

● અનિયત સંકલન (પૂર્વગ અથવા પ્રતિવિકલિત)

જો અંતરાલ $I \subset \mathbb{R}$ પર વ્યાખ્યાયિત કોઈ વિકલનીય વિધેય g એવું મળે, જેથી $\frac{d}{dx}(g(x)) = f(x)$, $\forall x \in I$, તો $g(x)$ ને $f(x)$ નો x વિશે પૂર્વગ (Primitive) અથવા પ્રતિવિકલિત (Antiderivative) અથવા અનિયત સંકલિત (Indefinite integral) કહે છે. તેને સંકેતમાં $\int f(x) dx$ દર્શાવાય છે. અહીં $f(x)$ ને સંકલનની ક્રિયાનો સંકલ્ય (Integrand) કહે છે.

● જો f તથા g બંને (a, b) પર વિકલનીય વિધેયો હોય તથા $f'(x) = g'(x)$, $\forall x \in (a, b)$, તો $f(x) = g(x) + c$ જ્યાં c અચળ છે.

● જો, $\frac{d}{dx}(F(x)) = f(x)$, તો $\int f(x) dx = F(x) + c$, જ્યાં c અચળ છે.

● f, g અંતરાલ (a, b) પર પ્રતિવિકલનીય વિધેયો હોય, તો

(i) $f + g$ પણ પ્રતિવિકલનીય છે અને $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$.

(ii) $f - g$ પણ પ્રતિવિકલનીય છે અને $\int (f(x) - g(x)) dx = \int f(x) dx - \int g(x) dx$.

(iii) $k \cdot f$ પણ પ્રતિવિકલનીય છે અને $\int k \cdot f(x) dx = k \int f(x) dx$, જ્યાં $k \in \mathbb{R}$.

● પ્રમાણિત સંકલિતો

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \in \mathbb{R} - \{-1\}$, $x \in \mathbb{R}^+$

(2) $\int \frac{1}{x} dx = \log |x| + c$, $x \in \mathbb{R} - \{0\}$

(3) $\int \cos x dx = \sin x + c$, $\forall x \in \mathbb{R}$

(4) $\int \sin x dx = -\cos x + c$, $\forall x \in \mathbb{R}$

(5) $\int \sec^2 x dx = \tan x + c$, $x \neq (2k-1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

(6) $\int \operatorname{cosec}^2 x dx = -\cot x + c$, $x \neq k\pi$, $k \in \mathbb{Z}$

(7) $\int \sec x \tan x dx = \sec x + c$, $x \neq (2k-1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

(8) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$, $x \neq k\pi$, $k \in \mathbb{Z}$

(9) $\int a^x dx = \frac{a^x}{\log_e a} + c$, $a \in \mathbb{R}^+ - \{1\}$, $x \in \mathbb{R}$

(10) $\int e^x dx = e^x + c$, $\forall x \in \mathbb{R}$

(11) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c = -\frac{1}{a} \cot^{-1} \frac{x}{a} + c$, $a \in \mathbb{R} - \{0\}$, $x \in \mathbb{R}$

$$\int \frac{1}{a + bx^2} dx = \frac{1}{\sqrt{ab}} \tan^{-1} \left(\sqrt{\frac{b}{a}} x \right) + c, (a, b > 0)$$

(12) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$, $a \in \mathbb{R} - \{0\}$

$$(13) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{x+a}{x-a} \right| + c, a \in \mathbb{R} - \{0\}$$

$$(14) \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c', x \in (-a, a), a > 0$$

$$(15) \int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + c, \forall x \in \mathbb{R}$$

$$(16) \int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + c, |x| > |a| > 0$$

$$(17) \int \frac{dx}{|x| \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c', |x| > |a| > 0$$

$$(18) \text{ to } (21) \text{ માટે, } I = (k\pi, (2k+1)\frac{\pi}{2}) \text{ અથવા } ((2k-1)\frac{\pi}{2}, k\pi), k \in \mathbb{Z}$$

$$(18) \int \tan x \, dx = \log |\sec x| + c = -\log |\cos x| + c$$

$$(19) \int \cot x \, dx = \log |\sin x| + c = -\log |\operatorname{cosec} x| + c$$

$$(20) \int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c = \log \left| \tan \frac{x}{2} \right| + c$$

$$(21) \int \sec x \, dx = \log |\sec x + \tan x| + c = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$$

$$(22) \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c, (x^2 > a^2)$$

$$(23) \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + c$$

$$(24) \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c, (a > 0)$$

$$(25) \int e^{ax} \sin (bx + k) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + k) - b \cos (bx + k)] + c, (a, b > 0)$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin (bx + k - \alpha) + c,$$

$$\text{જ્યાં, } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \alpha \in (0, 2\pi)$$

$$(26) \int e^{ax} \cos (bx + k) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos (bx + k) + b \sin (bx + k)] + c, (a, b > 0)$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos (bx + k - \alpha) + c,$$

$$\text{જ્યાં, } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \alpha \in (0, 2\pi)$$

(1) સંકલન માટે આદેશની રીત (Method of Substitution for Integration)

$g : [\alpha, \beta] \rightarrow \mathbb{R}$ સતત તથા (α, β) પર વિકલનીય છે. $g'(t)$ એ (α, β) પર સતત છે તથા $g'(t) \neq 0$, $\forall t \in (\alpha, \beta)$. વળી, g નો વિસ્તાર $[a, b]$ નો ઉપગણ હોય તથા $f : [a, b] \rightarrow \mathbb{R}$ સતત હોય, તો $x = g(t)$

$$\text{લેતાં, } \int f(x) dx = \int f(g(t)) \cdot g'(t) dt$$

- (2) $\int f(x) dx = F(x)$ હોય, તો $\int f(ax + b) dx = \frac{1}{a} F(ax + b)$, જ્યાં $f: I \rightarrow \mathbb{R}$ એ અંતરાલ I પર સતત વિધેય છે. ($a \neq 0$)

- (3) $\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$, ($n \neq -1, f(x) > 0$). અહીં f, f' સતત વિધેયો છે, $f'(x) \neq 0$.

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

- (4) જો f એ $[a, b]$ માં સતત તથા (a, b) માં વિકલનીય હોય તથા f' પણ સતત હોય અને $f'(x) \neq 0$, $\forall x \in (a, b)$ તથા $f(x) \neq 0, \forall x \in [a, b]$ તો $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$.

- (5) **ખંડશ: સંકલનનો નિયમ (Rule of Integration by Parts)** : જો f અને g એ કોઈ અંતરાલ (a, b) પર વિકલનીય હોય, f' અને g' એ (a, b) પર સતત હોય, તો

$$\int f(x) \cdot g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx.$$

જો $f(x) = u$ અને $g'(x) = v$ લઈએ, તો

$$\int u \cdot v dx = u \int v dx - \int \left(\frac{d}{dx}(u) \int v dx \right) dx.$$

- (6) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

- (7) **ત્રિકોણમિતિય આદેશો (Trigonometric Substitutions)** :

સંકલ્ય (Integrand)માં હોય	આદેશ (Substitution)
(i) $\sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan \theta$ અથવા $x = a \cot \theta$
(ii) $\sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec \theta$ અથવા $x = a \operatorname{cosec} \theta$
(iii) $\sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ અથવા $x = a \cos \theta$
(iv) $\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(v) $\sqrt{2ax - x^2}$	$x = 2a \sin^2 \theta$
$\sqrt{2ax - x^2} = \sqrt{a^2 - (x-a)^2}$	$x - a = a \sin \theta$ અથવા $x - a = a \cos \theta$
(vi) $\sqrt{2ax + x^2}$	$x = 2a \tan^2 \theta$
(vii) $\sqrt{a-x}, \sqrt{\frac{x}{a-x}}, \sqrt{x(a-x)}$	$x = a \sin^2 \theta, x = a \cos 2\theta$
(viii) $\sqrt{a+x}, \sqrt{\frac{x}{a+x}}, \sqrt{x(a+x)}$	$x = a \tan^2 \theta, x = a \cos 2\theta$
(ix) $\sqrt{\frac{x-\alpha}{\beta-x}}, \sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

(8) સંકલ્ય $\frac{1}{a+b\sin x}$, $\frac{1}{a+b\cos x}$ અથવા $\frac{1}{a+b\sin x+c\cos x}$ સ્વરૂપનો હોય, તો $\tan \frac{x}{2} = t$ આદેશ લઈશું.

$$\tan \frac{x}{2} = t \text{ લેતાં, } dx = \frac{2dt}{1+t^2}, \sin x = \frac{2t}{1+t^2} \text{ અને } \cos x = \frac{1-t^2}{1+t^2}$$

આમ, આપેલ સંકલ્ય t ના સંકલ્યમાં રૂપાંતરિત થશે.

(9) $\int \sin^m x \cos^n x dx$ સ્વરૂપનાં સંકલિતો, $m, n \in \mathbb{N}$

(i) m અને n બંને અયુગ્મ પૂર્ણાંક હોય, જો $m \geq n$ તો $\sin x = t$ અન્યથા $\cos x = t$ આદેશ લઈશું.

(ii) m અયુગ્મ અને n યુગ્મ હોય, તો $\cos x = t$ આદેશ લઈશું.

(iii) m યુગ્મ અને n અયુગ્મ હોય, તો $\sin x = t$ આદેશ લઈશું.

(iv) જો m અને n બંને યુગ્મ હોય, તો આપણે ગુણિત ખૂણાના ત્રિ-વિધેય સ્વરૂપમાં લાવવા માટે

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \text{ અને } \cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ નો ઉપયોગ કરીશું.}$$

(10) (a) $\int \frac{dx}{ax^2+bx+c}$ અથવા (b) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

અહીં જો આપણે ax^2+bx+c ને પૂર્ણ વર્ગના સ્વરૂપમાં વ્યક્ત કરીએ તો સંકલિત (a)નું સંકલન આપણે પ્રમાણિત સ્વરૂપ (11), (12) કે (13)નો ઉપયોગ કરી અને સંકલિત (b)નું સંકલન આપણે પ્રમાણિત સ્વરૂપ (14), (15) કે (16)નો ઉપયોગ કરી મેળવી શકીએ.

(11) (a) $\int \frac{Ax+B}{ax^2+bx+c} dx$ અથવા (b) $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx$

અહીં, આપણે પ્રથમ એવા બે અચળો m અને n શોધીશું કે જેથી,

$$Ax+B = m[(ax^2+bx+c) \text{ નો વિકલિત}] + n$$

$$= m(2ax+b) + n$$

$$= (2ma)x + (mb+n)$$

બંને બાજુએ x ના સહગુણકો અને અચળ પદ સરખાવતાં, $m = \frac{A}{2a}$ અને $n = B - mb$ મળે.

$$\text{હવે, (a) } \int \frac{Ax+B}{ax^2+bx+c} dx = \int \frac{m(2ax+b)+n}{ax^2+bx+c} dx$$

$$= m \int \frac{2ax+b}{ax^2+bx+c} dx + n \int \frac{1}{ax^2+bx+c} dx$$

$$= m \log |ax^2+bx+c| + n I_2$$

બાકી રહેલું સંકલિત I_2 આગળ દર્શાવેલી રીત (10)(a) મુજબ મેળવી શકાય.

$$(b) \int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx = \int \frac{m(2ax+b)+n}{\sqrt{ax^2+bx+c}} dx$$

$$= m \int (2ax+b)(ax^2+bx+c)^{-\frac{1}{2}} dx + n \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= 2m (ax^2+bx+c)^{\frac{1}{2}} + n I_2$$

બાકી રહેલું સંકલિત I_2 આગળ દર્શાવેલ રીત (10)(b) મુજબ મેળવી શકાય.

$$(12) \int \frac{dx}{a + b \cos^2 x}, \int \frac{dx}{a + b \sin^2 x} \text{ અથવા } \int \frac{dx}{a \cos^2 x + b \sin^2 x + c}$$

આ સંકલ્યમાં અંશ અને છેદને $\cos^2 x \neq 0$ વડે ભાગવું, ત્યાર બાદ છેદમાં જ્યાં $\sec^2 x$ હોય ત્યાં $1 + \tan^2 x$ લઈ છેદને $\tan^2 x$ સ્વરૂપમાં ફેરવો. હવે $\tan x = t$ આદેશ લેતાં, $\sec^2 x dx = dt$.

આમ, આપેલ સંકલ્ય t ના સ્વરૂપમાં રૂપાંતરિત થશે.

$$(13) \int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r} dx \text{ અથવા } \int \frac{ae^x + be^{-x} + c}{pe^x + qe^{-x} + r} dx$$

અહીં આપણે પ્રથમ એવા ત્રણ અચળો l, m અને n શોધીશું. કે જેથી,

$$\text{અંશ} = l (\text{છેદ}) + m (\text{છેદનું વિકલન}) + n.$$

અચળાંકો l, m તથા n નાં મૂલ્યો બંને બાજુ $\cos x, \sin x$ તથા અચળ (e^x, e^{-x}) સરખાવવાથી મેળવી શકાય. આમ કરવાથી પદાવલી $l \int dx + m \int \frac{\frac{d}{dx}(\text{છેદ})}{\text{છેદ}} + n \int \frac{dx}{\text{છેદ}}$ સ્વરૂપમાં થશે. જેનું સંકલિત મેળવી શકાય.

$$(14) \int \frac{a \tan x + b}{p \tan x + q} dx \text{ અથવા } \int \frac{ae^{2x} + b}{pe^{2x} + q} dx$$

અહીં આપણે પ્રથમ આપેલ સંકલ્યને $\int \frac{a \sin x + b \cos x}{p \sin x + q \cos x} dx$ અને $\int \frac{ae^x + be^{-x}}{pe^x + qe^{-x}} dx$ સ્વરૂપમાં ફેરવીશું. ત્યાર

બાદ,

$$\text{અંશ} = l (\text{છેદ}) + m (\text{છેદનું વિકલન})$$

l અને m ની કિંમત મેળવી સંકલિત મેળવવા (13)નો ઉપયોગ કરો.

$$(15) \int \frac{dx}{a \sin x \pm b \cos x}$$

$a = r \cos \alpha$ અને $b = r \sin \alpha$ નો ઉપયોગ કરી આપેલ સંકલિતને $\frac{1}{r} \int \frac{1}{\sin(x \pm \alpha)} dx$ સ્વરૂપમાં ફેરવો અને

$\int \operatorname{cosec}(x \pm \alpha) dx$ નો ઉપયોગ કરી સંકલિત મેળવી શકાય.

$$(16) (a) \int \sqrt{ax^2 + bx + c} dx \text{ અને } (B) \int (Ax + B) \sqrt{ax^2 + bx + c} dx$$

(a) પ્રથમ $ax^2 + bx + c$ ને પૂર્ણ વર્ગના સ્વરૂપમાં વ્યક્ત કરો, ત્યાર બાદ પ્રમાણિત રૂપ (22), (23) કે (24)નો ઉપયોગ કરી સંકલિત મેળવી શકાય.

(b) આપણે એવા બે અચળ m અને n શોધીશું કે જેથી,

$$(Ax + B) = m \left(\frac{d}{dx}(ax^2 + bx + c) \right) + n \text{ થાય.}$$

બંને બાજુ x ના સહગુણકો અને અચળ પદ સરખાવતાં $m = \frac{A}{2a}$ અને $n = B - mb$ મળે.

$$\begin{aligned} \int (Ax + B) \sqrt{ax^2 + bx + c} dx &= \int [m(2ax + b) + n] (ax^2 + bx + c)^{\frac{1}{2}} dx \\ &= \int m(2ax + b) (ax^2 + bx + c)^{\frac{1}{2}} dx + \int n \sqrt{ax^2 + bx + c} dx \end{aligned}$$

$$= \frac{m(ax^2 + bx + c)^{\frac{3}{2}}}{\frac{3}{2}} + n I_2$$

I_2 , (a)નો ઉપયોગ કરી મેળવી શકાય.

$$(17) \text{ (i) } \int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx, \int \frac{x^2 - 1}{x^4 + kx^2 + 1} dx, k \in \mathbb{Z}$$

આ સ્વરૂપના સંકલ્યમાં અંશ અને છેદને $x^2 \neq 0$ વડે ભાગતાં $\int \frac{\left(1 + \frac{1}{x^2}\right) dx}{x^2 + k + \frac{1}{x^2}}, \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{x^2 + k + \frac{1}{x^2}}$ મળશે.

અનુક્રમે હવે $\left(x - \frac{1}{x}\right) = t$ અથવા $\left(x + \frac{1}{x}\right) = t$ આદેશ લેતાં, આપેલ સંકલ્ય t નાં સ્વરૂપમાં રૂપાંતરિત થશે, જેનો સંકલિત પ્રમાણિત સંકલિતોનો ઉપયોગ કરી મેળવીશું :

$$(ii) \int \frac{x^2 + a^2}{x^4 + kx^2 + a^4} dx, \int \frac{x^2 - a^2}{x^4 + kx^2 + a^4} dx.$$

આ સ્વરૂપના સંકલ્યમાં અંશ અને છેદને $x^2 \neq 0$ વડે ભાગતા અને ત્યાર બાદ અનુક્રમે $x - \frac{a^2}{x} = t$ અથવા $x + \frac{a^2}{x} = t$ આદેશ લેતાં આપેલ સંકલ્ય t ના સ્વરૂપમાં રૂપાંતરિત થશે, જેનો સંકલિત મેળવવા પ્રમાણિત સંકલિતોનો ઉપયોગ કરીશું.

$$(18) \int \frac{dx}{P(x)\sqrt{Q(x)}} \text{ (જ્યાં, } P(x) \text{ અને } Q(x), \text{ સુરેખ કે દ્વિઘાત પદાવલીઓ છે.)}$$

નીચેનું કોષ્ટક જુદાં જુદાં સંકલ્યમાં કયો આદેશ લેવો તે દર્શાવે છે :

સંકલ્ય	આદેશ
$\int \frac{dx}{(ax + b)\sqrt{cx + d}}$	$cx + d = t^2$
$\int \frac{dx}{(ax^2 + bx + c)\sqrt{px + q}}$	$px + q = t^2$
$\int \frac{dx}{(px + q)\sqrt{ax^2 + bx + c}}$	$px + q = \frac{1}{t}$
$\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + d}}$	$x = \frac{1}{t}$ અને $c + dt^2 = u^2$

(19) આંશિક અપૂર્ણાંકની રીત (Method of Partial Fractions) :

(i) સંમેય વિધેય $\frac{p(x)}{q(x)}$ માં જો $p(x)$ ની ઘાત $<$ $q(x)$ ની ઘાત, તો $\frac{p(x)}{q(x)}$ ને ઉચિત સંમેય વિધેય (Proper rational function) કહે છે.

(ii) જો $p(x)$ ની ઘાત $\geq q(x)$ ની ઘાત હોય, તો $\frac{p(x)}{q(x)}$ ને અનુચિત સંમેય વિધેય (**Improper rational function**) કહેવાય છે. અનુચિત સંમેય વિધેયમાં $p(x)$ ને $q(x)$ વડે ભાગીશું કે જેથી શેષની ઘાત છેદની ઘાત કરતાં ઓછી થાય. આમ આપેલ અનુચિત સંમેય વિધેય તરીકે આપેલ સંકલ્પ $r(x) + \frac{p_1(x)}{q(x)}$ સ્વરૂપમાં મેળવી શકાય. અહીં $p_1(x)$ ની ઘાત $q(x)$ ની ઘાત કરતાં ઓછી હશે, તેથી $\frac{p_1(x)}{q(x)}$ ઉચિત સંમેય વિધેય થશે. આમ, આપેલ અનુચિત સંમેય વિધેયને વાસ્તવિક બહુપદી અને ઉચિત સંમેય વિધેયના સરવાળા તરીકે અનન્ય રીતે દર્શાવી શકાય. હવે આપણે ઉચિત સંમેય વિધેયનું સંકલન કેવી રીતે મેળવવું તે માટેની આંશિક અપૂર્ણાંકની રીતે શીખીશું.

ધારો કે $\frac{p(x)}{q(x)}$ ઉચિત સંમેય વિધેય છે. બહુપદી $q(x)$ ના અવયવોને આધારે નીચેના વિકલ્પો લઈશું :

વિકલ્પ 1 : વાસ્તવિક, સુરેખ અને અનાવૃત્ત અવયવો

ધારો કે $q(x)$ ને n વાસ્તવિક, સુરેખ અને અનાવૃત્ત અવયવો $x - \beta_1, x - \beta_2, \dots, x - \beta_n$ છે એટલે કે $q(x) = (x - \beta_1)(x - \beta_2) \dots (x - \beta_n)$ છે. આ સંજોગોમાં $\frac{p(x)}{q(x)}$ ને નીચેના સ્વરૂપમાં લખી શકાય :

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - \beta_1} + \frac{A_2}{x - \beta_2} + \dots + \frac{A_n}{x - \beta_n}. \text{ જ્યાં } A_1, A_2, \dots, A_n \text{ અજ્ઞાત અચળો છે. તે હંમેશાં મેળવી}$$

શકાય અને તે મેળવ્યા બાદ જમણી બાજુનાં પદોનું સરળતાથી સંકલન કરી શકાય.

વિકલ્પ 2 : વાસ્તવિક, સુરેખ આવૃત્ત અને અનાવૃત્ત અવયવો

એટલે કે $q(x) = (x - \beta_1)^2(x - \beta_2) \dots (x - \beta_n)$ હોય, તો

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - \beta_1} + \frac{A_2}{(x - \beta_1)^2} + \frac{A_3}{x - \beta_2} + \dots + \frac{A_n}{x - \beta_n} \text{ લો. જ્યાં } A_1, A_2, A_3, \dots, A_n \text{ અજ્ઞાત અચળો છે.}$$

તે હંમેશાં મેળવી શકાય.

વિકલ્પ 3 : એક વાસ્તવિક દ્વિઘાત અનાવૃત્ત અવયવ અને બીજા વાસ્તવિક સુરેખ અનાવૃત્ત અવયવો

એટલે કે $q(x) = (ax^2 + bx + c)(x - \beta_3)(x - \beta_4) \dots (x - \beta_n)$ હોય, તો

$$\frac{p(x)}{q(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3}{x - \beta_3} + \dots + \frac{A_n}{x - \beta_n} \text{ લો.}$$

જ્યાં $A_1, A_2, A_3, \dots, A_n$ અજ્ઞાત અચળો છે અને તે હંમેશાં મેળવી શકાય.

વિકલ્પ 4 : વાસ્તવિક દ્વિઘાત આવૃત્ત અવયવ અને બીજા વાસ્તવિક સુરેખ અનાવૃત્ત અવયવો

એટલે કે $q(x) = (ax^2 + bx + c)^2(x - \beta_5) \dots (x - \beta_n)$ હોય, તો

$$\frac{p(x)}{q(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3x + A_4}{(ax^2 + bx + c)^2} + \frac{A_5}{x - \beta_5} + \dots + \frac{A_n}{x - \beta_n}.$$

જ્યાં $A_1, A_2, A_3, \dots, A_n$ અજ્ઞાત અચળો છે અને તે હંમેશાં મેળવી શકાય.

નીચેનું કોષ્ટક આપેલ જુદાં જુદાં સ્વરૂપનાં સંમેય વિધેયને અનુરૂપ આંશિક અપૂર્ણાંકનું સ્વરૂપ કેવું લેવું તે દર્શાવે છે :

ક્રમ	ઉચિત સંમેય વિધેયનું સ્વરૂપ	આંશિક અપૂર્ણાંકનું સ્વરૂપ
1.	$\frac{p(x)}{(x-\alpha)(x-\beta)}$	$\frac{A}{x-\alpha} + \frac{B}{x-\beta}$
2.	$\frac{p(x)}{(x-\alpha)(x-\beta)(x-\gamma)}$	$\frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$
3.	$\frac{p(x)}{(x-\alpha)^2}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}$
4.	$\frac{p(x)}{(x-\alpha)^2(x-\beta)}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\beta)}$
5.	$\frac{p(x)}{(x-\alpha)(ax^2+bx+c)}$	$\frac{A}{x-\alpha} + \frac{Bx+C}{ax^2+bx+c}$
6.	$\frac{p(x)}{(x-\alpha)^3}$	$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\alpha)^3}$
7.	$\frac{p(x)}{(ax^2+bx+c)^2}$	$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$

બહુવિકલ્પી પ્રશ્નો :

(1) $\int \left(\frac{x}{p} + \frac{q}{x} + x^p + q^x + pqx \right) dx = \dots\dots$ $x \in R^+, (p \neq -1, 0), q > 0$

(A) $\frac{1}{p} - \frac{q}{x^2} + p \cdot x^{p-1} + q^x \cdot \log q + pq + c$

(B) $\frac{x^2}{2p} + \frac{q}{x^2} + p \cdot x^{p-1} + q^x \log q + pq \frac{x^2}{2} + c$

(C) $\frac{x^2}{2p} + q \log |x| + \frac{x^{p+1}}{p+1} + q^x \log q + pq \frac{x^2}{2} + c$

(D) $\frac{x^2}{2p} + q \log |x| + \frac{x^{p+1}}{p+1} + \frac{q^x}{\log q} + pq \frac{x^2}{2} + c$

ઉકેલ : $I = \int \left(\frac{x}{p} + \frac{q}{x} + x^p + q^x + pqx \right) dx$
 $= \frac{1}{p} \int x dx + q \int \frac{1}{x} dx + \int x^p dx + \int q^x dx + \int pqx dx$
 $= \frac{1}{p} \cdot \frac{x^2}{2} + q \log |x| + \frac{x^{p+1}}{p+1} + \frac{q^x}{\log q} + pq \frac{x^2}{2} + c$

જવાબ : (D)

(2) $\int e^{2a \log x} + e^{3x \log a} dx = \dots\dots$ $x, a \in R^+$

(A) $\frac{x^{2a}}{2a} + a^{3x} \log 3 + c$

(B) $\frac{x^{2a+1}}{2a+1} + a^{3x} \log a + c$

(C) $\frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{\log a} + c$

(D) $\frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{3 \log a} + c$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int e^{2a \log x} + e^{3x \log a} dx \\
 &= \int e^{\log x^{2a}} + e^{\log a^{3x}} dx \\
 &= \int (x^{2a} + a^{3x}) dx \\
 &= \frac{x^{2a+1}}{2a+1} + \frac{a^{3x}}{3 \log a} + c
 \end{aligned}$$

જવાબ : (D)

$$(3) \quad \int e^{x \log a} \cdot e^x dx = \dots\dots$$

$$(A) \quad \frac{a^x}{\log ae} + c \quad (B) \quad \frac{e^x}{1 + \log a} + c \quad (C) \quad \frac{a^x e^x}{\log a} + c \quad (D) \quad \frac{(ae)^x}{\log a + 1} + c$$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int e^{x \log a} \cdot e^x dx & a \in R^+, a \neq e^{-1} \\
 &= \int e^{\log a^x} \cdot e^x dx \\
 &= \int a^x e^x dx \\
 &= \int (ae)^x dx \\
 &= \frac{(ae)^x}{\log ae} + c \\
 &= \frac{(ae)^x}{\log a + \log e} + c \\
 &= \frac{(ae)^x}{\log a + 1} + c
 \end{aligned}$$

જવાબ : (D)

$$(4) \quad \int \frac{3^x + 4^x}{5^x} dx = \dots\dots + c$$

$$\begin{aligned}
 (A) \quad & \frac{3^x \log 3}{5^x \log 5} + \frac{4^x \log 4}{5^x \log 5} & (B) \quad & \left(\frac{3}{5}\right)^x (\log 3 - \log 5) + \left(\frac{4}{5}\right)^x (\log 4 - \log 5) \\
 (C) \quad & \frac{\left(\frac{3}{5}\right)^x}{\log 3 + \log 5} + \frac{\left(\frac{4}{5}\right)^x}{\log 4 + \log 5} & (D) \quad & \frac{\left(\frac{3}{5}\right)^x}{\log \left(\frac{3}{5}\right)} + \frac{\left(\frac{4}{5}\right)^x}{\log \left(\frac{4}{5}\right)}
 \end{aligned}$$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int \frac{3^x + 4^x}{5^x} dx \\
 &= \int \left(\frac{3^x}{5^x} + \frac{4^x}{5^x} \right) dx \\
 &= \int \left(\frac{3}{5} \right)^x + \left(\frac{4}{5} \right)^x dx \\
 &= \frac{\left(\frac{3}{5}\right)^x}{\log \left(\frac{3}{5}\right)} + \frac{\left(\frac{4}{5}\right)^x}{\log \left(\frac{4}{5}\right)} + c
 \end{aligned}$$

જવાબ : (D)

(5) જો $g'(x) = \sec^2 x + \operatorname{cosec}^2 x - 2$ અને $g\left(\frac{\pi}{4}\right) = 1$, તો $g(x) = \dots$

(A) $\tan x + \cot x - 2x + \frac{\pi}{2}$

(B) $\tan x - \cot x - 2x + \frac{\pi}{2} + 1$

(C) $\tan x + \cot x + \frac{\pi}{2} + 1$

(D) $\tan x - \cot x + 2x - \frac{\pi}{2} + 1$

ઉકેલ : આપેલ, $g'(x) = \sec^2 x + \operatorname{cosec}^2 x - 2$

$$\therefore g(x) = \int (\sec^2 x + \operatorname{cosec}^2 x - 2) dx$$

$$g(x) = \tan x - \cot x - 2x + c$$

હવે, $g\left(\frac{\pi}{4}\right) = 1$

$$\therefore \tan \frac{\pi}{4} - \cot \frac{\pi}{4} - 2\left(\frac{\pi}{4}\right) + c = 1$$

$$1 - 1 - \frac{\pi}{2} + c = 1$$

$$\therefore c = \frac{\pi}{2} + 1$$

$$\therefore g(x) = \tan x - \cot x - 2x + \frac{\pi}{2} + 1$$

જવાબ : (B)

(6) જો $f'(x) = 1 - \frac{3}{x^2}$ અને $f(1) = 6$, તો $f(2) = \dots$

(A) $\frac{11}{2}$

(B) $\frac{5}{2}$

(C) $\frac{3}{2}$

(D) $\frac{1}{2}$

ઉકેલ : $f'(x) = 1 - \frac{3}{x^2} = 1 - 3x^{-2}$

$$\therefore f(x) = \int (1 - 3x^{-2}) dx$$

$$\therefore f(x) = x - \frac{3x^{-1}}{-1} + c$$

$$\therefore f(x) = x + \frac{3}{x} + c$$

હવે, $f(1) = 6$

$$1 + 3 + c = 6$$

$$\therefore c = 2$$

$$\therefore f(x) = x + \frac{3}{x} + 2$$

$$\therefore f(2) = 2 + \frac{3}{2} + 2 = \frac{11}{2}$$

જવાબ : (A)

(7) $\int \frac{\sec x}{\sec x - \tan x} dx = \dots$

(A) $x - \log |\operatorname{cosec} x - \cot x + x| + c$

(B) $x + \log \left| \tan \frac{x}{2} \right| + c$

(C) $\sec x + \tan x + c$

(D) $\sec x \tan x + \sec^2 x + c$

ઉકેલ : $I = \int \frac{\sec x}{\sec x - \tan x} dx$

$$\begin{aligned}
&= \int \frac{\sec x (\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx \\
&= \int (\sec^2 x + \sec x \tan x) dx \\
&= \tan x + \sec x + c
\end{aligned}$$

જવાબ : (C)

(8) $\int \frac{\sin^2 x dx}{1 - \cos x} = \dots\dots$

(A) $x - \sin x + c$

(B) $x + \sin x + c$

(C) $\log |1 - \cos x| + c$

(D) $2 \log |\sin x| + c$

ઉકેલ : $I = \int \frac{\sin^2 x dx}{1 - \cos x}$

$$\begin{aligned}
&= \int \frac{1 - \cos^2 x}{1 - \cos x} dx \\
&= \int \frac{(1 + \cos x)(1 - \cos x)}{(1 - \cos x)} dx \\
&= \int (1 + \cos x) dx \\
&= x + \sin x + c
\end{aligned}$$

જવાબ : (B)

(9) $\int \frac{x^4 + 2}{x^2 + 1} dx = \dots\dots$

(A) $\frac{x^3}{3} + x + \tan^{-1} x + c$

(B) $\frac{x^3}{3} - x + 3\tan^{-1} x + c$

(C) $\frac{x^3}{3} - x + \frac{1}{3}\tan^{-1} x + c$

(D) $\frac{x^3}{3} + x + \frac{1}{3}\tan^{-1} x + c$

ઉકેલ : $I = \int \frac{x^4 + 2}{x^2 + 1} dx$

$$\begin{aligned}
&= \int \frac{(x^4 - 1) + 3}{(x^2 + 1)} dx \\
&= \int \left(x^2 - 1 + \frac{3}{x^2 + 1} \right) dx \\
&= \frac{x^3}{3} - x + 3\tan^{-1} x + c
\end{aligned}$$

જવાબ : (B)

(10) $\int x\sqrt{x+4} dx = \dots\dots + c$

(A) $2\sqrt{x+4} - 8(x+4)^{-\frac{1}{2}}$

(B) $\frac{1}{3}(x+4)^{\frac{3}{2}} - \frac{4}{3}(x+4)^{\frac{1}{2}}$

(C) $\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}$

(D) $\frac{2}{5}(x+4)^{\frac{5}{2}} + \frac{4}{3}(x+4)^{\frac{3}{2}}$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int x\sqrt{x+4} \, dx \\
 &= \int [(x+4) - 4] \sqrt{x+4} \, dx \\
 &= \int (x+4) \sqrt{x+4} \, dx - 4 \int \sqrt{x+4} \, dx \\
 &= \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}} - 4 \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}} + c
 \end{aligned}$$

જવાબ : (C)

$$(11) \int \frac{x-1}{\sqrt{x+4}} \, dx = \dots\dots$$

$$(A) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c$$

$$(B) \quad \frac{-2}{\sqrt{x+4}} + 10\sqrt{x+4} + c$$

$$(C) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} - \frac{5}{2}\sqrt{x+4} + c$$

$$(D) \quad \frac{2}{3}(x+4)^{\frac{3}{2}} + \frac{5}{2}\sqrt{x+4} + c$$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int \frac{x-1}{\sqrt{x+4}} \, dx \\
 &= \int \frac{(x+4) - 5}{\sqrt{x+4}} \, dx \\
 &= \int \left(\sqrt{x+4} - \frac{5}{\sqrt{x+4}} \right) \, dx \\
 &= \int (x+4)^{\frac{1}{2}} \, dx - 5 \int (x+4)^{-\frac{1}{2}} \, dx \\
 &= \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c
 \end{aligned}$$

જવાબ : (A)

$$(12) \int \frac{\cos 2x - 1}{\cos 2x + 1} \, dx = \dots\dots$$

$$(A) \quad \tan x - x + c$$

$$(B) \quad -\cot x - x + c$$

$$(C) \quad x + \tan x + c$$

$$(D) \quad x - \tan x + c$$

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int \frac{\cos 2x - 1}{\cos 2x + 1} \, dx \\
 &= -\int \tan^2 x \, dx \\
 &= -\int (\sec^2 x - 1) \, dx \\
 &= x - \tan x + c
 \end{aligned}$$

જવાબ : (D)

(13) જો $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = P \cos 4x + c$, તો $P = \dots$

(A) $-\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $-\frac{1}{8}$

(D) $\frac{1}{8}$

ઉકેલ :
$$\begin{aligned} \int \frac{\cos 4x + 1}{\cot x - \tan x} dx &= \int \frac{2 \cos^2 2x}{\left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right)} dx \\ &= \int \frac{2 \cos^2 2x \cdot \sin x \cdot \cos x}{\cos^2 x - \sin^2 x} dx \\ &= \int \frac{\cos^2 2x \cdot \sin 2x}{\cos 2x} dx \\ &= \frac{1}{2} \int 2 \sin 2x \cos 2x dx \\ &= \frac{1}{2} \int \sin 4x dx \\ &= \frac{1}{2} \left(-\frac{\cos 4x}{4} \right) + c \\ &= -\frac{1}{8} \cos 4x + c \end{aligned}$$

આથી, $P = -\frac{1}{8}$

જવાબ : (C)

(14) $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx = \dots$

(A) $-\cot \frac{x}{2} + c$

(B) $-\frac{1}{2} \cot \frac{x}{2} + c$

(C) $-\tan \frac{x}{2} + c$

(D) $\cot \frac{x}{2} + c$

ઉકેલ :
$$\begin{aligned} \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx &= \int \frac{\left(\frac{1}{\sin x}\right)}{\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)} dx \\ &= \int \frac{1}{1 - \cos x} dx \\ &= \int \frac{1}{2 \sin^2 \frac{x}{2}} dx \\ &= \frac{1}{2} \int \operatorname{cosec}^2 \frac{x}{2} dx \\ &= \frac{1}{2} \left(-\frac{\cot \frac{x}{2}}{\frac{1}{2}} \right) + c \\ &= -\cot \frac{x}{2} + c \end{aligned}$$

બીજી રીત :
$$\int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x) dx}{1}$$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

$$= -\frac{1 + \cos x}{\sin x} + c$$

$$= -\frac{2 \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= -\cot \frac{x}{2} + c$$

જવાબ : (A)

$$(15) \int \frac{\cos 2x - \cos 2\beta}{\cos x - \cos \beta} dx = \dots\dots$$

$$(A) \quad 2(\sin x + x \cos \beta) + c$$

$$(B) \quad 2(\sin x + \sin \beta) + c$$

$$(C) \quad 2(-\sin x + x \cos \beta) + c$$

$$(D) \quad -2(\sin x + \sin \beta) + c$$

$$\begin{aligned} \text{ઉકેલ : } & \int \frac{\cos 2x - \cos 2\beta}{\cos x - \cos \beta} dx \\ &= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \beta - 1)}{(\cos x - \cos \beta)} dx \\ &= \int \frac{2(\cos^2 x - \cos^2 \beta)}{(\cos x - \cos \beta)} dx \\ &= 2 \int (\cos x + \cos \beta) dx \\ &= 2(\sin x + x \cos \beta) + c \end{aligned}$$

જવાબ : (A)

$$(16) \int \frac{1 + \cos x}{\sin x \cos x} dx = \dots\dots$$

$$(A) \quad \log |\sin x| + \log |\cos x| + c$$

$$(B) \quad \log \left| \tan x \cdot \tan \frac{x}{2} \right| + c$$

$$(C) \quad \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(D) \quad \log \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + c$$

$$\begin{aligned} \text{ઉકેલ : } & \int \frac{1 + \cos x}{\sin x \cos x} dx \\ &= \int \frac{1}{\sin x \cos x} dx + \int \frac{1}{\sin x} dx \\ &= 2 \int \frac{1}{\sin 2x} dx + \int \operatorname{cosec} x dx \\ &= 2 \int \operatorname{cosec} 2x dx + \int \operatorname{cosec} x dx \\ &= 2 \frac{\log |\tan x|}{2} + \log \left| \tan \frac{x}{2} \right| + c \\ &= \log \left| \tan x \cdot \tan \frac{x}{2} \right| + c \end{aligned}$$

જવાબ : (B)

$$(17) \int \sec^2 x \cdot \operatorname{cosec}^2 x dx = \dots\dots$$

$$(A) \quad \cot x + \tan x + c$$

$$(B) \quad \cot x - \tan x + c$$

$$(C) \quad -\cot x - \tan x + c$$

$$(D) \quad -\cot x + \tan x + c$$

$$\begin{aligned} \text{ઉકેલ : } & \int \sec^2 x \cdot \operatorname{cosec}^2 x dx \\ &= \int \frac{1}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx \end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{\cos^2 x} + \int \frac{1}{\sin^2 x} dx \\
&= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\
&= \tan x - \cot x + c
\end{aligned}$$

જવાબ : (D)

$$(18) \int \frac{1 + \cos x}{\cos x (1 - \cos x)} dx = \dots\dots$$

$$(A) \quad 2 \log |\cos x| - \cot \frac{x}{2} + c$$

$$(B) \quad \log |\sec x + \tan x| - 2 \cot \frac{x}{2} + c$$

$$(C) \quad \log |\tan \frac{x}{2}| + 2 \cot \frac{x}{2} + c$$

$$(D) \quad \log |\sec x + \tan x| - \cot \frac{x}{2} + c$$

$$\begin{aligned}
\text{ઉકેલ : } &\int \frac{1 + \cos x}{\cos x (1 - \cos x)} dx \\
&= \int \frac{(1 - \cos x) + 2 \cos x}{\cos x (1 - \cos x)} dx \\
&= \int \sec x dx + \int \frac{2}{1 - \cos x} dx \\
&= \int \sec x dx + 2 \int \frac{1}{2 \sin^2 \frac{x}{2}} dx \\
&= \int \sec x dx + \int \operatorname{cosec}^2 \frac{x}{2} dx \\
&= \log |\sec x + \tan x| - 2 \cot \frac{x}{2} + c
\end{aligned}$$

જવાબ : (B)

$$(19) \quad \text{જો } \int \frac{2x^2 + 3}{(x^2 - 1)(x^2 + 4)} dx = a \log \left| \frac{x+1}{x-1} \right| + b \tan^{-1} \frac{x}{2} + c, \text{ તો } a = \dots\dots \text{ અને } b = \dots\dots$$

$$(A) \quad \frac{1}{2}, \frac{1}{2}$$

$$(B) \quad -\frac{1}{2}, \frac{1}{2}$$

$$(C) \quad \frac{1}{2}, -\frac{1}{2}$$

$$(D) \quad -\frac{1}{2}, -\frac{1}{2}$$

$$\begin{aligned}
\text{ઉકેલ : } &\int \frac{2x^2 + 3}{(x^2 - 1)(x^2 + 4)} dx \\
&= \int \frac{(x^2 + 4) + (x^2 - 1)}{(x^2 - 1)(x^2 + 4)} dx \\
&= \int \frac{1}{x^2 - 1} dx + \int \frac{1}{x^2 + 4} dx \\
&= \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c \\
&= -\frac{1}{2} \log \left| \frac{x+1}{x-1} \right| + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c
\end{aligned}$$

$$a = -\frac{1}{2}, b = \frac{1}{2}$$

જવાબ : (B)

$$(20) \int \frac{\sin^3 3x}{\cos^5 3x} dx = \dots\dots$$

- (A) $\tan^4 x + c$ (B) $\tan 6x + c$ (C) $\frac{1}{12} \tan^4 3x + c$ (D) $\frac{1}{3} \tan^4 3x + c$

$$\text{ઉકેલ : } \int \frac{\sin^3 3x}{\cos^5 3x} dx$$

$$= \int \tan^3 3x \cdot \sec^2 3x dx$$

$$\text{ધારીએ } \tan 3x = t, \text{ તો, } 3\sec^2 3x dx = dt$$

$$\therefore \sec^2 3x dx = \frac{1}{3} dt$$

$$I = \frac{1}{3} \int t^3 \cdot dt$$

$$= \frac{1}{3} \times \frac{t^4}{4} + c = \frac{1}{12} \tan^4 3x + c$$

જવાબ : (C)

$$(21) \int \frac{\cot x}{\sqrt{\sin x}} dx = \dots\dots$$

- (A) $\frac{-2}{\sqrt{\sin x}} + c$ (B) $\frac{-1}{2\sqrt{\sin x}} + c$ (C) $\frac{1}{2\sqrt{\sin x}} + c$ (D) $2\sqrt{\sin x} + c$

$$\text{ઉકેલ : } \int \frac{\cot x}{\sqrt{\sin x}} dx$$

$$= \int \frac{\cos x}{\sin x \cdot (\sin x)^{\frac{1}{2}}} dx$$

$$= \int (\sin x)^{\frac{-3}{2}} \cos x dx$$

$$= \frac{(\sin x)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} + c = \frac{(\sin x)^{\frac{-1}{2}}}{\frac{-1}{2}} + c = \frac{-2}{\sqrt{\sin x}} + c$$

જવાબ : (A)

$$(22) \text{ જો } f'(x) = \frac{1}{\cos^2 x \sqrt{1 + \tan x}} \text{ તો } f(0) = 4 \text{ તો } f(x) = \dots\dots$$

- (A) $\sqrt{1 + \tan x} + 4$ (B) $\frac{2}{3} (1 + \tan x)^{\frac{3}{2}}$
(C) $2(\sqrt{1 + \tan x} + 1)$ (D) $2(\sqrt{1 + \tan x} - 1)^{\frac{2}{3}}$

$$\text{ઉકેલ : } f'(x) = \frac{1}{\cos^2 x \sqrt{1 + \tan x}}$$

$$\therefore f(x) = \int \frac{\sec^2 x dx}{\sqrt{1 + \tan x}}$$

ધારો કે $\tan x = t$. અર્થાત્, $\sec^2 x \, dx = dt$

$$\therefore f(x) = \int \frac{dt}{\sqrt{1+t}}$$

$$\therefore = 2\sqrt{1+\tan x} + c$$

$$\text{હવે, } f(0) = 4$$

$$\therefore 2\sqrt{1+\tan 0} + c = 4$$

$$\therefore c = 2$$

$$\therefore f(x) = 2\sqrt{1+\tan x} + 2$$

$$= 2(\sqrt{\tan x + 1} + 1)$$

જવાબ : (C)

(23) જો $f(x) = \frac{1}{3-2\cos^2 x + 6\sin^2 x}$ અને તેનો પ્રતિવિકલિત $F(x) = \frac{1}{3} \tan^{-1}(g(x)) + c$ હોય, તો $g(x) = \dots\dots$

(A) $3\tan x$

(B) $\sqrt{3}\tan x$

(C) $2\tan x$

(D) $\tan x$

$$\text{ઉકેલ : } f(x) = \frac{1}{3-2\cos^2 x + 6\sin^2 x}$$

$$\therefore F(x) = \int \frac{dx}{3-2\cos^2 x + 6\sin^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3\sec^2 x - 2 + 6\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3(1+\tan^2 x) - 2 + 6\tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{9\tan^2 x + 1}$$

$$= \int \frac{dt}{9t^2 + 1}$$

(ધારો કે $\tan x = t$. અર્થાત્, $\sec^2 x \, dx = dt$)

$$= \frac{1}{3} \tan^{-1}(3t) + c$$

$$= \frac{1}{3} \tan^{-1}(3\tan x) + c$$

$$\therefore g(x) = 3\tan x$$

જવાબ : (A)

$$(24) \int \frac{e^{\sqrt{x}} \sin(e^{\sqrt{x}})}{\sqrt{x}} \, dx = \dots\dots$$

(A) $e^{\sqrt{x}} [\sin(e^{\sqrt{x}}) + \cos(e^{\sqrt{x}})] + c$

(B) $e^{\sqrt{x}} (\cos\sqrt{x} - \sin\sqrt{x}) + c$

(C) $-2\cos(e^{\sqrt{x}}) + c$

(D) $-2e^{\sqrt{x}}\cos(e^{\sqrt{x}}) + c$

ઉકેલ : $I = \int \frac{e^{\sqrt{x}} \sin(e^{\sqrt{x}})}{\sqrt{x}} dx$

ધારો કે $e^{\sqrt{x}} = t$

$\therefore e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} dx = dt$

$\therefore \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2dt$

$I = 2 \int \sin t dt = -2\cos t + c = -2\cos(e^{\sqrt{x}}) + c$

જવાબ : (C)

(25) જો $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \log |f(x)| + c$, તો $f(x) = \dots$

(A) $(\sin x + \cos x)^2$ (B) $(\sin x - \cos x)$ (C) $(\cos x + \sin x)$ (D) $\cos 2x$

ઉકેલ : $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$

$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

(ધારો કે $\cos x + \sin x = t$. આથી, $(-\sin x + \cos x)dx = dt$)

$\therefore I = \int \frac{dt}{t} = \log |t| + c = \log |\cos x + \sin x| + c$

$\therefore f(x) = \cos x + \sin x$

જવાબ : (C)

(26) $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx = \dots$

(A) $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) + c$

(B) $\log |\sin x + 1| + c$

(C) $\log |1 + \cos x| + \cos x + c$

(D) $\log |\sin x| - \sin x + c$

ઉકેલ : $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

$= \int \frac{(1 - \sin^2 x) \cos x}{\sin x (\sin x + 1)} dx$

$= \int \frac{(1 - \sin x) \cos x}{\sin x} dx$

ધારો કે $\sin x = t$. આથી, $\cos x dx = dt$

$= \int \frac{1-t}{t} dt = \int \frac{1}{t} dt - \int dt = \log |t| - t + c = \log |\sin x| - \sin x + c$

જવાબ : (D)

$$(27) \int \frac{\sin x}{\sin x - \cos x} dx = \dots\dots$$

$$(A) \quad \frac{x}{2} + \log |\cos x - \sin x| + c$$

$$(B) \quad \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c$$

$$(C) \quad \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| + c$$

$$(D) \quad \frac{x}{2} - \frac{1}{2} \log |\cos x + \sin x| + c$$

(IIT : 1978)

$$\begin{aligned} \text{ଉତ୍ତର} : \int \frac{\sin x}{\sin x - \cos x} dx \\ &= \frac{1}{2} \int \frac{\sin x - \cos x + \cos x + \sin x}{(\sin x - \cos x)} dx \\ &= \frac{1}{2} \int \left(1 + \frac{(\cos x + \sin x)}{\sin x - \cos x} \right) dx \\ &= \frac{1}{2} (x + \log |\sin x - \cos x|) + c \\ &= \frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c \end{aligned}$$

ସଠିକ୍ : (B)

$$(28) \int \frac{x^2}{(a+bx)^2} dx = P(a+bx) + \frac{Q}{(a+bx)} + R \log |a+bx| + c, \text{ ଯେଉଁ } P + Q + R = \dots\dots .$$

$$(A) \quad \frac{1+2a-a^2}{b^3}$$

$$(B) \quad \frac{(1+a)^2}{b^3}$$

$$(C) \quad \frac{1-2a-a^2}{b^3}$$

$$(D) \quad \frac{(1-a)^2}{3}$$

(IIT : 1979)

$$\text{ଉତ୍ତର} : I = \int \frac{x^2}{(a+bx)^2} dx \quad \text{ଏଠି } a+bx = t$$

$$\therefore b dx = dt \quad \text{ତେଣୁ} \quad dx = \frac{1}{b} dt$$

$$\begin{aligned} I &= \frac{1}{b^3} \int \frac{(t-a)^2}{t^2} dt \\ &= \frac{1}{b^3} \int \left(\frac{t^2 - 2at + a^2}{t^2} \right) dt \\ &= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt \\ &= \frac{1}{b^3} \left[t - 2a \log |t| + a^2 \left(\frac{-1}{t} \right) \right] + c \\ &= \frac{(a+bx)}{b^3} - \frac{2a}{b^3} \log |a+bx| - \frac{a^2}{b^3(a+bx)} + c \end{aligned}$$

$$\therefore P = \frac{1}{b^3}, Q = \frac{-a^2}{b^3}, R = -\frac{2a}{b^3}$$

$$\therefore P + Q + R = \frac{1-2a-a^2}{b^3}$$

ସଠିକ୍ : (C)

$$(29) \int \frac{dx}{\sqrt{\sin^3 x \cdot \cos x}} = \dots\dots$$

- (A) $\frac{2}{\sqrt{\tan x}} + c$ (B) $\frac{-2}{\sqrt{\tan x}} + c$ (C) $2\sqrt{\tan x} + c$ (D) $-2\sqrt{\tan x} + c$

$$\text{ઉકેલ : } \int \frac{dx}{\sqrt{\sin^3 x \cdot \cos x}}$$

$$= \int \frac{dx}{\sqrt{\frac{\sin^3 x}{\cos^3 x} \cdot \cos^4 x}}$$

$$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan^3 x}}$$

$$= \int \frac{dt}{\sqrt{t^3}}$$

(હાલે કે $\tan x = t$. આથી, $\sec^2 x \, dx = dt$)

$$= \int t^{-\frac{3}{2}} \, dt$$

$$= \frac{-2}{\sqrt{t}} + c$$

$$= \frac{-2}{\sqrt{\tan x}} + c$$

જવાબ : (B)

$$(30) \int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx = \dots\dots$$

- (A) $\frac{1}{a^2 + b^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$ (B) $\frac{1}{b^2 - a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$

- (C) $\frac{a^2}{2b^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$ (D) $\frac{b^2}{2a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$

$$\text{ઉકેલ : } I = \int \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \, dx$$

$$\text{હાલે કે } a^2 \cos^2 x + b^2 \sin^2 x = t$$

$$\therefore (-a^2 \sin 2x + b^2 \sin 2x) \, dx = dt$$

$$\therefore (b^2 - a^2) \sin 2x \, dx = dt$$

$$\therefore \sin 2x \, dx = \frac{1}{b^2 - a^2} \, dt$$

$$\therefore I = \frac{1}{b^2 - a^2} \int \frac{dt}{t}$$

$$= \frac{1}{b^2 - a^2} \log |t| + c$$

$$= \frac{1}{b^2 - a^2} \log |a^2 \cos^2 x + b^2 \sin^2 x| + c$$

જવાબ : (B)

$$(31) \int \frac{(7x^6 + 7^x \log 7)}{7^x + x^7} dx = \dots\dots$$

$$(A) 7^x \log |1 + x^7| + c$$

$$(B) (x + 1) \log |7^x + x^7| + c$$

$$(C) \log |7^x + x^7| + c$$

$$(D) \frac{1}{\log x} \log |7^x + x^7| + c$$

$$\text{ઉકેલ : } \int \frac{(7x^6 + 7^x \log 7)}{7^x + x^7} dx$$

$$\text{હોતું છે } 7^x + x^7 = t. \text{ આથી, } (7^x \log 7 + 7x^6) dx = dt$$

$$I = \int \frac{dt}{t} = \log |t| + c = \log |7^x + x^7| + c$$

જવાબ : (C)

$$(32) \int \log (\tan x) \cdot \sec x \operatorname{cosec} x dx = \dots\dots$$

$$(A) \log |\sec x \cdot \operatorname{cosec} x| + c$$

$$(B) \log |\log (\tan x)| + c$$

$$(C) \frac{1}{2} [\log (\tan x)]^2 + c$$

$$(D) \frac{1}{2} \log (\tan x) + c$$

$$\text{ઉકેલ : } I = \int \log (\tan x) \cdot \sec x \operatorname{cosec} x dx$$

$$\text{હોતું છે } \log (\tan x) = t$$

$$\therefore \left(\frac{1}{\tan x} \cdot \sec^2 x \right) dx = dt$$

$$\therefore \left(\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} \right) dx = dt$$

$$\therefore \sec x \operatorname{cosec} x dx = dt$$

$$I = \int t dt = \frac{t^2}{2} + c = \frac{1}{2} [\log (\tan x)]^2 + c$$

જવાબ : (C)

$$(33) \int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}} dx = \dots\dots \left(0 < x < \frac{\pi}{2}\right)$$

$$(A) \frac{1}{3} \cos^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$(B) \frac{1}{3} \sin^{-1} (\sin^{\frac{3}{2}} x) + c$$

$$(C) \frac{2}{3} \sin^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$(D) -\frac{2}{3} \sin^{-1} (\cos^{\frac{3}{2}} x) + c$$

$$\text{ઉકેલ : } I = \int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}} dx$$

$$= \int \frac{\sqrt{\cos x \cdot \sin x}}{\sqrt{1 - \cos^3 x}} dx$$

$$\text{હોતું છે } \cos^{\frac{3}{2}} x = t$$

$$\therefore \frac{-3}{2} \cdot \cos^{\frac{1}{2}} x \sin x dx = dt$$

$$\therefore \sqrt{\cos x} \cdot \sin x \, dx = -\frac{2}{3} dt$$

$$\therefore I = \frac{-2}{3} \int \frac{dt}{\sqrt{1-t^2}} = \frac{-2}{3} \sin^{-1}(t) + c = \frac{-2}{3} \sin^{-1}(\cos^{\frac{3}{2}} x) + c \quad \text{જવાબ : (D)}$$

$$(34) \int \frac{\log(x+1) - \log x}{x(x+1)} dx = \dots$$

$$(A) -\log \log \left(\frac{x+1}{x} \right) + c$$

$$(B) \log(x+1)^2 - \log x^2 + c$$

$$(C) -\log \left(\frac{x+1}{x} \right) + c$$

$$(D) \frac{-1}{2} \left[\log \left(\frac{x+1}{x} \right) \right]^2 + c$$

$$\text{ઉકેલ : } I = \int \frac{\log(x+1) - \log x}{x(x+1)} dx$$

$$\text{ધારીએ } \log(x+1) - \log x = t$$

$$\therefore \left(\frac{1}{x+1} - \frac{1}{x} \right) dx = dt$$

$$\therefore \frac{-1}{x(x+1)} dx = dt$$

$$\therefore I = - \int t \, dt$$

$$= -\frac{t^2}{2} + c = \frac{-1}{2} [\log(x+1) - \log x]^2 + c = \frac{-1}{2} \left[\log \left(\frac{x+1}{x} \right) \right]^2 + c \quad \text{જવાબ : (D)}$$

$$(35) \int \frac{e^{\log \left(1 + \frac{1}{x^2} \right)}}{x^2 + \frac{1}{x^2}} dx = \dots$$

$$(A) \frac{1}{2} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(B) \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(C) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) + c$$

$$(D) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{2x} \right) + c$$

$$\text{ઉકેલ : } I = \int \frac{e^{\log \left(1 + \frac{1}{x^2} \right)}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2} \right)}{\left(x - \frac{1}{x} \right)^2 + 2} dx$$

$$= \int \frac{dt}{t^2 + 2} \quad (\text{ପରିବର୍ତ୍ତନ } x - \frac{1}{x} = t. \text{ ଆଉ, } \left(1 + \frac{1}{x^2}\right) dx = dt)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + c = \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}} + c = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c \quad \text{ସ୍ୱତନ୍ତ୍ରତା : (C)}$$

$$(36) \int \frac{dx}{\sqrt{3} \sin x + \cos x} = \dots$$

$$(A) \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(B) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{2} \right) \right| + c$$

$$(C) \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

$$(D) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$\text{ଉତ୍ତର : } I = \int \frac{dx}{\sqrt{3} \sin x + \cos x}$$

$$= \frac{1}{2} \int \frac{dx}{\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x}$$

$$= \frac{1}{2} \int \frac{dx}{\sin \left(x + \frac{\pi}{6} \right)} = \frac{1}{2} \int \operatorname{cosec} \left(x + \frac{\pi}{6} \right) dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c \quad \text{ସ୍ୱତନ୍ତ୍ରତା : (D)}$$

$$(37) \int \frac{dx}{\cos x - \sin x} = \dots$$

$$(A) \frac{1}{2\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right| + c$$

$$(B) \frac{1}{\sqrt{2}} \log \left| \cot \left(\frac{x}{2} \right) \right| + c$$

$$(C) \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + c$$

$$(D) \frac{1}{2\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{3\pi}{8} \right) \right| + c$$

$$\text{ଉତ୍ତର : } \int \frac{dx}{\cos x - \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \sec \left(x + \frac{\pi}{4} \right) dx = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} + \frac{\pi}{4} \right) \right| + c = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + c$$

ସ୍ୱତନ୍ତ୍ରତା : (C)

$$(38) \int \frac{1}{1 + \sin x} dx = \tan(p(x)) + c \text{ ଯେଉଁ } p(x) = \dots$$

$$(A) x + \frac{\pi}{2}$$

$$(B) \frac{x}{2} + \frac{\pi}{4}$$

$$(C) \frac{x}{2} - \frac{\pi}{4}$$

$$(D) \frac{x}{2} - \frac{\pi}{8}$$

$$\text{ઉકેલ : } \int \frac{1}{1 + \sin x} dx$$

$$= \int \frac{1}{1 + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \frac{1}{2} \int \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\left(-\frac{1}{2}\right)} + c = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$$

જવાબ : (C)

$$(39) \int \sqrt{1 + \sin \frac{x}{2}} dx = \dots \quad 0 < x < \pi$$

$$(A) \quad 2\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right) + c$$

$$(B) \quad 4\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right) + c$$

$$(C) \quad 4\left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

$$(D) \quad 2\left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

$$\text{ઉકેલ : } \int \sqrt{1 + \sin \frac{x}{2}} dx$$

$$= \int \sqrt{\sin^2 \frac{x}{4} + \cos^2 \frac{x}{4} + 2 \sin \frac{x}{4} \cos \frac{x}{4}} dx$$

$$= \int \left(\sin \frac{x}{4} + \cos \frac{x}{4}\right) dx$$

$$\left(0 < \frac{x}{4} < \frac{\pi}{2}\right)$$

$$= 4 \left(\sin \frac{x}{4} - \cos \frac{x}{4}\right) + c$$

જવાબ : (C)

$$(40) \int \frac{\sin x dx}{\sqrt{1 + \sin x}} = \dots$$

$$(A) \quad 2\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right) + \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{4}\right) \right| + c$$

$$(B) \quad 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) + \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + c$$

$$(C) \quad 2\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) - \sqrt{2} \log \left| \tan \left(\frac{\pi}{8} + \frac{x}{4}\right) \right| + c$$

$$(D) \quad 2\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right) - \frac{1}{2} \log \left| \tan \left(\frac{x}{4} - \frac{\pi}{8}\right) \right| + c$$

$$\text{ઉકેલ : } \int \frac{\sin x dx}{\sqrt{1 + \sin x}}$$

$$(0 < x < \pi)$$

$$\begin{aligned}
&= \int \frac{(1 + \sin x) - (1)}{\sqrt{1 + \sin x}} dx \\
&= \int \sqrt{1 + \sin x} dx - \int \frac{1}{\sqrt{1 + \sin x}} dx \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \int \frac{dx}{\sin \frac{x}{2} + \cos \frac{x}{2}} \quad \left(1 + \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} \right) \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \frac{1}{\sqrt{2}} \int \frac{1}{\frac{1}{\sqrt{2}} \sin \frac{x}{2} + \frac{1}{\sqrt{2}} \cos \frac{x}{2}} dx \\
&= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx - \frac{1}{\sqrt{2}} \int \frac{dx}{\sin \left(\frac{x}{2} + \frac{\pi}{4} \right)} \\
&= \int \sin \frac{x}{2} dx + \int \cos \frac{x}{2} dx - \frac{1}{\sqrt{2}} \int \operatorname{cosec} \left(\frac{x}{2} + \frac{\pi}{4} \right) dx \\
&= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \frac{1}{\sqrt{2}} \times \left(\frac{1}{2} \right) \log \left| \tan \left(\frac{x}{4} + \frac{\pi}{8} \right) \right| + c \\
&= 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right) - \sqrt{2} \log \left| \tan \left(\frac{\pi}{8} + \frac{x}{4} \right) \right| + c \quad \text{જવાબ : (C)}
\end{aligned}$$

$$(41) \int \frac{(3x+2)^8}{(4x+3)^{10}} dx = A \left(\frac{3x+2}{4x+3} \right)^B + c, \text{ તો } A \cdot B = \dots$$

- (A) 10 (B) 1 (C) 8 (D) 9

$$\text{ઉકેલ : } \int \frac{(3x+2)^8}{(4x+3)^{10}} dx$$

$$= \int \left(\frac{3x+2}{4x+3} \right)^8 \times \frac{1}{(4x+3)^2} dx$$

$$\text{ધારીએ } \frac{3x+2}{4x+3} = t. \text{ તો, } \frac{(4x+3)3 - (3x+2)4}{(4x+3)^2} dx = dt$$

$$\therefore \frac{1}{(4x+3)^2} dx = dt$$

$$\therefore I = \int t^8 dt = \frac{t^9}{9} + c$$

$$= \frac{1}{9} \left(\frac{3x+2}{4x+3} \right)^9 + c$$

$$\therefore A = \frac{1}{9}, B = 9$$

$$\therefore AB = 1$$

જવાબ : (B)

$$(42) \int \frac{dx}{\sqrt{1 + \sec^2 x}} = \dots \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$(A) \sin^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

$$(B) \sin^{-1} \left(\frac{\cos x}{\sqrt{2}} \right) + c$$

$$(C) \cos^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

$$(D) \cos^{-1} \left(\frac{\cos x}{\sqrt{2}} \right) + c$$

$$\text{ઉકેલ : } \int \frac{dx}{\sqrt{1 + \sec^2 x}} \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$I = \int \frac{\cos x}{\sqrt{\cos^2 x + 1}} dx$$

$$= \int \frac{\cos x}{\sqrt{2 - \sin^2 x}} dx$$

$$\text{હાલે } \sin x = t. \text{ આથી, } \cos x dx = dt$$

$$\therefore I = \int \frac{dt}{\sqrt{2 - t^2}}$$

$$= \sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + c$$

$$= \sin^{-1} \left(\frac{\sin x}{\sqrt{2}} \right) + c$$

જવાબ : (A)

$$(43) \int \sqrt{1 + \operatorname{cosec} x} dx = \dots \quad \left(0 < x < \frac{\pi}{2}\right)$$

$$(A) 2\sin^{-1} \sqrt{\sin x} + c$$

$$(B) 2\sin^{-1} \sqrt{\cos x} + c$$

$$(C) 2\cos^{-1} \sqrt{\cos x} + c$$

$$(D) 2\sin^{-1} \sqrt{\sin x} + c$$

$$\text{ઉકેલ : } I = \int \sqrt{1 + \operatorname{cosec} x} dx$$

$$= \int \frac{\sqrt{1 + \sin x}}{\sqrt{\sin x}} dx$$

$$\text{હાલે } \sin x = t^2. \text{ આથી, } \cos x dx = 2t dt$$

$$I = \int \frac{\sqrt{1 + t^2}}{t} \cdot \frac{2t dt}{\sqrt{1 - t^4}} = 2 \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= 2 \sin^{-1} (t) + c$$

$$= 2\sin^{-1} \sqrt{\sin x} + c$$

જવાબ : (D)

$$(44) \quad \int \sqrt{\frac{a-x}{a+x}} dx = \dots\dots (a > 0)$$

$$(A) \quad \sin^{-1}x + \sqrt{a^2 - x^2} + c$$

$$(B) \quad a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + c$$

$$(C) \quad \frac{a}{2} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + c$$

$$(D) \quad \frac{1}{a} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + c$$

$$\text{ઉકેલ : } I = \int \sqrt{\frac{a-x}{a+x}} dx$$

$$= \int \frac{a-x}{\sqrt{a^2 - x^2}} dx$$

(અંશ અને છેદને $\sqrt{a-x}$ વડે ગુણવ)

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx - \int \frac{x dx}{\sqrt{a^2 - x^2}}$$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx + \frac{1}{2} \int \frac{-2x dx}{\sqrt{a^2 - x^2}}$$

$$= a \int \frac{1}{\sqrt{a^2 - x^2}} dx + \frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} \times \frac{(a^2 - x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + (a^2 - x^2)^{\frac{1}{2}} + c$$

$$= a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + c$$

$$\text{બીજી રીત : } = \int \sqrt{\frac{a-x}{a+x}} dx = \int \tan \frac{\theta}{2} (-a \sin \theta) d\theta \quad (x = a \cos \theta \text{ લેતાં } 0 < \theta < \pi)$$

$$= a \int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} (-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}) d\theta$$

$$= -a \int (1 - \cos \theta) d\theta$$

$$= -a (\theta - \sin \theta) + c' = -a \cos^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c'$$

$$= a \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c$$

જવાબ : (B)

$$(45) \quad \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx = Px \cos^{-1}x + Q\sqrt{1-x^2} + c, \text{ તલ } P + Q = \dots$$

- (A) $\frac{1}{2}$ (B) 0 (C) $-\frac{1}{2}$ (D) -1

$$\text{ઉકેલ : } I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

$$\text{ધારૂં કે } x = \cos 2\theta. \quad \text{અર્થ઼, } dx = -2\sin 2\theta d\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\begin{aligned} I &= \int \tan^{-1} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} (-2\sin 2\theta) d\theta \\ &= -2 \int \tan^{-1} \sqrt{\tan^2 \theta} \cdot \sin 2\theta d\theta \\ &= -2 \int \theta \cdot \sin 2\theta d\theta \\ &= -2 \left[\theta \cdot \left(\frac{-\cos 2\theta}{2} \right) + \int \left(1 \cdot \frac{\cos 2\theta}{2} \right) d\theta \right] \quad (\text{ખંડશ઼ સંકલન}) \\ &= -2 \left[-\frac{\theta}{2} \cos 2\theta + \frac{1}{2} \frac{\sin 2\theta}{2} \right] + c \\ &= \theta \cdot \cos 2\theta - \frac{1}{2} \sin 2\theta + c \\ &= \frac{1}{2} \cos^{-1}x \cdot x - \frac{1}{2} \sqrt{1-x^2} + c \quad (0 < 2\theta < \pi) \\ &= \frac{x}{2} \cos^{-1}x - \frac{1}{2} \sqrt{1-x^2} + c \end{aligned}$$

$$\therefore P = \frac{1}{2}, Q = -\frac{1}{2} \quad \text{અર્થ઼, } P + Q = 0$$

જવાબ : (B)

$$(46) \quad \int \tan^4 x dx = \alpha x + \beta \tan x + \gamma \tan^3 x + c, \text{ તલ } \alpha + \beta + \gamma = \dots$$

- (A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) -1

$$\begin{aligned} \text{ઉકેલ : } I &= \int \tan^4 x dx \\ &= \int \tan^2 x (\sec^2 x - 1) dx \\ &= \int (\tan^2 x \sec^2 x - \tan^2 x) dx \\ &= \int (\tan x)^2 \sec^2 x dx - \int (\sec^2 x - 1) dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + c \end{aligned}$$

$$\therefore \alpha = 1, \beta = -1, \gamma = \frac{1}{3}$$

$$\therefore \alpha + \beta + \gamma = \frac{1}{3}$$

જવાબ : (C)

$$(47) \int \frac{1}{\cos x + \sin x + 1} dx = \dots\dots$$

$$(A) \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(B) \frac{1}{2} \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(C) 2 \log \left| 1 + \tan \frac{x}{2} \right| + c$$

$$(D) \frac{1}{2} \log \left| 1 - \tan \frac{x}{2} \right| + c$$

$$\text{ઉકેલ : } I = \int \frac{1}{\cos x + \sin x + 1} dx$$

$$\text{ધારીએ } \tan \frac{x}{2} = t. \text{ ત્યજી } \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \text{ ત્યજી } dx = \frac{2dt}{1+t^2}$$

$$I = \int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 1} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{1-t^2+2t+1+t^2}$$

$$= 2 \int \frac{dt}{2t+2} = \int \frac{dt}{t+1} = \log |t+1| + c = \log \left| 1 + \tan \frac{x}{2} \right| + c$$

જવાબ : (A)

$$(48) \int \frac{dx}{\sin x - \cos x + \sqrt{2}} = \dots\dots$$

$$(A) \frac{1}{\sqrt{2}} \tan \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(B) -\frac{1}{\sqrt{2}} \tan \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(C) \frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$(D) -\frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

$$\text{ઉકેલ : } I = \int \frac{dx}{\sin x - \cos x + \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x \right) + 1}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \cos \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{2 \sin^2 \left(\frac{x}{2} + \frac{\pi}{8} \right)}$$

$$= \frac{1}{2\sqrt{2}} \times \int \operatorname{cosec}^2 \left(\frac{\pi}{8} + \frac{x}{2} \right) dx$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{1}{2} \right) \left[-\cot \left(\frac{\pi}{8} + \frac{x}{2} \right) \right] + c$$

$$= -\frac{1}{\sqrt{2}} \cot \left(\frac{\pi}{8} + \frac{x}{2} \right) + c$$

જવાબ : (D)

$$(49) \int \frac{\operatorname{cosec} x}{\cos^2 \left(1 + \log \tan \frac{x}{2}\right)} dx = \dots$$

$$(A) \sec^2 \left[1 + \log \left(\tan \frac{x}{2}\right)\right] + c$$

$$(B) -\tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

$$(C) \sin^2 \left[1 + \log \left(\tan \frac{x}{2}\right)\right] + c$$

$$(D) \tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

$$\text{ઉકેલ : } I = \int \frac{\operatorname{cosec} x}{\cos^2 \left(1 + \log \tan \frac{x}{2}\right)} dx$$

$$\text{ધારીએ કે, } 1 + \log \tan \frac{x}{2} = u. \text{ આથી, } \frac{1}{\tan \frac{x}{2}} \cdot \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = du$$

$$\therefore \operatorname{cosec} x dx = du$$

$$I = \int \frac{1}{\cos^2 u} du$$

$$= \int \sec^2 u du = \tan u + c = \tan \left(1 + \log \tan \frac{x}{2}\right) + c$$

જવાબ : (D)

$$(50) \int \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x dx = \dots$$

$$(A) -3(\tan x)^{\frac{-1}{3}} + c$$

$$(B) 3(\tan x)^{\frac{-1}{3}} + c$$

$$(C) -3(\tan x)^{\frac{1}{3}} + c$$

$$(D) (\tan x)^{\frac{-1}{3}} + c$$

$$\text{ઉકેલ : } I = \int \sec^{\frac{2}{3}} x \cdot \operatorname{cosec}^{\frac{4}{3}} x dx$$

$$= \int \frac{1}{\cos^{\frac{2}{3}} x \cdot \sin^{\frac{4}{3}} x} dx$$

$$= \int \frac{\left(\frac{1}{\cos^2 x}\right)}{\frac{\cos^{\frac{2}{3}} x}{\cos^2 x} \cdot \sin^{\frac{4}{3}} x} dx$$

$$= \int \frac{\sec^2 x}{\tan^{\frac{4}{3}} x} dx$$

$$= \int t^{-\frac{4}{3}} \cdot dt.$$

$$(\text{ધારીએ કે, } \tan x = t. \text{ આથી, } \sec^2 x dx = dt)$$

$$= \frac{t^{-\frac{1}{3}}}{-\frac{1}{3}} + c$$

$$= -3(\tan x)^{-\frac{1}{3}} + c$$

જવાબ : (A)

$$(51) \int \cos^{-\frac{3}{7}} x \cdot \sin^{-\frac{11}{7}} x dx = \dots$$

$$(A) \log |\sin^{\frac{4}{7}} x| + c$$

$$(B) \frac{-7}{4} \tan^{\frac{4}{7}} x + c$$

$$(C) \frac{-7}{4} \cot^{\frac{4}{7}} x + c$$

$$(D) \log |\cos^{\frac{3}{7}} x| + c$$

$$\text{ઉકેલ : } I = \int \cos^{\frac{-3}{7}} x \cdot \sin^{\frac{-11}{7}} x \, dx$$

$$= \int \frac{1}{\sin^{\frac{11}{7}} x \cdot \cos^{\frac{3}{7}} x} \, dx$$

$$= \int \frac{dx}{\tan^{\frac{11}{7}} x \cdot \cos^{\frac{3}{7}} x \cdot \cos^{\frac{11}{7}} x} \quad (\text{અંશ અને છેદને } \cos^{\frac{11}{7}} x \text{ વડે ભાગતાં})$$

$$= \int \frac{\sec^2 x}{(\tan x)^{\frac{11}{7}}} \, dx$$

$$= \frac{(\tan x)^{\frac{-11}{7}+1}}{\frac{-11}{7}+1} + c = \frac{-7}{4} (\tan x)^{\frac{-4}{7}} + c = \frac{-7}{4} (\cot x)^{\frac{4}{7}} + c$$

જવાબ : (C)

$$(52) \int \frac{(1+x)e^x}{\cos^2(xe^x)} \, dx = f(x) + c, \text{ તો } f(x) = \dots$$

- (A) $\cos(x \cdot e^x)$ (B) $2\tan^{-1}x$ (C) $\tan(x \cdot e^x)$ (D) $\sin(x \cdot e^x)$

$$\text{ઉકેલ : } I = \int \frac{(1+x)e^x}{\cos^2(xe^x)} \, dx$$

$$\text{ધારો કે, } x \cdot e^x = t$$

$$\therefore (x \cdot e^x + e^x) \, dx = dt. \text{ આથી, } (x+1)e^x \, dx = dt$$

$$I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt = \tan t + c = \tan(x \cdot e^x) + c$$

$$\therefore f(x) = \tan(x \cdot e^x)$$

જવાબ : (C)

$$(53) \int x \cdot (x^x)^x \cdot (2 \log x + 1) \, dx = \dots$$

- (A) $(x^x)^x + c$ (B) $\log(x)^x + c$ (C) $x^x + c$ (D) $x(x)^x + c$

$$\text{ઉકેલ : } I = \int x \cdot (x^x)^x \cdot (2 \log x + 1) \, dx$$

$$\text{ધારો કે, } (x^x)^x = t. \text{ આથી, } x^{x^2} = t$$

$$\therefore x^2 \cdot \log x = \log t$$

$$\therefore (x^2 \cdot \frac{1}{x} + \log x \cdot 2x) \, dx = \frac{1}{t} \, dt$$

$$x(1 + 2 \log x) \cdot (x^x)^x \, dx = dt$$

$$\therefore I = \int dt = t + c = (x^x)^x + c$$

જવાબ : (A)

$$(54) \int \frac{\sin x}{\sin(x - \frac{\pi}{4})} \, dx = \dots$$

$$(A) \frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c$$

$$(B) \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c$$

$$(C) \frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}} \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + c$$

$$(D) \frac{x}{2} + \frac{1}{\sqrt{2}} \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + c$$

$$\text{ઉકેલ : } \int \frac{\sin x}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{\sin \left[\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\right]}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{\sin \left(x - \frac{\pi}{4}\right) \cos \frac{\pi}{4} + \cos \left(x - \frac{\pi}{4}\right) \sin \frac{\pi}{4}}{\sin \left(x - \frac{\pi}{4}\right)} dx$$

$$= \int \frac{1}{\sqrt{2}} dx + \frac{1}{\sqrt{2}} \int \cot \left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \log \left| \sin \left(x - \frac{\pi}{4}\right) \right| + c$$

જવાબ : (B)

$$(55) \int \frac{\sin 2x}{\sin 5x \sin 3x} dx = \dots\dots$$

$$(A) \quad 3 \log |\sin 3x| - 5 \log |\sin 5x| + c$$

$$(B) \quad \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + c$$

$$(C) \quad \frac{1}{3} \log |\sin 3x| + \frac{1}{5} \log |\sin 5x| + c$$

$$(D) \quad \log |\sin 3x| - \log |\sin 5x| + c$$

$$\text{ઉકેલ : } \int \frac{\sin 2x}{\sin 5x \sin 3x} dx$$

$$= \int \frac{\sin (5x - 3x)}{\sin 5x \sin 3x} dx$$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$$

$$= \int (\cot 3x - \cot 5x) dx$$

$$= \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + c$$

જવાબ : (B)

$$(56) \int (x+1)(x+2)^7(x+3) dx = \dots\dots$$

$$(A) \quad \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c$$

$$(B) \quad \frac{(x+1)^2}{2} - \frac{(x+2)^8}{8} - \frac{(x+3)^2}{2} + c$$

$$(C) \quad \frac{(x+2)^{10}}{10} + c$$

$$(D) \quad \frac{(x+1)^2}{2} + \frac{(x+2)^8}{8} + \frac{(x+3)^2}{2} + c$$

$$\text{ઉકેલ : } \int (x+1)(x+2)^7(x+3) dx$$

$$\text{ધારીએ કે, } x+2 = t. \text{ આથી, } dx = dt$$

$$I = \int (t-1)(t^7)(t+1) dt$$

$$\begin{aligned}
&= \int (t^2 - 1) t^7 dt \\
&= \int (t^9 - t^7) dt \\
&= \frac{t^{10}}{10} - \frac{t^8}{8} + c \\
&= \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c
\end{aligned}$$

જવાબ : (A)

$$(57) \int \frac{dx}{\cos(x-\alpha) \cos(x-\beta)} = \dots$$

- (A) $\frac{1}{\cos(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c$ (B) $\frac{1}{\cos(\alpha-\beta)} \log \left| \frac{\sin(x-\alpha)}{\sin(x-\beta)} \right| + c$
- (C) $\frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c$ (D) $\frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\sin(x-\alpha)}{\sin(x-\beta)} \right| + c$

$$\text{ઉદા : } I = \int \frac{dx}{\cos(x-\alpha) \cos(x-\beta)}$$

$$\begin{aligned}
&= \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin[(x-\beta)-(x-\alpha)]}{\cos(x-\alpha) \cos(x-\beta)} dx \\
&= \frac{1}{\sin(\alpha-\beta)} \int \frac{\sin(x-\beta) \cos(x-\alpha) - \cos(x-\beta) \sin(x-\alpha)}{\cos(x-\alpha) \cos(x-\beta)} dx \\
&= \frac{1}{\sin(\alpha-\beta)} \int [\tan(x-\beta) - \tan(x-\alpha)] dx \\
&= \frac{1}{\sin(\alpha-\beta)} \int [\log |\sec(x-\beta)| - \log |\sec(x-\alpha)|] + c \\
&= \frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\sec(x-\beta)}{\sec(x-\alpha)} \right| + c \\
&= \frac{1}{\sin(\alpha-\beta)} \log \left| \frac{\cos(x-\alpha)}{\cos(x-\beta)} \right| + c
\end{aligned}$$

જવાબ : (C)

$$(58) \int \operatorname{cosec} \left(x - \frac{\pi}{6}\right) \operatorname{cosec} \left(x - \frac{\pi}{3}\right) dx = \dots$$

- (A) $2 \log \left| \frac{\sin \left(x - \frac{\pi}{3}\right)}{\cos \left(x - \frac{\pi}{6}\right)} \right| + c$ (B) $2 \log \left| \frac{\cos \left(x - \frac{\pi}{3}\right)}{\cos \left(x - \frac{\pi}{6}\right)} \right| + c$
- (C) $2 \log \left| \frac{\sin \left(x - \frac{\pi}{3}\right)}{\sin \left(x - \frac{\pi}{6}\right)} \right| + c$ (D) $2 \log \left| \frac{\cos \left(x - \frac{\pi}{6}\right)}{\cos \left(x - \frac{\pi}{3}\right)} \right| + c$

$$\text{ઉકેલ : } I = \int \operatorname{cosec} \left(x - \frac{\pi}{6} \right) \operatorname{cosec} \left(x - \frac{\pi}{3} \right) dx$$

$$= \int \frac{1}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\sin \left(\frac{\pi}{6} \right)} \int \frac{\sin \left[\left(x - \frac{\pi}{6} \right) - \left(x - \frac{\pi}{3} \right) \right]}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\sin \frac{\pi}{6}} \int \frac{\sin \left(x - \frac{\pi}{6} \right) \cos \left(x - \frac{\pi}{3} \right) - \cos \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{\left(\frac{1}{2} \right)} \int \left[\cot \left(x - \frac{\pi}{3} \right) - \cot \left(x - \frac{\pi}{6} \right) \right] dx$$

$$= 2 \left[\log \left| \sin \left(x - \frac{\pi}{3} \right) \right| - \log \left| \sin \left(x - \frac{\pi}{6} \right) \right| \right] + c$$

$$= 2 \log \left| \frac{\sin \left(x - \frac{\pi}{3} \right)}{\sin \left(x - \frac{\pi}{6} \right)} \right| + c$$

જવાબ : (C)

$$(59) \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \dots$$

$$(A) \sqrt{2} \sin^{-1} (\sin x - \cos x) + c$$

$$(B) \sqrt{2} \sin^{-1} (\sin x + \cos x) + c$$

$$(C) \sin^{-1} (\sin x - \cos x) + c$$

$$(D) \sin^{-1} (\sin x + \cos x) + c \quad (\text{IIT : 1988})$$

$$\text{ઉકેલ : } I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{(\sin x + \cos x)}{\sqrt{2 \sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$\text{ધારો કે, } \sin x - \cos x = t \quad \text{અથવા } (\cos x + \sin x) dx = dt$$

$$I = \sqrt{2} \int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \sqrt{2} \sin^{-1} (t) + c$$

$$= \sqrt{2} \sin^{-1} (\sin x - \cos x) + c$$

જવાબ : (A)

$$(60) \int \tan x \tan 2x \tan 3x \, dx = \dots$$

$$(A) \log \left| \frac{\cos x \sqrt{\cos 2x}}{\sqrt[3]{\cos 3x}} \right| + c$$

$$(B) \log \left| \frac{3\sqrt{\cos 3x}}{\cos x \sqrt{\cos 2x}} \right| + c$$

$$(C) \log \left| \frac{\sin x \sqrt{\sin 2x}}{\sqrt[3]{\sin 3x}} \right| + c$$

$$(D) \log \left| \frac{3\sqrt{\sin 3x}}{\sin x \sqrt{\sin 2x}} \right| + c$$

$$\text{ଓଡ଼ିଆ : } \tan 3x = \tan (2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$\therefore \tan 3x (1 - \tan 2x \tan x) = \tan 2x + \tan x$$

$$\therefore \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

$$\therefore I = \int (\tan 3x \tan 2x \tan x) \, dx$$

$$= \int (\tan 3x - \tan 2x - \tan x) \, dx$$

$$= \frac{-1}{3} \log |\cos 3x| + \frac{1}{2} \log |\cos 2x| + \log |\cos x| + c$$

$$= \log \left| \frac{\cos x \sqrt{\cos 2x}}{\sqrt[3]{\cos 3x}} \right| + c$$

ଞ୍ଜାମ : (A)

$$(61) \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \cdot \frac{dx}{x} = \dots$$

$$(A) \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - \sin^{-1} \sqrt{x} + c$$

$$(B) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c$$

$$(C) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - \cos^{-1} \sqrt{x} + c$$

$$(D) 2 \log \left| \frac{1-\sqrt{1-x}}{\sqrt{x}} \right| - 2 \cos^{-1} \sqrt{x} + c$$

(IIT : 1997)

$$\text{ଓଡ଼ିଆ : } I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \cdot \frac{dx}{x}$$

$$\text{ଉତ୍ତର : } x = \sin^2 \theta$$

$$(0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = 2 \sin \theta \cos \theta \, d\theta$$

$$I = \int \sqrt{\frac{1-\sin \theta}{1+\sin \theta}} \times \frac{2 \sin \theta \cos \theta \, d\theta}{\sin^2 \theta}$$

$$= 2 \int \sqrt{\frac{1-\sin \theta}{1+\sin \theta}} \times \frac{1-\sin \theta}{1-\sin \theta} \times \frac{\cos \theta}{\sin \theta} \, d\theta$$

$$= 2 \int \frac{(1-\sin \theta)}{\sin \theta} \, d\theta$$

$$(0 < \theta < \frac{\pi}{2})$$

$$\begin{aligned}
&= 2 \int (\operatorname{cosec} \theta - 1) d\theta \\
&= 2 (\log |\operatorname{cosec} \theta - \cot \theta| - \theta) + c \\
&= 2 \log \left| \frac{1}{\sqrt{x}} - \frac{\sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c \\
&= 2 \log \left| \frac{1 - \sqrt{1-x}}{\sqrt{x}} \right| - 2 \sin^{-1} \sqrt{x} + c
\end{aligned}$$

જવાબ : (B)

(62) જો $\int \frac{x^2 dx}{\sqrt{1-x^2}} = P \cos^{-1} x + Qx\sqrt{1-x^2} + c$, તો $P + Q = \dots$

- (A) 0 (B) $\frac{-1}{2}$ (C) $\frac{1}{2}$ (D) -1

ઉકેલ : $I = \int \frac{x^2 dx}{\sqrt{1-x^2}}$

ધારો કે, $x = \sin \theta$ ($0 < \theta < \frac{\pi}{2}$)

$\therefore dx = \cos \theta d\theta$

$I = \frac{\sin^2 \theta}{\cos \theta} \times \cos \theta d\theta$

$= \int \sin^2 \theta d\theta$

$= \int \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$

$= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] + c'$

$= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + c'$

$= \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + c'$

$= -\frac{1}{2} \cos^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + c$ ($\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$, $c' + \frac{\pi}{4} = c$)

$P = -\frac{1}{2}, Q = -\frac{1}{2}$

$\therefore P + Q = \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1$

જવાબ : (D)

(63) $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx = \dots$

- (A) $e^x \sqrt{\frac{1+x}{1-x}} + c$ (B) $\frac{e^x}{\sqrt{1-x}} + c$ (C) $\sqrt{\frac{1+x}{1-x^2}} + c$ (D) $\frac{x e^x}{\sqrt{1-x^2}} + c$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx \\
&= \int \frac{e^x[(1-x^2)+1]}{(1-x)\sqrt{1-x^2}} dx \\
&= \int \left[\frac{e^x(1-x^2)}{(1-x)\sqrt{1-x^2}} + \frac{e^x}{(1-x)\sqrt{1-x^2}} \right] dx \\
&= \int e^x \left\{ \frac{\sqrt{1-x^2}}{1-x} + \frac{1}{(1-x)\sqrt{1-x^2}} \right\} dx \\
&= \int e^x \left\{ \frac{\sqrt{1+x}}{\sqrt{1-x}} + \frac{d}{dx} \left(\frac{\sqrt{1+x}}{\sqrt{1-x}} \right) \right\} dx \\
&= e^x \sqrt{\frac{1+x}{1-x}} + c
\end{aligned}$$

જવાબ : (A)

$$(64) \int \frac{(\log x - 1)^2}{(1 + (\log x)^2)^2} dx = \dots$$

$$(A) \frac{xe^x}{1+x^2} + c \quad (B) \frac{\log x}{1+(\log x)^2} + c \quad (C) \frac{x}{x^2+1} + c \quad (D) \frac{x}{1+(\log x)^2} + c$$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{(\log x - 1)^2}{(1 + (\log x)^2)^2} dx \\
&\text{ધારે } \log x = t. \text{ તો, } x = e^t. \text{ તો, } dx = e^t dt \\
&= \int \left(\frac{t-1}{1+t^2} \right)^2 e^t dt \\
&= \int \frac{t^2 - 2t + 1}{(1+t^2)^2} e^t dt \\
&= \int \left[\frac{t^2+1}{(t^2+1)^2} - \frac{2t}{(t^2+1)^2} \right] e^t dt \\
&= \int \left[\frac{1}{t^2+1} - \frac{2t}{(1+t^2)^2} \right] e^t dt \\
&= \int \left\{ \frac{1}{t^2+1} + \frac{d}{dx} \frac{1}{(t^2+1)} \right\} e^t dt \\
&= \frac{e^t}{t^2+1} + c = \frac{x}{1+(\log x)^2} + c
\end{aligned}$$

જવાબ : (D)

$$(65) \quad \int \frac{xe^x}{\sqrt{1+e^x}} dx = (Px + Q)\sqrt{1+e^x} + R \log \left(\frac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1} \right) + c, \text{ ଯେଉଁ } P + Q + R = \dots$$

(A) 2

(B) -2

(C) 0

(D) 4

$$\text{ଉତ୍ତର : } I = \int \frac{xe^x}{\sqrt{1+e^x}} dx$$

$$\text{ଧାର୍ଯ୍ୟ କରୁଛୁ } 1 + e^x = t^2$$

$$\therefore e^x dx = 2t dt$$

$$\text{କାରଣ, } e^x = t^2 - 1$$

$$x = \log_e (t^2 - 1)$$

$$I = \int \frac{[\log(t^2 - 1)] 2t}{t} dt$$

$$= 2 \int \log(t^2 - 1) dt$$

$$= 2 \left[\log(t^2 - 1) \int 1 dt - \int \left(\frac{d}{dt} \log(t^2 - 1) \int 1 dt \right) dt \right]$$

$$= 2 \left[t \log(t^2 - 1) - 2 \int \frac{t^2}{t^2 - 1} dt \right]$$

$$= 2t \log(t^2 - 1) - 4 \int \left(1 + \frac{1}{t^2 - 1} \right) dt$$

$$= 2t \log(t^2 - 1) - 4t - 4 \times \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + c$$

$$= 2\sqrt{1+e^x} \cdot x - 4\sqrt{e^x+1} - 2 \log \left| \frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} \right| + c$$

$$= \sqrt{1+e^x} (2x - 4) + 2 \log \left| \frac{\sqrt{e^x+1}+1}{\sqrt{e^x+1}-1} \right| + c$$

$$\therefore P = 2, Q = -4, R = 2$$

$$\therefore P + Q + R = 0$$

ଠିକ୍ ଉତ୍ତର : (C)

$$(66) \quad \int \tan^{-1} \sqrt{x} dx = \dots$$

$$(A) \quad x \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

$$(B) \quad (x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

$$(C) \quad \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

$$(D) \quad x \tan^{-1} \sqrt{x} + \sqrt{x} + c$$

ઉકેલ : $I = \int \tan^{-1} \sqrt{x} \, dx$

ધારો કે $\theta = \tan^{-1} \sqrt{x}$ એટલે કે $x = \tan^2 \theta$ ($0 < \theta < \frac{\pi}{2}$)

$\therefore dx = 2 \tan \theta \sec^2 \theta \, d\theta$

$I = 2 \int \theta \cdot \tan \theta \cdot \sec^2 \theta \, d\theta$

ધારો કે $\tan \theta = t$ ($0 < \theta < \frac{\pi}{2} \Rightarrow t > 0$)

$\therefore \sec^2 \theta \, d\theta = dt$

$I = 2 \int t \cdot \tan^{-1} t \, dt$

$= 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{t^2 + 1} \, dt \right]$

$= t^2 \tan^{-1} t - \int \frac{(t^2 + 1) - (1)}{t^2 + 1} \, dt$

$= t^2 \tan^{-1} t - t + \tan^{-1} t + c$

$= (t^2 + 1) \tan^{-1} t - t + c$

$= \theta \sec^2 \theta - \tan \theta + c$

$= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$

જવાબ : (B)

(67) જો $I = \int \frac{e^x \, dx}{e^{4x} + e^{2x} + 1}$, $J = \int \frac{e^{-x} \, dx}{e^{-4x} + e^{-2x} + 1}$, તો $J - I = \dots$

(A) $\frac{1}{2} \log \left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right) + c$

(B) $\frac{1}{2} \log \left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right) + c$

(C) $\frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + c$

(D) $\frac{1}{2} \log \left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1} \right) + c$ (IIT : 2008)

ઉકેલ : $J - I = \int \left(\frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx$

$= \int \frac{\frac{1}{e^x}}{\frac{1}{e^{4x}} + \frac{1}{e^{2x}} + 1} - \int \frac{e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{e^{3x} \, dx}{1 + e^{2x} + e^{4x}} - \int \frac{e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1} \, dx$

$= \int \frac{(e^{2x} - 1) e^x}{e^{4x} + e^{2x} + 1} \, dx$ (ધારો કે $e^x = t$. આથી, $e^x \, dx = dt$, $t > 0$)

$$= \int \frac{t^2 - 1}{t^4 + t^2 + 1} dt$$

$$= \int \frac{\left(1 - \frac{1}{t^2}\right)}{t^2 + 1 + \frac{1}{t^2}} dt$$

ધારો કે $t + \frac{1}{t} = u$ આથી, $\left(1 - \frac{1}{t^2}\right) dt = du$

$$\therefore J - I = \int \frac{du}{u^2 - 1}$$

$$= \frac{1}{2} \log \left(\frac{u-1}{u+1} \right) + c \quad (u > 1)$$

$$= \frac{1}{2} \log \left(\frac{t^2 - t + 1}{t^2 + t + 1} \right) + c$$

$$= \frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + c$$

જવાબ : (C)

(68) જો $f(t) = \int \frac{t^2 dt}{(1+t^2)(1+\sqrt{1+t^2})}$ અને $f(0) = 0$, તો $f(1) = \dots$

(A) $\log(\sqrt{2} + 1)$

(B) $\log(\sqrt{2} + 1) - \frac{\pi}{4}$

(C) $\log(\sqrt{2} + 1) + \frac{\pi}{2}$

(D) $\log(\sqrt{2} + 1) + \frac{\pi}{4}$

ઉકેલ : $f(t) = \int \frac{t^2 dt}{(1+t^2)(1+\sqrt{1+t^2})}$

ધારો કે $t = \tan \theta$ આથી $dt = \sec^2 \theta d\theta$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$f(t) = \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{(1 + \tan^2 \theta)(1 + \sqrt{1 + \tan^2 \theta})}$$

$$= \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^2 \theta (1 + \sec \theta)}$$

$$= \int \frac{\tan^2 \theta d\theta}{1 + \sec \theta}$$

$$= \int \frac{\sin^2 \theta}{\cos \theta (1 + \cos \theta)} d\theta$$

$$= \int \frac{(1 - \cos^2 \theta) d\theta}{\cos \theta (1 + \cos \theta)}$$

$$\begin{aligned}
&= \int \frac{1 - \cos \theta}{\cos \theta} d\theta \\
&= \int (\sec \theta - 1) d\theta \\
&= \log |\sec \theta + \tan \theta| - \theta + c \\
&= \log |\sqrt{1+t^2} + t| - \tan^{-1} t + c \\
f(0) &= \log (\sqrt{1+0} + 0) - \tan^{-1} 0 + c \\
\therefore c &= 0 \\
f(t) &= \log |t + \sqrt{1+t^2}| - \tan^{-1} t \\
f(1) &= \log |1 + \sqrt{2}| - \tan^{-1} 1 \\
f(1) &= \log (\sqrt{2} + 1) - \frac{\pi}{4}
\end{aligned}$$

જવાબ : (B)

$$(69) \int \frac{2 - \sin x}{2 + \cos x} dx = \dots\dots$$

$$(A) \quad \frac{-4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c$$

$$(B) \quad \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c$$

$$(C) \quad \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) - \log (2 + \cos x) + c$$

$$(D) \quad 0$$

$$\text{ઉકેલ : } I = \int \frac{2 - \sin x}{2 + \cos x} dx$$

$$\therefore I = \int \frac{2}{2 + \cos x} dx - \int \frac{\sin x}{2 + \cos x} dx$$

$$\therefore I = I_1 - I_2$$

$$\text{જાણી, } I_1 = \int \frac{2}{2 + \cos x} dx,$$

$$\text{ધારીએ કે } \tan \frac{x}{2} = t. \quad \text{આથી } \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{2}{2 + \left(\frac{1-t^2}{1+t^2} \right)} \cdot \frac{2dt}{1+t^2} \\
&= \int \frac{2(1+t^2)}{2+2t^2+1-t^2} \times \frac{2dt}{1+t^2}
\end{aligned}$$

$$= 4 \int \frac{1}{t^2 + 3} dt$$

$$I_1 = 4 \times \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c_1$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + c_1$$

$$I_2 = \int \frac{\sin x}{2 + \cos x} dx$$

$$= - \int \frac{\frac{d}{dx}(2 + \cos x)}{(2 + \cos x)} dx$$

$$I_2 = - \log |2 + \cos x| + c_2$$

$$\text{ଏଠା, } I = I_1 - I_2$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log |2 + \cos x| + c_1 - c_2$$

$$= \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + \log (2 + \cos x) + c, \quad \text{ଝାଲି, } c_1 - c_2 = c$$

ଝାଲିଆ : (B)

$$(70) \int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx = \dots\dots$$

$$(A) \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(B) \frac{1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(C) \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

$$(D) \frac{1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c$$

(IIT : 2012)

$$\text{ଓଢ଼େଇ : } I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$$

$$\text{ଏଠାରେ ଓଢ଼େଇ } \sec x + \tan x = t$$

$$\therefore (\sec x \tan x + \sec^2 x) dx = dt$$

$$\therefore \sec x (\tan x + \sec x) dx = dt$$

$$\therefore \sec x (t) dx = dt$$

$$\therefore \sec x dx = \frac{1}{t} dt$$

$$\text{ଏଠା, } \sec x + \tan x = t$$

$$\therefore \sec x - \tan x = \frac{1}{t}$$

$$\therefore 2\sec x = t + \frac{1}{t}$$

$$\therefore \sec x = \frac{1}{2} \left(t + \frac{1}{t} \right)$$

$$\begin{aligned}
\therefore I &= \int \frac{\sec x \cdot (\sec x \, dx)}{(\sec x + \tan x)^{\frac{9}{2}}} \\
&= \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right) \cdot dt}{\frac{9}{t^2} \cdot t} \\
&= \frac{1}{2} \int \frac{(t^2 + 1)}{t^2 \cdot t^2} dt \\
&= \frac{1}{2} \int \left(\frac{1}{t^2} + \frac{1}{t^3} \right) dt \\
&= -\frac{1}{2} \left[\frac{2}{7t^2} + \frac{2}{11t^3} \right] \\
&= -\frac{1}{t^2} \left[\frac{t^2}{7} + \frac{1}{11} \right] + c \\
&= \frac{-1}{(\sec x + \tan x)^{\frac{11}{2}}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + c
\end{aligned}$$

જવાબ : (C)

(71) જો $\int \left(\frac{2 \sin \theta + \cos \theta}{7 \sin \theta - 5 \cos \theta} \right) d\theta = A\theta + B \log |7 \sin \theta - 5 \cos \theta| + c$, તો $A + B = \dots$

(A) $\frac{1}{17}$ (B) $\frac{3}{37}$ (C) $\frac{13}{37}$ (D) $\frac{23}{47}$

ઉકેલ : $I = \int \left(\frac{2 \sin \theta + \cos \theta}{7 \sin \theta - 5 \cos \theta} \right) d\theta$

ધારો કે, $2 \sin \theta + \cos \theta = P(7 \sin \theta - 5 \cos \theta) + Q \frac{d}{d\theta} (7 \sin \theta - 5 \cos \theta)$

$$= 7P \sin \theta - 5P \cos \theta + Q (7 \cos \theta + 5 \sin \theta)$$

$$2 \sin \theta + \cos \theta = (7P + 5Q) \sin \theta + (7Q - 5P) \cos \theta$$

$$\therefore 7P + 5Q = 2, \quad 7Q - 5P = 1$$

ઉકેલતાં, $P = \frac{9}{74}, Q = \frac{17}{74}$

$$\begin{aligned}
I &= \int \left[\frac{9}{74} + \frac{17}{74} \left(\frac{7 \cos \theta + 5 \sin \theta}{7 \sin \theta - 5 \cos \theta} \right) \right] d\theta \\
&= \frac{9\theta}{74} + \frac{17}{74} \log |7 \sin \theta - 5 \cos \theta| + c
\end{aligned}$$

$$\therefore A = \frac{9}{74}, B = \frac{17}{74}$$

$$\therefore A + B = \frac{26}{74} = \frac{13}{37}$$

જવાબ : (C)

$$(72) \quad \int \frac{\sin x - 8 \cos x}{2 \sin x - \cos x} dx = \dots$$

$$(A) \quad 2x + 3 \log |2 \sin x - \cos x| + c$$

$$(B) \quad 2x - 3 \log |2 \sin x - \cos x| + c$$

$$(C) \quad x + 3 \log |\sin x - 8 \cos x| + c$$

$$(D) \quad 2x - 3 \log |\sin x - 8 \cos x| + c$$

$$\text{ଓଡ଼ିଆ : } I = \int \frac{\sin x - 8 \cos x}{2 \sin x - \cos x} dx$$

$$\text{ଧରିବା, } \sin x - 8 \cos x = m(2 \sin x - \cos x) + n \left[\frac{d}{dx}(2 \sin x - \cos x) \right]$$

$$\sin x - 8 \cos x = 2m \sin x - m \cos x + 2n \cos x + n \sin x$$

$$\sin x - 8 \cos x = (2m + n) \sin x + (-m + 2n) \cos x$$

$$2m + n = 1, -m + 2n = -8$$

$$\text{ଓଡ଼ିଆ, } m = 2, n = -3$$

$$\therefore I = \int \left[2 - 3 \left(\frac{2 \cos x + \sin x}{2 \sin x - \cos x} \right) \right] dx$$

$$= 2x - 3 \log |2 \sin x - \cos x| + c$$

ଞ୍ଜାଳ : (B)

$$(73) \quad \int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx = \dots$$

$$(A) \quad \frac{x}{(\log x)^2} + x \log (\log x) + c$$

$$(B) \quad \frac{1}{\log x} + \log (\log x) + c$$

$$(C) \quad \log (\log x) - \frac{1}{\log x} + c$$

$$(D) \quad x \log (\log x) - \frac{x}{\log x} + c$$

$$\text{ଓଡ଼ିଆ : } I = \int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx$$

$$\text{ଧରିବା, } \log x = t \quad \text{ଆଉ, } x = e^t. \quad \text{ଆଉ, } dx = e^t dt$$

$$\therefore I = \int \left\{ \log (t) + \frac{1}{t^2} \right\} e^t dt$$

$$= \int \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right) e^t dt$$

$$= \int \left(\log t + \frac{1}{t} \right) e^t dt - \int \left\{ \frac{1}{t} + \left(-\frac{1}{t^2} \right) \right\} e^t dt$$

$$= e^t \log t - e^t \frac{1}{t} + c$$

$$= x \log (\log x) - \frac{x}{\log x} + c$$

ଞ୍ଜାଳ : (D)

$$(74) \int e^{3x} \left(\frac{2 + 3 \sin 2x}{1 + \cos 2x} \right) dx = \dots$$

- (A) $\frac{2}{3} e^{3x} \sin 2x + c$ (B) $\frac{3}{2} e^{3x} \tan x + c$ (C) $e^{3x} \tan x + c$ (D) $2e^{3x} \tan x + c$

$$\begin{aligned} \text{ઉકેલ : } I &= \int e^{3x} \left(\frac{2 + 3 \sin 2x}{1 + \cos 2x} \right) dx \\ &= \int e^{3x} \left(\frac{2 + 6 \sin x \cos x}{2 \cos^2 x} \right) dx \\ &= \int e^{3x} (\sec^2 x + 3 \tan x) dx \\ &= \int e^{3x} (3 \tan x + \sec^2 x) dx \\ &= e^{3x} \tan x + c \end{aligned}$$

$$\left[\int e^{kx} (kf(x) + f'(x)) dx = e^{kx} \cdot f(x) + c \right]$$

જવાબ : (C)

$$(75) \int \frac{e^{-\frac{x}{2}} \sqrt{1 - \sin x}}{(1 + \cos x)} dx = \dots \quad \left(\frac{\pi}{2} < x < \pi \right)$$

- (A) $-e^{-\frac{x}{2}} \tan \frac{x}{2} + c$ (B) $e^{-\frac{x}{2}} \sec \frac{x}{2} + c$
 (C) $-\frac{1}{2} e^{-\frac{x}{2}} \sec \frac{x}{2} + c$ (D) $-2e^{-\frac{x}{2}} \sec \frac{x}{2} + c$

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{e^{-\frac{x}{2}} \sqrt{1 - \sin x}}{(1 + \cos x)} dx \\ &= \int \frac{e^{-\frac{x}{2}} \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{1 + \cos x} dx \end{aligned}$$

$$= \int \frac{e^{-\frac{x}{2}} \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)}{2 \cos^2 \frac{x}{2}} dx$$

$$\left(\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \right)$$

$$= \int e^{-\frac{x}{2}} \left(\frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} - \frac{1}{2} \sec \frac{x}{2} \right) dx$$

$$= \int e^{-\frac{x}{2}} \left(-\frac{1}{2} \sec \frac{x}{2} + \frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} \right) dx$$

$$= e^{-\frac{x}{2}} \sec \frac{x}{2} + c \quad \left[\int e^{kx} (kf(x) + f'(x)) dx = e^{kx} \cdot f(x) + c \right]$$

જવાબ : (B)

$$(76) \quad \int \frac{dx}{\sqrt{2ax - x^2}} = \dots \quad (a > 0)$$

$$(A) \quad \frac{1}{a} \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

$$(B) \quad \frac{1}{a} \log |(x-a) + \sqrt{2ax - x^2}| + c$$

$$(C) \quad \sin^{-1} \left(\frac{x-a}{a} \right) + c$$

$$(D) \quad \log |(x-a) + \sqrt{2ax - x^2}| + c$$

$$\begin{aligned} \text{उत्तर : I} &= \int \frac{dx}{\sqrt{2ax - x^2}} \\ &= \int \frac{dx}{\sqrt{a^2 - a^2 + 2ax - x^2}} \\ &= \int \frac{dx}{\sqrt{(a)^2 - (x-a)^2}} \\ &= \sin^{-1} \left(\frac{x-a}{a} \right) + c \end{aligned}$$

जवाब : (C)

$$(77) \quad \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right) dx = \dots$$

$$(A) \quad 2(x+1) \tan^{-1} \left\{ \frac{2}{3}(x+1) \right\} + \frac{3}{4} \log \left| \sqrt{4 + 9(x+1)^2} \right| + c$$

$$(B) \quad (x+1) \tan^{-1} \left\{ \frac{2}{3}(x+1) \right\} - \frac{3}{2} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c$$

$$(C) \quad (x+1) \tan^{-1} \left\{ \frac{3}{2(x+1)} \right\} + \frac{2}{3} \log \left| \sqrt{4 + 9(x+1)^2} \right| + c$$

$$(D) \quad (x+1) \tan^{-1} \left\{ \frac{3}{2(x+1)} \right\} - \frac{4}{3} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c$$

(IIT : 2000)

$$\begin{aligned} \text{उत्तर : I} &= \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right) dx \\ &= \int \sin^{-1} \left(\frac{x+1}{\sqrt{x^2 + 2x + \frac{13}{4}}} \right) dx \\ &= \int \sin^{-1} \left(\frac{x+1}{\sqrt{(x+1)^2 + \left(\frac{3}{2}\right)^2}} \right) dx \end{aligned}$$

ધારેલે છે, $x + 1 = \frac{3}{2} \tan \theta$. આથી, $dx = \frac{3}{2} \sec^2 \theta \, d\theta$

$$\begin{aligned}
 I &= \int \sin^{-1} \left(\frac{\frac{3}{2} \tan \theta}{\frac{3}{2} \sec \theta} \right) \frac{3}{2} \sec^2 \theta \, d\theta \\
 &= \int \sin^{-1} (\sin \theta) \cdot \frac{3}{2} \sec^2 \theta \, d\theta, & (-\frac{\pi}{2} < \theta < \frac{\pi}{2}) \\
 &= \frac{3}{2} \int \theta \cdot \sec^2 \theta \, d\theta \\
 &= \frac{3}{2} [\theta \tan \theta - \int 1 \cdot \tan \theta \, d\theta] + c' \\
 &= \frac{3}{2} [\theta \tan \theta - \log | \sec \theta |] + c' \\
 &= \frac{3}{2} \left[\tan^{-1} \left\{ \frac{2}{3} (x+1) \right\} \cdot \frac{2}{3} (x+1) - \log \left| \sqrt{1 + \left\{ \frac{2}{3} (x+1) \right\}^2} \right| \right] + c' \\
 &= (x+1) \tan^{-1} \left\{ \frac{2}{3} (x+1) \right\} - \frac{3}{2} \log \left| \sqrt{9 + 4(x+1)^2} \right| + c \quad \text{જ્યાં, } c = c' + \frac{3}{2} \log 3
 \end{aligned}$$

જવાબ : (B)

(78) $\int x \sin^{-1} \left(\sqrt{\frac{2a-x}{4a}} \right) dx = \dots\dots$ $(a > 0)$

(A) $\left(\frac{x^2 - 2a^2}{4} \right) \cos^{-1} \left(\frac{x}{2a} \right) - \frac{x}{8} \sqrt{4a^2 - x^2} + c$

(B) $\left(\frac{x^2 - 2a^2}{2} \right) \cos^{-1} \left(\frac{x}{2a} \right) + \frac{x}{4} \sqrt{2a^2 - x^2} + c$

(C) $\left(\frac{x^2 - 2a^2}{4} \right) \sin^{-1} \left(\frac{x}{2a} \right) - \frac{x}{2} \sqrt{4a^2 - x^2} + c$

(D) $\left(\frac{x^2 - 4a^2}{2} \right) \sin^{-1} \left(\frac{x}{2a} \right) + \frac{x}{8} \sqrt{2a^2 - x^2} + c$

ઉકેલ : $I = \int x \sin^{-1} \left(\sqrt{\frac{2a-x}{4a}} \right) dx$ $(a > 0)$

ધારેલે છે, $x = 2a \cos \theta$. આથી, $dx = -2a \sin \theta \, d\theta$ $(0 < \theta < \pi)$

$$\sqrt{\frac{2a-x}{4a}} = \sqrt{\frac{2a-2a \cos \theta}{4a}} = \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2}} = \sin \frac{\theta}{2} \quad (0 < \frac{\theta}{2} < \frac{\pi}{2})$$

$$\begin{aligned}
 \therefore I &= \int 2a \cos \theta \cdot \sin^{-1} \left(\sin \frac{\theta}{2} \right) (-2a \sin \theta) \, d\theta \\
 &= -a^2 \int \theta \cdot \sin 2\theta \, d\theta \\
 &= -a^2 \left[\theta \cdot \left(\frac{-\cos 2\theta}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2\theta}{2} \right) d\theta \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{a^2}{2} \left[\theta \cdot \cos 2\theta - \frac{\sin 2\theta}{2} \right] + c \\
&= \frac{a^2}{2} \left[\cos^{-1} \left(\frac{x}{2a} \right) \left(\frac{x^2 - 2a^2}{2a^2} \right) - \frac{x}{4a^2} \sqrt{4a^2 - x^2} \right] + c \\
&\left(\begin{array}{l} \because \cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{x}{2a} \right)^2 - \left(1 - \frac{x^2}{4a^2} \right) = \frac{x^2 - 2a^2}{2a^2} \\ \text{અને } \sin 2\theta = 2 \sin \theta \cdot \cos \theta = 2 \sqrt{1 - \frac{x^2}{4a^2}} \cdot \frac{x}{2a} = \frac{x}{2a^2} \sqrt{4a^2 - x^2} \end{array} \right) \\
&= \left(\frac{x^2 - 2a^2}{4} \right) \cos^{-1} \left(\frac{x}{2a} \right) - \frac{x}{8} \sqrt{4a^2 - x^2} + c \quad \text{જવાબ : (A)}
\end{aligned}$$

$$(79) \quad \int \cos 2\theta \log \left| \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right| d\theta = \dots$$

$$(A) \quad \frac{\cos 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| - \frac{1}{2} \log |\sin 2\theta| + c$$

$$(B) \quad \frac{\cos 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + \frac{1}{2} \log |\sin 2\theta| + c$$

$$(C) \quad \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + \frac{1}{2} \log |\cos 2\theta| + c$$

$$(D) \quad \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| - \frac{1}{2} \log |\cos 2\theta| + c \quad (\text{IIT : 1994})$$

$$\begin{aligned}
\text{ઉત્તર : } I &= \int \cos 2\theta \log \left| \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right| d\theta \\
&= \int \cos 2\theta \cdot \log \left| \frac{1 + \tan \theta}{1 - \tan \theta} \right| d\theta \\
&= \int \cos 2\theta \cdot \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| d\theta \\
&= \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| \frac{\sin 2\theta}{2} - \int 2 \sec 2\theta \cdot \left(\frac{\sin 2\theta}{2} \right) d\theta \\
&\left[\int \sec \theta d\theta = \log \left| \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right| + c \right. \\
&\quad \left. \int \sec 2\theta d\theta = \frac{1}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + c \right. \\
&\therefore \frac{d}{d\theta} \left\{ \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| + c \right\} = 2 \sec 2\theta \left. \right] \\
&\therefore I = \frac{\sin 2\theta}{2} \log \left| \tan \left(\frac{\pi}{4} + \theta \right) \right| - \int \tan 2\theta d\theta \\
&= \frac{1}{2} \sin 2\theta \log \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\} + \frac{1}{2} \log |\cos 2\theta| + c \quad \text{જવાબ : (C)}
\end{aligned}$$

$$(80) \int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx = \dots$$

$$(A) e^{\sin x} (x - \cos x) + c$$

$$(B) e^{\sin x} (2x + \cot x) + c$$

$$(C) e^{\sin x} (x - \sec x) + c$$

$$(D) e^{\sin x} (2x + \tan x) + c$$

$$\text{ઉકેલ : } I = \int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx$$

$$= \int e^{\sin x} (x \cos x - \sec x \tan x) dx$$

$$= \int e^{\sin x} x \cos x dx - \int e^{\sin x} \sec x \tan x dx$$

$$I = I_1 - I_2$$

$$\text{હવે, } I_1 = \int x \cdot (e^{\sin x} \cos x) dx$$

$$= x \int e^{\sin x} \cos x dx - \int (1 \cdot \int e^{\sin x} \cdot \cos x dx) dx$$

$$= x e^{\sin x} - \int e^{\sin x} dx + c_1 \quad \left(\int e^{\sin x} \cos x dx = e^{\sin x} \right)$$

$$\text{અને } I_2 = \int e^{\sin x} (\sec x \tan x) dx$$

$$= e^{\sin x} \cdot \sec x - \int (e^{\sin x} \cos x \sec x) dx$$

$$I_2 = e^{\sin x} \cdot \sec x - \int e^{\sin x} dx + c_2$$

$$I = I_1 - I_2$$

$$= (x \cdot e^{\sin x} - \int e^{\sin x} dx + c_1 - e^{\sin x} \cdot \sec x + \int e^{\sin x} dx - c_2)$$

$$= x e^{\sin x} - e^{\sin x} \sec x + c \quad (c_1 - c_2 = c)$$

$$= e^{\sin x} (x - \sec x) + c$$

જવાબ : (C)

$$(81) \text{ જો } I = \int \sin^{-1} x dx \text{ અને } J = \int \sin^{-1} \sqrt{1-x^2} dx, \text{ તો } \dots \quad (x > 0)$$

$$(A) J = \frac{\pi}{2} I$$

$$(B) I + J = \frac{\pi}{2}$$

$$(C) I + J = \frac{\pi}{2} x$$

$$(D) I - J = \frac{\pi}{2} x$$

$$\text{ઉકેલ : } J = \int \sin^{-1} \sqrt{1-x^2} dx = \int \cos^{-1} x dx$$

$$\therefore I + J = \int (\sin^{-1} x + \cos^{-1} x) dx = \frac{\pi}{2} \int dx = \frac{\pi}{2} x$$

જવાબ : (C)

નોંધ : જો $x < 0$ તો $I - J = \frac{\pi}{2} x$. જો શરત ની અપેક્ષા હોય, તો જવાબ (C), (D)

$$(82) \int \frac{dt}{t^n(1+t^n)^{\frac{1}{n}}} = a \left(\frac{1}{t^n} + 1 \right)^b + c, \text{ તો } a \cdot b = \dots$$

$$(A) 1$$

$$(B) -1$$

$$(C) \frac{1}{n}$$

$$(D) -\frac{1}{n}$$

(IIT : 2007)

$$\begin{aligned}\text{ઉકેલ : I} &= \int \frac{dt}{t^n(1+t^n)^{\frac{1}{n}}} \\ &= \int \frac{dt}{t^{n+1}\left(\frac{1}{t^n}+1\right)^{\frac{1}{n}}}\end{aligned}$$

$$\text{ધારીએ, } \frac{1}{t^n} + 1 = x$$

$$\therefore -n \cdot t^{-n-1} dt = dx$$

$$-n \left(\frac{1}{t^{n+1}} \right) dt = dx$$

$$\frac{dt}{t^{n+1}} = \frac{-1}{n} dx$$

$$\therefore I = \frac{-1}{n} \int x^{\frac{-1}{n}} dx$$

$$= \frac{-1}{n} \frac{x^{-\frac{1}{n}+1}}{-\frac{1}{n}+1} + c$$

$$= \frac{1}{1-n} x^{1-\frac{1}{n}} + c$$

$$= \frac{1}{1-n} \left(\frac{1}{t^n} + 1 \right)^{\frac{n-1}{n}} + c$$

$$\therefore a = \frac{1}{1-n}, \quad b = \frac{n-1}{n}$$

$$\therefore a \cdot b = \left(\frac{1}{1-n} \right) \left(\frac{n-1}{n} \right) = -\frac{1}{n}$$

$$\therefore ab = -\frac{1}{n}$$

જવાબ : (D)

$$(83) \quad \text{જો } \int \frac{(1+x^n)^{\frac{1}{n}}}{x^{n+2}} dx = a \left(1 + \frac{1}{x^n} \right)^b + c, \text{ તો } a + b = \dots \quad (\text{જ્યાં } n = 4)$$

$$(A) \quad \frac{6}{5}$$

$$(B) \quad \frac{16}{15}$$

$$(C) \quad \frac{21}{20}$$

$$(D) \quad \frac{11}{10}$$

$$\text{ઉકેલ : I} = \int \frac{(1+x^n)^{\frac{1}{n}}}{x^{n+2}} dx$$

$$= \int \frac{\left(\frac{1}{x^n} + 1 \right)^{\frac{1}{n}}}{x^{n+1}} dx$$

ધારો કે, $1 + \frac{1}{x^n} = t$

$$-n \cdot x^{-n-1} dx = dt$$

$$\frac{1}{x^{n+1}} dx = \frac{-1}{n} dt$$

$$I = \frac{-1}{n} \int t^{\frac{1}{n}} dt$$

$$= \frac{-1}{n} \frac{t^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c$$

$$= \frac{-1}{n+1} (t)^{\frac{n+1}{n}} + c$$

$$= \frac{-1}{n+1} \left(1 + \frac{1}{x^n}\right)^{\frac{n+1}{n}} + c$$

$$\therefore n = 4, a = \frac{-1}{n+1} = \frac{-1}{5}, b = \frac{n+1}{n} = \frac{5}{4}$$

$$\therefore a + b = \frac{21}{20}$$

જવાબ : (C)

(84) જો $\int \left(x + \sqrt{x^2 + 1}\right)^{-5} dx = P \left(x + \sqrt{x^2 + 1}\right)^{-4} + Q \left(x + \sqrt{x^2 + 1}\right)^{-6} + c$, તો $P + Q = \dots$

(A) $\frac{-5}{3}$

(B) $\frac{-5}{12}$

(C) $\frac{-5}{6}$

(D) $\frac{-5}{24}$

ઉકેલ : $I = \int \left(x + \sqrt{x^2 + 1}\right)^{-5} dx$

ધારો કે, $x + \sqrt{x^2 + 1} = t$

$$\frac{1}{t} = \frac{1}{x + \sqrt{x^2 + 1}} \times \frac{x - \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}}$$

$$= \frac{x - \sqrt{x^2 + 1}}{x^2 - x^2 - 1}$$

$$\frac{1}{t} = -x + \sqrt{x^2 + 1}$$

$$t + \frac{1}{t} = 2\sqrt{x^2 + 1}. \text{ આથી, } \sqrt{x^2 + 1} = \frac{1}{2} \left(t + \frac{1}{t}\right)$$

હવે, $x + \sqrt{x^2 + 1} = t$

$$\therefore \left(1 + \frac{2x}{2\sqrt{x^2 + 1}}\right) dx = dt$$

$$\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) dx = dt$$

$$t dx = \sqrt{x^2 + 1} dt$$

$$dx = \frac{\sqrt{x^2 + 1}}{t} dt$$

$$\therefore dx = \frac{1}{2} \left(t + \frac{1}{t} \right) \frac{dt}{t}$$

$$I = \frac{1}{2} \int \left(t + \frac{1}{t} \right) t^{-6} dt$$

$$= \frac{1}{2} \int (t^{-5} - t^{-7}) dt$$

$$= \frac{1}{2} \left[\frac{t^{-4}}{-4} + \frac{t^{-6}}{-6} \right] + c$$

$$= \frac{-1}{8} t^{-4} - \frac{1}{12} t^{-6} + c$$

$$= \frac{-1}{8} (x + \sqrt{x^2 + 1})^{-4} - \frac{1}{12} (x + \sqrt{x^2 + 1})^{-6} + c$$

$$\therefore P = \frac{-1}{8}, Q = \frac{-1}{12}$$

$$\therefore P + Q = \left(\frac{-1}{8} \right) + \left(\frac{-1}{12} \right) = \frac{-5}{24}$$

જવાબ : (D)

$$(85) \int x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx = A \cos^{-1} x^2 + B \sqrt{1-x^4} + C x^2 \sqrt{1-x^4} + D \text{ તો } A + B + C = \dots$$

(A) 0

(B) $\frac{-1}{2}$

(C) $\frac{1}{2}$

(D) -1

ઉકેલ : $I = \int x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx$

ધારો કે, $x^2 = \sin \theta$ ($0 < \theta < \frac{\pi}{2}$)

$$2x dx = \cos \theta d\theta. \text{ આથી, } x dx = \frac{1}{2} \cos \theta d\theta$$

$$I = \int x^3 \frac{(1+x^2)}{\sqrt{1-x^4}} dx$$

$$= \int \frac{x^2(1+x^2)}{\sqrt{1-x^4}} (x dx)$$

$$= \frac{1}{2} \int \frac{\sin \theta (1 + \sin \theta)}{\cos \theta} \cos \theta d\theta$$

$$\begin{aligned}
I &= \frac{1}{2} \int (\sin\theta + \sin^2\theta) d\theta \\
&= \frac{1}{2} \int \left(\sin\theta + \frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{-1}{2} \cos\theta + \frac{1}{4} \int (1 - \cos 2\theta) d\theta \\
&= \frac{-1}{2} \cos\theta + \frac{1}{4} \theta - \frac{\sin 2\theta}{4 \times 2} + c \\
&= \frac{-1}{2} \cos\theta + \frac{\theta}{4} - \frac{\sin\theta \cos\theta}{4} + c \\
&= \frac{-1}{2} \sqrt{1 - x^4} + \frac{1}{4} \sin^{-1}x^2 - \frac{1}{4} x^2 \sqrt{1 - x^4} + c
\end{aligned}$$

$$A = \frac{-1}{4}, B = \frac{-1}{2}, C = \frac{-1}{4} \quad (\sin^{-1}x^2 = \frac{\pi}{2} - \cos^{-1}x^2 \text{ ເປັນ})$$

$$A + B + C = \left(\frac{-1}{4}\right) + \left(\frac{-1}{2}\right) + \left(\frac{-1}{4}\right) = -1 \quad \text{જવાબ : (D)}$$

$$(86) \int \frac{dx}{(4 + 3x^2)\sqrt{3 - 4x^2}} = \dots$$

$$(A) \quad \frac{1}{5} \tan^{-1} \frac{2x}{\sqrt{3 - 4x^2}} + c$$

$$(B) \quad \frac{1}{10} \tan^{-1} \left(\frac{5x}{2\sqrt{3 - 4x^2}} \right) + c$$

$$(C) \quad \frac{1}{5} \tan^{-1} \frac{5x}{2\sqrt{3 - 4x^2}} + c$$

$$(D) \quad \frac{1}{10} \tan^{-1} \frac{5x}{\sqrt{3 - 4x^2}} + c$$

$$\text{ઉકેલ : } I = \int \frac{1}{(4 + 3x^2)\sqrt{3 - 4x^2}} dx$$

$$\text{ધારો કે, } x = \frac{\sqrt{3}}{2} \sin\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = \frac{\sqrt{3}}{2} \cos\theta d\theta$$

$$\therefore I = \int \frac{\frac{\sqrt{3}}{2} \cos\theta d\theta}{\left(4 + \frac{9}{4} \sin^2\theta\right) \sqrt{3 - 3\sin^2\theta}}$$

$$I = 2 \int \frac{d\theta}{9\sin^2\theta + 16}$$

$$= 2 \int \frac{\sec^2\theta d\theta}{16 + 25 \tan^2\theta}$$

$$= 2 \times \int \frac{dt}{(5t)^2 + (4)^2} \quad (\text{ધારો કે, } \tan\theta = t \text{ અને } \sec^2\theta d\theta = dt)$$

$$= 2 \times \frac{1}{5} \times \frac{1}{4} \tan^{-1} \left(\frac{5t}{4} \right) + c$$

$$= \frac{1}{10} \tan^{-1} \left(\frac{5}{4} \tan \theta \right) + c$$

$$= \frac{1}{10} \tan^{-1} \left(\frac{5x}{2\sqrt{3-4x^2}} \right) + c$$

જવાબ : (B)

$$(87) \int \frac{dx}{5+4\cos x} = \dots\dots$$

$$(A) \quad \frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + c$$

$$(B) \quad \frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + c$$

$$(C) \quad \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + c$$

$$(D) \quad \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \left(\frac{x}{2} \right) \right) + c$$

$$\text{ઉકેલ : } I = \int \frac{dx}{5+4\cos x}$$

$$\text{હાલે } \frac{x}{2}, \tan \frac{x}{2} = t$$

$$\therefore dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{5(1+t^2) + 4(1-t^2)}$$

$$= 2 \int \frac{dt}{(t^2) + (3)^2}$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{t}{3} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + c$$

જવાબ : (D)

$$(88) \quad \text{જો } \int \frac{dx}{\sin^4 x + \cos^4 x} = \frac{1}{\sqrt{2}} \tan^{-1} (f(x)) + c \text{ હો, } \dots\dots$$

$$(A) \quad f(x) = \tan x - \cot x$$

$$(B) \quad f\left(\frac{\pi}{4}\right) = 0$$

$$(C) \quad f(x) = \tan x + \cot x$$

$$(D) \quad f(x) = \frac{1}{2} (\tan x - \cot x)$$

$$\text{ઉકેલ : } I = \int \frac{dx}{\sin^4 x + \cos^4 x}$$

$$\begin{aligned}
I &= \int \frac{\sec^4 x \, dx}{\tan^4 x + 1} && (\text{અંશ અને છેદને } \cos^4 x \text{ વડે ભાગતાં}) \\
&= \int \frac{(1 + \tan^2 x) \sec^2 x}{1 + \tan^4 x} \, dx \\
&= \int \frac{1 + t^2}{1 + t^4} \, dt && (\text{ધારેલ છે, } \tan x = t, \text{ અર્થાત્, } \sec^2 x \, dx = dt) \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)} \, dt \\
&= \int \frac{du}{u^2 + (\sqrt{2})^2} && (\text{ધારેલ છે, } t - \frac{1}{t} = u, \text{ અર્થાત્, } \left(1 + \frac{1}{t^2}\right) dt = du) \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} \right) + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{(\tan x - \cot x)}{\sqrt{2}} \right) + c \\
\therefore f(x) &= \frac{\tan x - \cot x}{\sqrt{2}} \text{ . તેથી } f\left(\frac{\pi}{4}\right) = 0 && \text{જવાબ : (B)}
\end{aligned}$$

$$(89) \quad \int \left[\log (\log x) + \frac{1}{(\log x)^2} \right] dx = x [f(x) - g(x)] + c, \text{ તો } \dots$$

$$(A) \quad f(x) = \log (\log x), g(x) = \frac{1}{\log x}$$

$$(B) \quad f(x) = \log x, g(x) = \frac{1}{\log x}$$

$$(C) \quad f(x) = \frac{1}{\log x}, g(x) = \log (\log x)$$

$$(D) \quad f(x) = \frac{1}{x \log x}, g(x) = \frac{1}{\log x}$$

$$\text{ઉકેલ : } \int \log (\log x) \, dx$$

$$= \log (\log x) \int 1 \, dx - \int \left(\frac{d}{dx} \log (\log x) \int 1 \, dx \right) dx$$

$$= x \cdot \log (\log x) - \int \left(\frac{1}{\log x} \cdot \frac{1}{x} \cdot x \right) dx$$

$$= x \cdot \log (\log x) - \int \frac{1}{\log x} \, dx$$

$$= x \cdot \log (\log x) - x \cdot \frac{1}{\log x} + \int \left(\frac{(-1)}{(\log x)^2} \cdot \frac{1}{x} \cdot x \right) dx$$

$$\int \log (\log x) dx = x \cdot \log (\log x) - x \cdot \frac{1}{\log x} - \int \frac{1}{(\log x)^2} dx$$

$$\begin{aligned} \therefore \int \log (\log x) dx + \int \frac{1}{(\log x)^2} dx &= x \cdot \log (\log x) - x \cdot \frac{1}{\log x} \\ &= x \left[\log (\log x) - \frac{1}{\log x} \right] \end{aligned}$$

$$\therefore f(x) = \log (\log x), g(x) = \frac{1}{\log x} \quad \text{જવાબ : (A)}$$

નોંધ : માંગેલ વિકલ્પો પ્રમાણે આ જવાબ છે. $f(x) = \log \log x + \sin x$, $g(x) = \frac{1}{\log x} + \sin x$ પણ લઈ

શકાય. ખરેખર તો $f(x) = \log \log x + h(x)$, $g(x) = \frac{1}{\log x} + h(x)$

$$(90) \int \frac{x \tan^{-1} x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2} f(x) + k \log (x + \sqrt{x^2+1}) + c, \text{ તો } \dots$$

$$(A) f(x) = \tan^{-1} x, k = -1$$

$$(B) f(x) = \tan^{-1} x, k = 1$$

$$(C) f(x) = 2 \tan^{-1} x, k = -1$$

$$(D) f(x) = 2 \tan^{-1} x, k = 1$$

$$\text{ઉકેલ : } \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

$$\therefore \int \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{1+x^2}} dx = \frac{1}{2} (2\sqrt{1+x^2}) = \sqrt{1+x^2}$$

$$\text{હવે, } I = \int (\tan^{-1} x) \frac{x}{\sqrt{1+x^2}} dx$$

$$= (\tan^{-1} x) \int \frac{x}{\sqrt{1+x^2}} dx - \int \left(\frac{d}{dx} (\tan^{-1} x) \int \frac{x}{\sqrt{1+x^2}} dx \right) dx$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \int \left(\frac{1}{1+x^2} \cdot \sqrt{1+x^2} \right) dx$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \int \frac{dx}{\sqrt{1+x^2}}$$

$$= \sqrt{1+x^2} \cdot \tan^{-1} x - \log |x + \sqrt{1+x^2}| + c$$

$$\text{તેથી, } f(x) = \tan^{-1} x, k = -1$$

$$\text{જવાબ : (A)}$$

(91) જો $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = F(x)$ અને $F(1) = 0$, તો $x > 0$ માટે $F(x) = \dots$

(A) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1+x^2}} \right) + \frac{\pi}{2}$

(B) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1-x^2}} \right) - \frac{\pi}{2\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}x}{\sqrt{1-x^2}} \right) + \frac{\pi}{2\sqrt{2}}$

(D) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{1-x^2}}{\sqrt{2}x} \right)$

ઉકેલ : $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

$$= \int \frac{-t dt}{(1+t^2)\sqrt{t^2-1}}$$

(ધારો કે, $x = \frac{1}{t}$, $dx = \frac{-1}{t^2} dt$)

$$= - \int \frac{u du}{(u^2+2)u}$$

(ધારો કે, $t^2 - 1 = u^2$, આથી, $2t dt = 2u du$)

$$= - \int \frac{du}{(u)^2 + (\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{t^2-1}}{\sqrt{2}} \right) + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} \sqrt{1-x^2}}{x} \right) + c$$

હવે, $F(1) = 0$. આથી, $c = 0$

$$\therefore F(x) = \frac{-1}{\sqrt{2}} \left[\tan^{-1} \frac{\sqrt{1-x^2}}{\sqrt{2}x} \right]$$

$$\therefore F(x) = \frac{-1}{\sqrt{2}} \left[\frac{\pi}{2} - \cot^{-1} \frac{\sqrt{1-x^2}}{\sqrt{2}x} \right]$$

$$= \frac{-\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{\sqrt{1-x^2}} \quad (\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \cot^{-1}x = \tan^{-1} \left(\frac{1}{x} \right), x > 0)$$

જવાબ : (B)