2x + 3y) \right]$

$\frac{dy}{dx} = 2x + 3y$

$\therefore \frac{dy}{dx} - 3y = 2x$ is a linear differential equation.

Integrating factor $= e^{-\int 3dx} = e^{-3x}$

$\therefore ye^{-3x} = \int 2xe^{-3x} dx$
 $= -\frac{2}{3} xe^{-3x} + \frac{2}{3} \int e^{-3x} dx$

$\therefore ye^{-3x} = -\frac{2}{3} xe^{-3x} - \frac{2}{9} e^{-3x} + c$

$\therefore y = -\frac{2}{3} x - \frac{2}{9} + ce^{3x}$

Now, the curve is passing through point $(1, 2)$. So, $2 = -\frac{2}{3} - \frac{2}{9} + ce^3$

$\therefore c = \frac{26}{9} e^{-3}$

$\therefore y = -\frac{2}{3} x - \frac{2}{9} + \frac{26}{9} e^{3(x-1)}$

$\therefore 9y = -6x - 2 + 26 e^{3(x-1)}$

$\therefore 6x + 9y + 2 = 26 e^{3(x-1)}$

Ans. : (A)

(20) The solution of the differential equation $\frac{dy}{dx} = \frac{1}{xy[x^2 \sin y^2 + 1]}$ is

(A) $x^2 (\cos y^2 - \sin y^2 - 2c e^{-y^2}) = 2$ (B) $y^2 (\cos x^2 - (\sin y^2 - 2c e^{-y^2})) = 2$

(C) $x^2 (\cos y^2 - \sin y^2 - e^{-y^2}) = 4$ (D) None of these

Sol. : $\frac{dx}{dy} = xy[x^2 \sin y^2 + 1]$ is the differential equation.

$$\therefore \frac{1}{x^3} \frac{dx}{dy} - \frac{1}{x^2} y = y \sin y^2$$

Taking $-\frac{1}{x^2} = u$, $\frac{2dx}{x^3} = du$

$$\frac{du}{dy} + 2uy = 2y \sin y^2 \text{ is a linear differential equation.}$$

Integrating factor is e^{y^2} .

$$ue^{y^2} = \int 2y \sin y^2 e^{y^2} dy + c$$

$$= \int \sin t e^t dt + c = \frac{e^t}{2} (\sin t - \cos t) + c$$

$$= \frac{1}{2} e^{y^2} (\sin y^2 - \cos y^2) + c \quad (y^2 = t)$$

$$\therefore 2u = (\sin y^2 - \cos y^2) + 2c e^{-y^2}$$

$$\therefore 2 = x^2 (\cos y^2 - \sin y^2 - 2c e^{-y^2})$$

Ans. : (A)

(21) A curve passes through the origin. Lines are drawn parallel to axes from a point on the curve. The rectangle formed by these lines and axes form a partition of the area of region of curve. If the area of one part is twice the area of the other part, the family of curve is a family of

(A) circles (B) parabolas (C) hyperbolas (D) ellipses

Sol. : Suppose $P(x, y)$ is a point on the curve. As shown in the figure, $\overleftrightarrow{PN}$ and $\overleftrightarrow{PM}$ are parallel to X-axis and Y-axis.

$$\text{Area of the region POM} = \int_0^x y dx$$

$$\text{Area of the region PON} = xy - \int_0^x y dx$$

Given $2(\text{POM}) = \text{PON}$

$$\therefore 2 \int_0^x y dx = xy - \int_0^x y dx$$

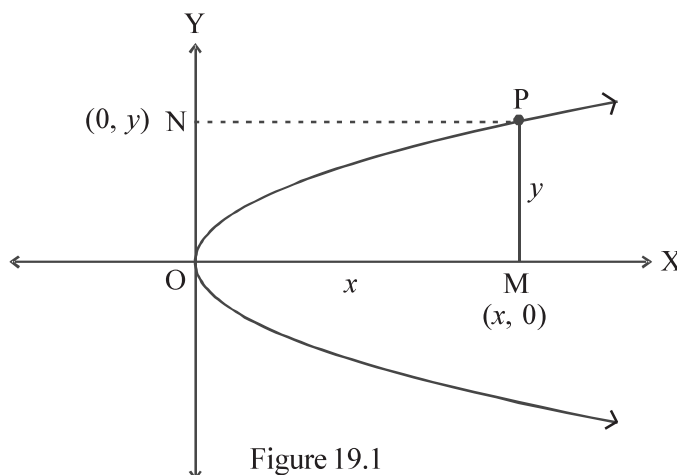


Figure 19.1

$$\therefore \int_0^x y \, dx = xy.$$

Differentiating, $3y = x \frac{dy}{dx} + y$

$$\therefore 2y = x \frac{dy}{dx}$$

$$\therefore \frac{dy}{y} = 2 \frac{dx}{x}$$

$$\therefore \log |y| = 2 \log |x| + \log c \quad \text{So, } y = x^2 c$$

Ans. : (B)

Note : If $PON = 2POM$?

(22) The differential equation of family of the curves $x^2 + y^2 = 2ax$ is (a is an arbitrary constant)

(A) $2y \frac{dy}{dx} = x^2$

(B) $2xy \frac{dy}{dx} + x^2 - y^2 = 0$

(C) $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$

(D) $x + y \frac{dy}{dx} = y$

Sol. : $x^2 + y^2 = 2ax$

$$\therefore 2x + 2y \frac{dy}{dx} = 2a$$

$$\therefore x + y \frac{dy}{dx} = a$$

$$\therefore x^2 + y^2 = 2x \left(x + y \frac{dy}{dx} \right)$$

$$\therefore 2xy \frac{dy}{dx} + x^2 - y^2 = 0$$

Ans. : (B)

(23) The differential equation of the family of non-vertical lines is

(A) $\frac{d^2 y}{dx^2} = 0$

(B) $\frac{dy}{dx} = x$

(C) $y = x \frac{dy}{dx}$

(D) $x = y \frac{dy}{dx}$

Sol. : The family of non-vertical lines is $y = mx + c$ m, c are arbitrary constants

$$\therefore \frac{dy}{dx} = m$$

$$\therefore \frac{d^2 y}{dx^2} = 0$$

(we have two arbitrary constants) **Ans. : (A)**

(24) The differential equation of the curve $y = e^{-x} + ax + b$ is (a, b are arbitrary constants)

(A) $e^x \frac{d^2 y}{dx^2} = 1$

(B) $\frac{d^2 y}{dx^2} + e^x \frac{dy}{dx} + 2y = 0$

(C) $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = e^x$

(D) $\frac{d^2 y}{dx^2} + \frac{2}{x} \left(\frac{dy}{dx} \right) = 0$

Sol. : $y = e^{-x} + ax + b$

$$\therefore \frac{dy}{dx} = -e^{-x} + a$$

$$\therefore \frac{d^2y}{dx^2} = e^{-x}$$

$$\therefore e^x \frac{d^2y}{dx^2} = 1 \text{ is the required differential equation.}$$

Ans. : (A)

(25) The differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$ is

(A) $\frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - y$

(B) $\frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 2y$

(C) $\frac{d^2y}{dx^2} = \frac{dy}{dx} - 2y$

(D) $\frac{d^2y}{dx^2} = 2 \frac{dy}{dx} + y$

Sol. : $y = e^x (A \cos x + B \sin x)$

$$\therefore y_1 = e^x (-A \sin x + B \cos x) + e^x (A \cos x + B \sin x)$$

$$\therefore y_1 = e^x (-A \sin x + B \cos x) + y$$

$$\therefore y_2 = e^x (-A \cos x - B \sin x) + e^x (-A \sin x + B \cos x) + y_1$$

$$\therefore y_2 = -y + y_1 - y + y_1$$

$$\therefore y_2 = 2y_1 - 2y$$

Ans. : (B)

Second method : $ye^{-x} = A \cos x + B \sin x$

$$\therefore -ye^{-x} + e^{-x} \frac{dy}{dx} = -A \sin x + B \cos x$$

$$\therefore ye^{-x} - e^{-x} \frac{dy}{dx} - e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} = -A \cos x - B \sin x = -ye^{-x}$$

$$\therefore y - y_1 - y_1 + y_2 = -y$$

$$\therefore y_2 - 2y_1 + 2y = 0$$

(26) The differential equation of the family of curves whose general solution is $(x-h)^2 + (y-k)^2 = a^2$ (a is a fixed constant) (h, k are arbitrary constants)

(A) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2 \frac{d^2y}{dx^2}$

(B) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

(C) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

(D) $(y-k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -1$ (IIT : 1992)

$$\text{Sol. : } (x-h)^2 + (y-k)^2 = a^2 \quad (\text{i})$$

$$\therefore (x-h) + (y-k) \frac{dy}{dx} = 0 \quad (\text{ii})$$

$$\therefore 1 + (y-k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad (\text{iii})$$

$$\text{From (i) and (ii), } (y-k)^2 \left(\left(\frac{dy}{dx}\right)^2 + 1 \right) = a^2 \quad (\text{iv})$$

$$\frac{\left(\left(\frac{dy}{dx}\right)^2 + 1 \right)^2}{\left(\frac{d^2y}{dx^2} \right)^2} \left(\left(\frac{dy}{dx}\right)^2 + 1 \right) = a^2 \quad (\text{from (iii) and (iv)})$$

$$\left[\left(\frac{dy}{dx}\right)^2 + 1 \right]^3 = a^2 \left(\frac{d^2y}{dx^2} \right)^2 \quad \text{Ans. : (B)}$$

(27) The integrating factor of the differential equation $\frac{dy}{dx} = 1 - e^x y$ is

- (A) e^{x^2} (B) e^{e^x} (C) e^{-x} (D) e^{-e^x}

$$\text{Sol. : } \frac{dy}{dx} + e^x y = 1$$

$$\text{I.F.} = e^{\int P dx} = e^{\int e^x dx} = e^{e^x} \quad \text{Ans. : (B)}$$

(28) The general solution of the differential equation $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$ is

$$x, y \in \left(0, \frac{\pi}{4}\right).$$

- (A) $\tan x \cot y = c$ (B) $\cot x \cot y = c$ (C) $\cot x \tan y = c$ (D) $\sec x \sec y = c$

$$\text{Sol. : } \sec^2 x \tan y dx = -\sec^2 y \tan x dy$$

$$\therefore \int \frac{1}{\tan x} \cdot \sec^2 x dx = -\int \frac{1}{\tan y} \cdot \sec^2 y dy$$

$$\therefore \log |\tan x| = -\log |\tan y| + \log |c_1|$$

$$\therefore \log |\tan x \tan y| = \log |c_1|$$

$$\therefore \tan x \tan y = \pm c_1$$

$$\therefore \pm \frac{1}{c_1} = \cot x \cot y$$

$$\therefore \cot x \cot y = c \quad (\text{Take } \pm \frac{1}{c_1} = c) \quad \text{Ans. : (B)}$$

(29) The length of the subnormal of the curve $x^2 = 4y$ at point $(-2, 1)$ is

- (A) 1 (B) 2 (C) 4 (D) $\frac{1}{4}$

Sol. : $x^2 = 4y$

$$2x = 4 \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x}{2}. \text{ So, } \left(\frac{dy}{dx} \right)_{(-2,1)} = -\frac{2}{2} = -1$$

$$\begin{aligned} \therefore \text{Length of subnormal} &= \left| y \cdot \left(\frac{dy}{dx} \right)_{(-2,1)} \right| \\ &= |1(-1)| = 1 \end{aligned}$$

Ans. : (A)

(30) The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$; where $y(1)=1$ is

- (A) $y = x \log x + x^2$ (B) $y = x e^{x-1}$
(C) $y = x \log x + x$ (D) $y = x + \log x$

Sol. : $\frac{dy}{dx} - \frac{y}{x} = 1$

$$\therefore \frac{xdy - ydx}{x^2} = \frac{dx}{x}$$

$$\therefore \frac{d}{dx} \left(\frac{y}{x} \right) = \frac{1}{x}$$

$$\frac{y}{x} = \log x + c$$

Let $x = 1, y = 1$

$$\therefore 1 = \log 1 + c. \quad \text{So, } c = 1$$

$$\therefore \frac{y}{x} = \log x + 1. \quad \text{So, } y = x \log x + x$$

Ans. : (C)

(31) The differential equation of the family of non-vertical lines passing through the origin is

- (A) $x \frac{dy}{dx} + y = 0$ (B) $x \frac{dy}{dx} - y = 0$
(C) $\frac{dy}{dx} = m$ (D) None of these

Sol. : Required family of lines is $y = mx$, where m is an arbitrary constant. (1)

$$\therefore \frac{dy}{dx} = m$$

From equation (1), $y = x \frac{dy}{dx}$. So, $x \frac{dy}{dx} - y = 0$

Ans. : (B)

- (32) If the tangent to the curves of a family of curves make an angle of measure $\frac{\pi}{4}$ with hyperbola $xy = c^2$, then the differential equation of the family of curves is

(A) $\frac{dy}{dx} = -\frac{c^2}{x^2}$ (B) $\frac{dy}{dx} = \frac{x^2 + c^2}{c^2 - x^2}$ (C) $\frac{dy}{dx} = \frac{c^2 - x^2}{x^2 + c^2}$ (D) None of these

Sol. : $xy = c^2$

$$\therefore y + x \frac{dy}{dx} = 0$$

$$m_1 = -\frac{y}{x}$$

Suppose m_2 is the slope of the tangent to the curves of the family of curves.

$$\text{Now, } \tan \frac{\pi}{4} = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad \text{So, } \pm 1 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Case (I)

$$1 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\therefore m_1 - m_2 = 1 + m_1 m_2$$

$$\therefore -\frac{y}{x} - \frac{dy}{dx} = 1 - \frac{y}{x} \frac{dy}{dx}$$

$$\frac{-y}{x} - 1 = \frac{dy}{dx} - \frac{y}{x} \frac{dy}{dx}$$

$$\frac{-(y+x)}{x} = \left(\frac{x-y}{x} \right) \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-(y+x)}{x-y} = \frac{x+y}{y-x} = \frac{x + \frac{c^2}{x}}{\frac{c^2}{x} - x} \quad \left(y = \frac{c^2}{x} \right)$$

$$\frac{dy}{dx} = \frac{x^2 + c^2}{c^2 - x^2}$$

Case (II)

$$\text{If } -1 = \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ then } \frac{dy}{dx} = -\left(\frac{c^2 - x^2}{x^2 + c^2} \right)$$

Ans. : (B)

- (33) The particular solution of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$ with initial condition $y(1) = 1$ is

(A) $xy^2 = 1$ (B) $x^2y = 1$ (C) $x = 2y^2$ (D) $x^2 = 2y$

Sol. : $\frac{dy}{dx} + \frac{2y}{x} = 0$

$$\therefore \frac{dy}{y} = -\frac{2}{x} dx$$

$$\therefore \log |y| = -2 \log |x| + \log(c)$$

$$\therefore yx^2 = c. \text{ Now, } x=1, y=1 \Rightarrow c=1. \text{ So, } x^2y=1$$

Ans. : (B)

(34) $\frac{dy}{dx} + y \tan x = \sec x$ has the solution

(A) $y = \sin x + c \cos x$

(B) $y = \sin x - c \cos x$

(C) $y = \tan x + \cos x + c$

(D) None of these

(IIT : 1990)

Sol. : $\frac{dy}{dx} + y \tan x = \sec x$

I.F. $e^{\int \tan x dx} = e^{\log |\sec x|} = \sec x$

$$\therefore y \sec x = \int \sec^2 x dx$$

(Linear equation)

$$\therefore y \sec x = \tan x + c$$

$$y = \sin x + c \cos x$$

Ans. : (A)

(35) If the differential equation $\frac{dy}{dx} = \frac{ax+b}{cy+e}$ represents a line, then the value of c and a are respectively. ($e \neq 0$)

(A) $a=0, c=0$

(B) $a=1, c=0$

(C) $a=-1, c=1$

(D) $a \in R$

Sol. : $(cy+e) dy = (ax+b) dx$

$$\frac{cy^2}{2} + ey = \frac{ax^2}{2} + bx + c_1$$

$$a=0, c=0 \Rightarrow ey = bx + c_1 \text{ represents a line.}$$

Note : Slope of a line $\frac{dy}{dx}$ is constant. So, $a=c=0$.

Ans. : (A)

(36) The integrating factor of differential equation $\frac{dy}{dx} (1+x) - xy = 1-x$ is

(A) $x e^x$

(B) $(1+x)e^{-x}$

(C) $1+x$

(D) $\log(1+x)$

Sol. : $\frac{dy}{dx} - \frac{x}{1+x} y = \frac{1-x}{1+x}$

I.F. $e^{\int P dx} = e^{\int -\frac{x}{1+x} dx} = e^{\int \left(\frac{1}{1+x} - 1\right) dx}$

$$= e^{\log(1+x) - x}$$

$$= e^{\log(1+x)} \cdot e^{-x}$$

$$= (1+x) e^{-x}$$

Ans. : (B)

(37) The solution of the differential equation $\frac{dy}{dx} + P(x)y = 0$ is

- (A) $y = ce^{-\int P dx}$ (B) $y = ce^{\int P dx}$ (C) $x = ce^{\int P dy}$ (D) $x = ce^{-\int P dy}$

Sol. : $\frac{dy}{dx} = -P(x)y$

$$\therefore \int \frac{1}{y} dy = -\int P(x) dx$$

$$\therefore \log y = -\int P(x) dx + \log c$$

$$\therefore \frac{y}{c} = e^{-\int P(x) dx}$$

$$\therefore y = ce^{-\int P(x) dx}$$

Ans. : (A)

(38) If the differential equation $(x^2 + 4y^2 + 4xy) dy = (2x + 4y + 1) dx$ has solution

$$v - 2\log \left| v^2 + 4v + 2 \right| + \frac{3}{\sqrt{2}} \log \left| \frac{v+2-\sqrt{2}}{v+2+\sqrt{2}} \right| = x + c, \text{ then}$$

- (A) $v = x + 2y$ (B) $v = 2x + y$ (C) $v = x + y$ (D) None of these

Sol. : $\frac{dy}{dx} = \frac{2x + 4y + 1}{x^2 + 4y^2 + 4xy}$

$$\therefore \frac{dy}{dx} = \frac{2(x + 2y) + 1}{(x + 2y)^2} \quad (1)$$

Let, $x + 2y = v$

$$\therefore 1 + 2 \frac{dy}{dx} = \frac{dv}{dx} \cdot \text{ So, } \frac{dy}{dx} = \frac{1}{2} \left[\frac{dv}{dx} - 1 \right]$$

$$\therefore \text{ From equation (1), } \frac{1}{2} \left[\frac{dv}{dx} - 1 \right] = \frac{2v + 1}{v^2}$$

$$\therefore \frac{dv}{dx} = \frac{4v + 2}{v^2} + 1 = \frac{v^2 + 4v + 2}{v^2}$$

$$\therefore \frac{v^2 dv}{v^2 + 4v + 2} = dx$$

$$\therefore \frac{v^2 + 4v + 2 - 4v - 2}{v^2 + 4v + 2} dv = dx$$

$$\therefore \frac{v^2 + 4v + 2 - 2(2v + 4) + 6}{v^2 + 4v + 2} dv = dx$$

$$\therefore \int 1 dv - 2 \int \frac{2v + 4}{v^2 + 4v + 2} dv + 6 \int \frac{1}{v^2 + 4v + 4 - 2} dv = \int dx$$

$$\therefore v - 2\log \left| v^2 + 4v + 2 \right| + \frac{3}{\sqrt{2}} \log \left| \frac{v+2-\sqrt{2}}{v+2+\sqrt{2}} \right| = x + c$$

Ans. : (A)

(39) The solution of the differential equation $\frac{dy}{dx} = e^{y+x} + e^{y-x}$ is

(A) $e^{-y} = e^{-x} - e^x + c$

(B) $e^x = e^y - e^{-x} + c$

(C) $e^{-y} = e^x - e^{-x} + c$

(D) $-e^{-x} = e^{-y} + e^x + c$

Sol. : $\frac{dy}{dx} = e^y \cdot e^x + e^y \cdot e^{-x}$

$$\therefore \frac{dy}{dx} = e^y (e^x + e^{-x})$$

$$\therefore e^{-y} dy = (e^x + e^{-x}) dx$$

$$\therefore \frac{e^{-y}}{(-1)} = e^x + \frac{e^{-x}}{(-1)} - c$$

$$\therefore -e^{-y} = e^x - e^{-x} - c$$

$$\therefore e^{-y} = e^{-x} - e^x + c$$

Ans. : (A)

(40) The solution of the differential equation $(x+y-1) dx + (2x+2y-3) dy = 0$ is

(A) $2y - x - \log(3x+3y-4) = c_4$

(B) $y + 2x + \log(x+y-2) = c$

(C) $2y + x + \log(x+y-2) = c$

(D) $2y + 2x + \log(x+y-2) = c$

Sol. : $(2x+2y-3) dy = -(x+y-1) dx$

$$\therefore \frac{dy}{dx} = -\frac{x+y-1}{2x+2y-3}$$

Let, $x+y-1=t$

$$\therefore 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{dt}{dx} - 1 = \frac{-t}{2t-1} \quad \text{So, } \frac{dt}{dx} = \frac{-t}{2t-1} + 1$$

$$= \frac{-t+2t-1}{2t-1}$$

$$\therefore \frac{dt}{dx} = \frac{t-1}{2t-1}$$

$$\therefore \frac{2t-1}{t-1} dt = dx$$

$$\therefore \frac{2t-2+1}{t-1} dt = dx$$

$$\therefore \left(2 + \frac{1}{t-1} \right) dt = dx$$

$$\therefore 2t + \log|t-1| = x + c_1$$

$$\therefore 2x + 2y - 2 + \log |x + y - 2| = x + c_1$$

$$\therefore x + 2y + \log |x + y - 2| = c, \text{ where } c = c_1 + 2$$

Ans. : (C)

(41) The solution of the differential equation $(2x - 10y^3) \frac{dy}{dx} + y = 0$ is

(A) $x + y = ce^{2x}$ (B) $y^2 = 2x^3 + c$ (C) $xy^2 = 2y^5 + c$ (D) $x(y^2 + xy) = 0$

Sol. : $\frac{dy}{dx} = -\frac{y}{2x - 10y^3}$

$$\therefore \frac{dx}{dy} = 10y^2 - \frac{2x}{y}$$

$$\therefore \frac{dx}{dy} + \frac{2}{y}x = 10y^2$$

$$\text{I.F. } e^{\int \frac{2}{y} dy} = e^{\frac{2}{y}} = e^{2 \log y} = y^2$$

$$\therefore y^2 \frac{dx}{dy} + 2xy = 10y^4$$

$$\therefore \frac{d}{dy}(y^2x) = 10y^4$$

$$\therefore y^2x = \frac{10y^5}{5} + c = 2y^5 + c$$

Ans. : (C)

(42) The general solution of the differential equation $y dx - x dy - 3x^2 y^2 e^{x^3} dx = 0$ is

(A) $x = ye^{x^3} + cy$ (B) $y = xe^{x^3} + cx$ (C) $ye^{x^3} = cx$ (D) $\frac{y}{x} e^{x^3} = 0$

Sol. : $y dx - x dy - 3x^2 y^2 e^{x^3} dx = 0$

$$\therefore \frac{y dx - x dy}{y^2} - 3x^2 e^{x^3} dx = 0$$

$$\therefore d\left(\frac{x}{y}\right) - d\left(e^{x^3}\right) = 0$$

$$\therefore \frac{x}{y} = e^{x^3} + c \text{ So, } x = ye^{x^3} + cy$$

Ans. : (A)

(43) The solution of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1, x \neq 0$ is

(A) $ye^{2\sqrt{x}} = 2\sqrt{x} + c$

(B) $y = 2\sqrt{x} e^{2\sqrt{x}} + c$

(C) $2\sqrt{x} e^{2\sqrt{x}} = y + c$

(D) $y = \sqrt{x} e^{2\sqrt{x}} + c$

Sol. : $\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$

$\therefore \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ So, $P(x) = \frac{1}{\sqrt{x}}$.

I.F. $e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$

$\therefore y e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}} dx + c = \int \frac{1}{\sqrt{x}} dx + c$

$\therefore y e^{2\sqrt{x}} = 2\sqrt{x} + c$

Ans. : (A)

(44) The solution of the differential equation $\cos y dx + (1 + 2e^{-x}) \sin y dy = 0$, where $y(0) = \frac{\pi}{4}$, is

(A) $e^x + 2 = 3\sqrt{2} \cos y$

(B) $e^x - 2 = 3\sqrt{2} \cos y$

(C) $e^x + 2 = 3\sqrt{2} \sec y$

(D) $e^x - 2 = 3\sqrt{2} \sec y$

Sol. : $\cos y dx + (1 + 2e^{-x}) \sin y dy = 0$

$\therefore \cos y dx = -(1 + 2e^{-x}) \sin y dy$

$\therefore \frac{1}{1 + 2e^{-x}} dx = -\tan y dy$

$\therefore \int \frac{e^x}{e^x + 2} dx = -\int \tan y dy$

$\therefore \log |e^x + 2| = \log |\cos y| + \log c$. So, $(e^x + 2) = c \cos y$

$y(0) = \frac{\pi}{4}$.

So, $e^0 + 2 = c \cos \frac{\pi}{4}$.

So, $3 = \frac{c}{\sqrt{2}}$.

So, $c = 3\sqrt{2}$

$\therefore e^x + 2 = 3\sqrt{2} \cos y$

Ans. : (A)

(45) The solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$ is

(A) $2x e^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$

(B) $x e^{\tan^{-1}y} = e^{\tan^{-1}y} + c$

(C) $x e^{2\tan^{-1}y} = e^{\tan^{-1}y} + c$

(D) $x - 2 = c e^{-\tan^{-1}y}$

Sol. : Here, $(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

$$\therefore (1+y^2) = -\left(x - e^{\tan^{-1}y}\right) \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{1+y^2}{x - e^{\tan^{-1}y}}$$

$$\therefore \frac{dx}{dy} = -\frac{x - e^{\tan^{-1}y}}{1+y^2}$$

$$\therefore \frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\text{I.F. } e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$$\therefore x e^{\tan^{-1}y} = \int e^{2\tan^{-1}y} \cdot \frac{1}{1+y^2} dy$$

$$\therefore x e^{\tan^{-1}y} = \frac{1}{2} e^{2\tan^{-1}y} + c_1$$

$$\therefore 2x e^{\tan^{-1}y} = e^{2\tan^{-1}y} + 2c_1$$

$$\therefore 2x e^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$$

Ans. : (A)

(46) The solution of the differential equation $x \frac{dy}{dx} = y(\log y - \log x + 1)$ is

(A) $x \log\left(\frac{y}{x}\right) = cy$ (B) $y \log\left(\frac{x}{y}\right) = cx$ (C) $\log\left(\frac{x}{y}\right) = cy$ (D) $\log\left(\frac{y}{x}\right) = cx$

Sol. : $x \frac{dy}{dx} = y(\log y - \log x + 1)$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\log \frac{y}{x} + 1 \right)$$

Suppose $\frac{y}{x} = v$. So, $y = xv$.

So, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = v(\log v + 1)$$

$$\therefore x \frac{dv}{dx} = v \log v$$

$$\therefore \int \frac{dv}{v \cdot \log v} = \int \frac{dx}{x}$$

$$\therefore \log |\log v| = \log |x| + \log |c|$$

$$\therefore \log v = xc$$

$$\therefore \log \frac{y}{x} = cx$$

Ans. : (D)

(47) The solution of the differential equation $y dx + (x + x^2 y) dy = 0$ is

- (A) $\frac{1}{xy} + \log y = c$ (B) $-\frac{1}{xy} + \log y = c$ (C) $-\frac{1}{xy} = c$ (D) $\log y = cx$

Sol. : $y dx + (x + x^2 y) dy = 0$

$$\therefore y dx + x dy + x^2 y dy = 0$$

$$\therefore \frac{y dx + x dy}{(xy)^2} = -\frac{dy}{y}$$

$$\therefore \frac{d(xy)}{(xy)^2} = \frac{-dy}{y}$$

Integrating on both the sides,

$$\therefore -\frac{1}{xy} = -\log y + c$$

$$\therefore -\frac{1}{xy} + \log y = c$$

Ans. : (B)

(48) The solution of the differential equation $\cos x dy = y(\sin x - y) dx$ is

- (A) $y \sec x = \tan x + c$ (B) $y \tan x = \sec x + c$
 (C) $\tan x = (\sec x + c)y$ (D) $\sec x = (\tan x + c)y$

Sol. : $\cos x dy = y(\sin x - y) dx$

$$\cos x \frac{dy}{dx} = y \sin x - y^2$$

$$\therefore \frac{dy}{dx} = y \tan x - y^2 \sec x$$

$$\frac{1}{y^2} \frac{dy}{dx} = \frac{1}{y} \tan x - \sec x$$

Let, $\frac{1}{y} = v$

$$\therefore \frac{-1}{y^2} \frac{dy}{dx} = \frac{dv}{dx} \cdot \quad \text{So, } -\frac{dv}{dx} = v \tan x - \sec x$$

$$\therefore \frac{dv}{dx} + v \tan x = \sec x$$

I.F. $e^{\int \tan x dx} = e^{\log |\sec x|} = \sec x$

$$\therefore v \sec x = \int \sec^2 x dx + c$$

$$\therefore v \sec x = \tan x + c$$

$$\therefore \frac{1}{y} \sec x = \tan x + c$$

$$\therefore \sec x = y(\tan x + c)$$

Ans. : (D)

(49) The solution of the differential equation $2ye^{\frac{x}{y}} dx - \left(y + 2xe^{\frac{x}{y}}\right) dy = 0$ is

- (A) $2ce^{\frac{x}{y}} = \log y$ (B) $2ce^{\frac{x}{y}} = \log x$ (C) $2ce^{\frac{y}{x}} = \log y$ (D) $2ce^{\frac{y}{x}} = \log x$

Sol. : $2ye^{\frac{x}{y}} dx - \left(y + 2xe^{\frac{x}{y}}\right) dy = 0$

$$\therefore \frac{dx}{dy} = \frac{y + 2xe^{\frac{x}{y}}}{2ye^{\frac{x}{y}}} = \frac{1 + 2\frac{x}{y}e^{\frac{x}{y}}}{2e^{\frac{x}{y}}}$$

Let, $\frac{x}{y} = v$. So, $x = vy$.

$$\frac{dx}{dy} = v + y \cdot \frac{dv}{dy}. \text{ Substitute into the equation, } y \frac{dv}{dy} = \frac{1 + 2ve^v}{2e^v} - v$$

$$\therefore y \frac{dv}{dy} = \frac{1 + 2ve^v - 2ve^v}{2e^v} = \frac{1}{2e^v}$$

$$\therefore 2e^v dv = \frac{1}{y} dy$$

$$\therefore 2e^v = \log |y| - \log c_1$$

$$\therefore 2ce^v = \log y \quad (\text{Let } \log c_1 + 2e^v = 2ce^v)$$

$$\therefore 2ce^{\frac{x}{y}} = \log y$$

Ans. : (A)

: Questions asked in JEE Main and Advanced :

(50) The equation of the curve passing through the point $\left(1, \frac{\pi}{6}\right)$ and having the slope of the tangent

$\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$ at any point (x, y) on the curve is

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$

(B) $\cos\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$

(D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Sol. : The slope of the curve is $\frac{dy}{dx} = \frac{y}{x} + \sec \frac{y}{x}$.

Let $\frac{y}{x} = v, y = vx$.

So, $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = v + \sec v$$

$$\therefore \int \cos v dv = \int \frac{1}{x} dx$$

$$\therefore \sin v = \log x + c$$

$$\therefore \sin \frac{y}{x} = \log x + c$$

The curve passes through the point $\left(1, \frac{\pi}{6}\right)$.

$$\sin \frac{\pi}{6} = \log 1 + c$$

$$c = \frac{1}{2}$$

$$\therefore \sin \frac{y}{x} = \log x + \frac{1}{2}$$

Ans. : (A)

(51) If $y(x)$ is the solution of the differential equation $(x \log x) \frac{dy}{dx} + y = 2x \log x$ ($x \geq 1$), then

$y(e) = \dots\dots$

- (A) e (B) 0 (C) 2 (D) $2e$

Sol. : $(x \log x) \frac{dy}{dx} + y = 2x \log x$

Case (1) $x = 1 \Rightarrow y = 0$.

Case (2) Let $x > 1$

$$\frac{dy}{dx} + \frac{y}{x \log x} = 2$$

$$\text{I.F.} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$$

$$\therefore y \log x = 2 \int \log x dx + c$$

$$\therefore y \log x = 2[x \log x - x] + c \quad (1)$$

Substitute $x = 1, y = 0$ in (1), $0 \log 1 = 2[1 \log 1 - 1] + c$

$$\therefore c = 2$$

$$\therefore y \log x = 2[x \log x - x] + 2$$

$$\therefore y \log e = 2[e \log e - e] + 2 = 2 \quad (\text{Substitute } x = e)$$

$$\therefore y(e) = 2 \quad \text{Ans. : (C)}$$

(52) The solution of the differential equation $2x \frac{dy}{dx} - y = 0$; $y(1) = 2$ represents

- (A) a line (B) a parabola (C) a circle (D) an ellipse

Sol. : $2x \frac{dy}{dx} - y = 0$

$$\therefore \frac{2}{y} dy = \frac{dx}{x}$$

Integrating on both the sides,

$$\therefore 2 \log |y| = \log |x| + \log(c)$$

$$\therefore y^2 = xc$$

$$y(1) = 2. \text{ So, } x = 1 \Rightarrow y = 2$$

$$4 = 1 \cdot c \quad \text{So, } c = 4$$

$$y^2 = 4x. \text{ This equation represents a parabola.}$$

Ans. : (B)

- (53) A curve passes through (1, 2). If the product of the slope of the tangent at any point (x, y) of the curve and y-coordinate of the point is equal to its x-coordinate, then the curve is
 (A) a circle (B) a parabola (C) a rectangular hyperbola (D) an ellipse

Sol. : Here, $\frac{dy}{dx} \cdot y = x$

$$\therefore y \, dy = x \, dx$$

$$\text{Integrate, } \int y \, dy = \int x \, dx$$

$$\therefore \frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2} \quad (1)$$

$$\therefore y^2 - x^2 = c$$

It is passing through (1, 2),

$$\therefore 4 - 1 = c \quad \text{So, } c = 3$$

$$\therefore y^2 - x^2 = 3$$

It represents a rectangular hyperbola.

Ans. : (C)

- (54) The rate of decay of a radioactive body at any time is proportional to its mass present at that time. If the mass of body is 100 gms after one hour and 80 gms after two hours, then the initial mass of radioactive body is
 (A) 100 (B) 120 (C) 125 (D) 105

Sol. : Suppose initial mass is m_0 .

If the mass of radioactive body at time t is m , $\frac{dm}{dt} \propto m$

$$\text{Let } \frac{dm}{dt} = -\lambda m. \quad \text{So, } \int \frac{1}{m} dm = -\lambda \int dt$$

$$\therefore \log m = -\lambda t + c \quad (1)$$

$$\therefore t = 0 \Rightarrow m = m_0. \quad \text{So, } \log m_0 = c$$

$$\text{From (1) , } \log m = -\lambda t + \log m_0$$

$$\therefore \log \frac{m}{m_0} = -\lambda t$$

$$\therefore m = m_0 e^{-\lambda t}$$

$$\text{If } t = 1, \text{ then } m = 100 \text{ grams. So, } 100 = m_0 e^{-\lambda}$$

$$\text{If } t = 2, \text{ then } m = 80 \text{ grams. So, } 80 = m_0 e^{-2\lambda}$$

$$\therefore \frac{100}{80} = \frac{m_0 e^{-\lambda}}{m_0 e^{-2\lambda}} \Rightarrow e^{\lambda} = \frac{5}{4}$$

$$100 = m_0 \frac{4}{5}$$

$$\therefore m_0 = 125 \text{ grams}$$

Ans. : (C)

Short method :

Add 25 % of 80 means 20,
to get 100.

Add 25 % of 100 means 25,
to get 125.

$$\text{So, } \frac{x}{100} = \frac{100}{80} \Rightarrow x = 125$$

- (55) In an experiment of culture of bacteria in a laboratory, the rate of increase of bacteria is proportional to the number of bacteria present at that time. If at the end of one hour the number of bacteria gets doubled, then the number of bacteria at the end of 3 hours is
 (A) 4 (B) 8 (C) 16 (D) 32

Sol. : Suppose the number of bacteria at time t is x .

$$\therefore \text{Rate of change } \frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = \lambda x. \text{ So, } \frac{dx}{x} = \lambda dt \quad (\lambda \neq 0 \text{ is constant})$$

$$\text{Integrate, } \log|x| = \lambda t + \log c$$

$$\therefore \log \frac{x}{c} = \lambda t. \text{ So, } \frac{x}{c} = e^{\lambda t}. \text{ So, } x = c e^{\lambda t}$$

Suppose at $t = 0$, the number of bacteria is x_0 .

$$\therefore x_0 = c e^0 \text{ So, } c = x_0$$

$$\therefore x = x_0 e^{\lambda t} \quad (1)$$

After 1 hour, number of bacteria are doubled. So, $x = 2x_0$

$$\therefore \text{From the result (1), } 2x_0 = x_0 e^{\lambda}$$

$$\therefore 2 = e^{\lambda}. \quad \text{So, } \lambda = \log_e 2$$

$$\therefore x = x_0 e^{t \log_e 2}$$

$$\text{At the end of } t = 3 \text{ hours } x = x_0 e^{3 \log 2}$$

$$\therefore x = 8x_0$$

$\therefore$ Number of bacteria will be 8 times.

Ans. : (B)

Note : After 1 hour it is doubled, after 2 hours it is 4 times, after 3 hours it is 8 times.

- (56) The population of a city increases at the rate of 1 % per year. How many years will it take to double the population ?

- (A) 30 years (B) 301 years (C) 35 years (D) 30.1 years

Sol. : Let p_0 be the population of the city at present and suppose after t years it will be p ,

$$\text{Rate of change} = \frac{dp}{dt}$$

$$\therefore \frac{dp}{dt} = \frac{1}{100} p$$

$$\therefore \int \frac{1}{p} dp = \frac{1}{100} \int 1 dt$$

$$\therefore \log p = \frac{1}{100} t + \log c$$

$$\text{If } t = 0, \text{ then } p = p_0$$

$$\therefore \log p_0 = \log c$$

$$\therefore c = p_0$$

$$\therefore \log p = \frac{1}{100} t + \log p_0$$

$$\therefore \log \frac{p}{p_0} = \frac{1}{100} t$$

$$\therefore p = p_0 e^{\frac{t}{100}}$$

To get population doubled, $p = 2 p_0$

$$\therefore 2 p_0 = p_0 e^{\frac{t}{100}} . \quad \text{So, } 2 = e^{\frac{t}{100}}$$

$$\therefore \log 2 = \frac{t}{100} . \quad \text{So, } t = 100 \times 0.3010 = 30.1 \text{ years.} \quad \text{Ans. : (D)}$$

(57) The degree of homogeneous function $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$ is

- (A) 1 (B) 2 (C) 3 (D) not defined

Sol. : $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$

$$= \frac{x^3 \left(1 - \left(\frac{y}{x} \right)^3 \right)}{x^2 \left(1 + \left(\frac{y}{x} \right)^2 \right)}$$

$$= x \phi \left(\frac{y}{x} \right)$$

$\therefore$ Degree is 1.

Ans. : (A)

(58) The differential equation of the family of rectangular hyperbolas $x^2 - y^2 = a^2$ is

- (A) $y_2 = 0$ (B) $xy + y_2 = 0$ (C) $xy + y_1 = 0$ (D) $y y_1 = x$

Sol. : $x^2 - y^2 = a^2$

$$\therefore 2x - 2y \frac{dy}{dx} = 0$$

$$\therefore x - y y_1 = 0$$

$$\therefore x = y y_1$$

Ans. : (D)

(59) The order and the degree of the differential equation of the family of parabolas whose major axis is X-axis are and respectively .

- (A) 1, 2 (B) 3, 2 (C) 2, 3 (D) 2, 1

(A.I.EEE :2003)

Sol. : Family of parabola is $y^2 = 4a(x - b)$; where a, b are an arbitrary constants.

$$2y \frac{dy}{dx} = 4a$$

∴ Here, there are two arbitrary constants. So, differentiate two times.

$$\therefore 2 \left(\frac{dy}{dx} \right)^2 + 2y \frac{d^2y}{dx^2} = 0$$

∴ Order is 2 and degree is 1.

Ans. : (D)

- (60) The order of the differential equation whose general solution is $y = (c_1 + c_2) \sin(x + c_3) - c_4 e^{x+c_5}$ is

(IIT : 1998)

- (A) 5 (B) 4 (C) 2 (D) 3

Sol. : $y = (c_1 + c_2) \sin(x + c_3) - c_4 e^{c_5} e^x$

$$y = k_1 \sin(x + c_3) - k_2 e^x$$

∴ Here, we have 3 arbitrary constants. So the order of the differential equation is 3. **Ans. : (D)**

- (61) At present, a firm is manufacturing 2000 items. It is estimated that rate of change of production p w.r.t. additional number of workers x is given by $\frac{dp}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is

- (A) 2500 (B) 3000 (C) 3500 (D) 4500

(JEE MAIN : 2013)

Sol. : $\frac{dp}{dx} = 100 - 12\sqrt{x}$

$$\therefore dp = (100 - 12\sqrt{x}) dx$$

$$\therefore p = 100x - 12 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore p = 100x - 8x^{\frac{3}{2}} + c$$

If $x = 0$, then $p = 2000$.

So, $c = 2000$

$$\therefore p = 100x - 8x^{\frac{3}{2}} + 2000$$

$$x = 25 \Rightarrow p = 2500 - 8 \cdot 25^{\frac{3}{2}} + 2000 = 4500 - 1000 = 3500$$

Ans. : (C)

- (62) Let the population of rats surviving at time t be governed by the differential equation $\frac{dp(t)}{dt} = 0.5 p(t) - 450$. If $p(0) = 850$, then at what time will they extinct ?

- (A) $2 \log 18$ (B) $\log 9$ (C) $\frac{1}{2} \log 18$ (D) $\log 18$

Sol. : $\frac{dp(t)}{dt} = \frac{1}{2} p(t) - 450$

$$\frac{dp(t)}{dt} = \frac{p(t) - 900}{2}$$

$$\therefore 2 \int \frac{dp(t)}{p(t) - 900} = \int dt$$

$$\therefore 2 \log |p(t) - 900| = t + c$$

If $t = 0$, then $p(t) = 850$. So, $2 \log 50 = c$

$$\therefore 2 \log |p(t) - 900| = t + 2 \log 50$$

Now, let $p(t) = 900$

$$\therefore 2 \log 900 = t + 2 \log 50$$

$$\therefore t = 2 (\log 900 - \log 50)$$

$$\therefore t = 2 \log \left(\frac{900}{50} \right) = 2 \log 18$$

Ans. : (A)

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

- (A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.
- (B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.
- (C) Statement 1 is true, statement 2 is false.
- (D) Statement 1 is false, statement 2 is true.

(63) **Statement 1 :** $y(1 + \log x + cx) = 1$ is the solution of the differential equation $xy' + y = y^2 \log x$.

Statement 2 : Given differential equation is linear and its integrating factor is x .

Sol. : $xy' + y = y^2 \log x \Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \frac{1}{y} = \frac{\log x}{x}$

This is not linear differential equation but we can convert it in this form.

Let, $\frac{1}{y} = z$. So, $\frac{-1}{y^2} \frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore \frac{-dz}{dx} + \frac{1}{x} z = \frac{\log x}{x} \Rightarrow \frac{dz}{dx} - \frac{1}{x} z = -\frac{\log x}{x}$$

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

$$\therefore z \frac{1}{x} = - \int \frac{\log x}{x^2} dx + c$$

$$\therefore \frac{1}{y} \frac{1}{x} = - \left[\frac{-1}{x} \log x + 1 \int \frac{1}{x^2} dx \right] + c$$

$$= - \left[\frac{-1}{x} \log x - \frac{1}{x} \right] + c$$

$$\therefore \frac{1}{y} = [1 + \log x] + cx$$

$$\therefore 1 = y(1 + \log x + cx)$$

Statement 1 is true and statement 2 is false.

Ans. : (C)

