

MTHE A-3 Number Theory (New/Old Course)

કુલ સમય :- ૬૦ મીનીટ

કુલ પ્રશ્નો : ૫૦

વિદ્યાર્થીએ ટીક કરવાના પ્રશ્નો : ૩૫

નોંધ : વિદ્યાર્થીએ માત્ર ૩૫ પ્રશ્નો જ ટીક કરવાના રહેશે. જો વધારે ટીક કરેલ હશે તો પ્રથમ ૩૫ પ્રશ્નો જ ગણતરીમાં લેવામાં આવશે.

1. For integers a, b , not both zero, determine one of the false statement among the following:
 - (a) If $ax + by = 2$ for some integers x, y then $\gcd(a, b) = 2$.
 - (b) If a divides $b \cdot c$ and $\gcd(a, b) = 1$ then a divides c .
 - (c) The square of an odd integer is of the form $8k + 1$.
 - (d) $\gcd(a, b) \cdot \text{lcm}(a, b) = |a \cdot b|$.
2. If $\gcd(a, b) = 1$ then _____.
 - (a) $\gcd(a + b, a - b) = 1$.
 - (b) $\gcd(a^n, b^2) = 1$ for any positive integer n .
 - (c) $\gcd(a + 2b, 2a + b) = 1$.
 - (d) $\gcd(a + b, a^2 + b^2) = 1$.
3. Determine one of the false statements among the following:
 - (a) If there is no prime divisor $p \leq \sqrt{N}$ of an integer $N > 1$ then N is a prime.
 - (b) If $n > 1$ is a prime then $2^n - 1$ is a prime.
 - (c) There are infinitely many primes of the form $4k + 3$.
 - (d) For integer $N > 2$, the common difference of an Arithmetic Progression of N -primes is divisible by every prime $q < N$.
4. Determine one of the false statements among the following:
 - (a) If f is a multiplicative function and $F(n) = \sum_{d|n} f(d)$, then F is also a multiplicative function.
 - (b) $\prod_{d|n} d = n^{\frac{\tau(n)}{2}}$.
 - (c) If $n > 1$ is a composite number then $\tau(n) > n + \sqrt{n}$.
 - (d) If $f(n) = n^2 + 1$ is a multiplicative function.
5. Determine one of the false statements among the following:
 - (a) For any integer $n \geq 3$, $\sum_{k=1}^n \mu(k!) = 1$.
 - (b) For $n > 1$, $\varphi(n) = n - 1$ if and only if n is a prime.
 - (c) All of the functions $\tau, \sigma, \mu, \varphi$ are multiplicative functions.
 - (d) For a real number x , $[x] + [-x] = 0$.
6. Determine one of the false statements among the following:
 - (a) If $a \equiv b \pmod{n}$ then $\gcd(a, n) = \gcd(b, n)$.
 - (b) If $a \equiv b \pmod{n}$ then $a \equiv r \pmod{n}$ where r is the least non-negative remainder upon the division of b by n .
 - (c) An integer $n > 1$ is divisible by 11 if and only if the sum of the digits of n (in the decimal representation of n) is divisible by 11.
 - (d) If $a \equiv b \pmod{n}$ then $a^k \equiv b^k \pmod{n}$ for any positive integer k .
7. Determine one of the false statements among the following:
 - (a) If $a_1, a_2, a_3, \dots, a_n$ form a complete set of residues modulo n then $a_1^2, a_2^2, a_3^2, \dots, a_n^2$ form a complete set of residues modulo n .
 - (b) If $\gcd(a, n) = 1$ then the integers $k, k + a, k + 2a, k + 3a, \dots, k + (n - 1)a$ form a complete set of residues modulo n .
 - (c) Any set of n -consecutive integers form a complete set of residues modulo n .
 - (d) If $a_1, a_2, a_3, \dots, a_n$ form a complete set of residues modulo n and $\gcd(b, n) = 1$ then $ba_1, ba_2, ba_3, \dots, ba_n$ form a complete set of residues modulo n .
8. "If $\gcd(a, n) = 1$ then _____" is the false statement.
 - (a) $a^{\varphi(n)} \equiv 1 \pmod{n}$.
 - (b) $a^n \equiv 1 \pmod{n}$.
 - (c) $a^{n-1} \equiv 1 \pmod{n}$ whenever n is prime.
 - (d) $ab \equiv ac \pmod{n}$ implies $b \equiv c \pmod{n}$.

9. (i) Euler's Theorem (I) If a prime p with $p \nmid a$ then $a^{p-1} \equiv 1 \pmod{p}$
(ii) Fermat's Little Theorem (II) If a prime p then $(p-1)! \equiv -1 \pmod{p}$
(iii) Wilson's Theorem (III) If $\gcd(a, n) = 1$ then $a^{\varphi(n)} \equiv 1 \pmod{n}$

Determine correct Answer.

- (a) (i) $\leftrightarrow$ (II), (ii) $\leftrightarrow$ (I), (iii) $\leftrightarrow$ (III) b. (i) $\leftrightarrow$ (III), (ii) $\leftrightarrow$ (I), (iii) $\leftrightarrow$ (II)
c. (i) $\leftrightarrow$ (II), (ii) $\leftrightarrow$ (III), (iii) $\leftrightarrow$ (I) d. (i) $\leftrightarrow$ (III), (ii) $\leftrightarrow$ (II), (iii) $\leftrightarrow$ (I)

10. For the results:

I. $n = \sum_{d|n} \varphi(d)$

II. If $\gcd(n, m) = 1$, then $n^{\varphi(m)} + m^{\varphi(n)} \equiv 1 \pmod{mn}$

Determine one of the true statements among the following:

- (a) (I) is true and (II) is true.
(b) (I) is true and (II) is false.
(c) (I) is false and (II) is true.
(d) (I) is false and (II) is false.

11. If p is the only prime for which $3p + 1$ is a perfect square then $p = \underline{\hspace{2cm}}$.

- (a) 3 b. 7 c. 5 d. 11

12. For integer $n > 2$, there are n -consecutive positive integers which are all $\underline{\hspace{2cm}}$.

- (a) Composite numbers. B. Primes. c. Perfect squares. D. Cubes.

13. $2^n \equiv 3 \pmod{4}$ is true if

- (a) $n \equiv 2 \pmod{4}$ b. $n \equiv 1 \pmod{4}$ c. $n \equiv 3 \pmod{4}$ d. $n \equiv 0 \pmod{4}$

14. Which of the following congruence equation is not solvable?

- (a) $42x \equiv 133 \pmod{111}$. B. $6x \equiv 15 \pmod{21}$.
c. $34x \equiv 60 \pmod{98}$. D. $25x \equiv 15 \pmod{20}$.

15. What is the units digit of 3^{100} ?

- (a) 9 b. 1 c. 5 d. 3

16. The congruence equation $x^2 + 1 \equiv 0 \pmod{p}$, where p is an odd prime, has a solution if and only if $\underline{\hspace{2cm}}$.

- (a) $p \equiv 3 \pmod{4}$ b. $p \equiv 7 \pmod{8}$ c. $p \equiv 3 \pmod{8}$ d. $p \equiv 1 \pmod{4}$

17. Given a prime p , which of the following congruence is not true?

- (a) $(p-1)! \equiv p-1 \pmod{p}$
(b) $(p-1)! \equiv p-1 \pmod{1+2+3+\dots+(p-1)}$
(c) $(p-1)! \equiv -1 \pmod{p^2}$
(d) $(p-1)! a + a^p \equiv p \pmod{p}$

18. Let $\text{ord}_n(a) = r$, then exactly one of the following is not necessarily true.

- (a) $a \equiv b \pmod{n}$ implies $\text{ord}_n(a) = r$
(b) $\varphi(n) \equiv 0 \pmod{r}$
(c) $\text{ord}_n(a) = \varphi(n)$
(d) $\text{ord}_n(a^k) = \frac{r}{\gcd(k, r)}$ for positive integer k .

19. Which of the following has no primitive roots?

- (a) 18 b. 15 c. 16 d. 29

20. Let p be an odd prime and $\gcd(a, p) = 1$, which of the following is not true?

- (a) $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ if and only if a is not a quadratic residue of p .
(b) $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ if and only if a is a quadratic residue of p .
(c) $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ if and only if a is a quadratic residue of p .
(d) a is a primitive root of p if and only if a is a quadratic residue of p .

21. Let p be an odd prime and $\gcd(a, p) = 1$, which of the following is not true?

- (a) $a \equiv b \pmod{n}$ implies $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$
(b) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$ c. $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$ d. $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 1$

22. For an integer $n > 1$, which of the following is true?
- n is perfect if $\varphi(n) = 2n$.
 - n is Fermat number if $2^{2^m} - 1 = n$ for some integer $m \geq 0$.
 - n is perfect if $\sigma(n) = 2n$.
 - n is Fermat number if $2^m + 1 = n$ for some integer $m > 0$.
23. Which of the following is true?
- 1 is a quadratic residue of 31
 - 2 is a quadratic non residue of 31
 - 2 is a quadratic non residue of 31
 - 3 is a quadratic non residue of 31
24. Let p be an odd prime $p \equiv 3 \pmod{4}$, which of the following is not true?
- p is a sum of five squares.
 - p is a sum of four squares.
 - p is a sum of six squares.
 - p is a sum of two squares.
25. Which of the following is not a pythagorean triplet?
- For odd integer $n > 2$, $n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)$
 - For integers $a > 3, d > 0$; $a, a + d, a + 2d$
 - For even integer $n > 2$, $n, \left(\frac{n^2}{4} - 1\right), \left(\frac{n^2}{4} + 1\right)$
 - For integer $t > 0$, $2t + 1, 2t(t + 1), 2t(t + 1) + 1$
26. If x, y, z is a primitive pythagorean triple in which $z - x = 2$ then _____ .
- $x = r^2 - 1, y = 2r, z = r^2 + 1$ for some even integer $r > 0$
 - $x = 98, y = 20, z = 102$
 - $x = 24, y = 10, z = 26$
 - $x = r^2 - s^2, y = 2rs, z = r^2 + s^2$ for some integer $r > s > 0$
27. The primitive solutions of the equation $x^2 + y^2 = z^2$ are given by $x = r^2 - s^2, y = 2rs, z = r^2 + s^2$ for some integers r, s where
- $r > s > 0, \gcd(r, s) = 1$ and $r \equiv s \pmod{2}$
 - $r > s > 0$ and $\gcd(r, s) = 1$
 - $r > s > 0, \gcd(r, s) = 1$ and $r \not\equiv s \pmod{2}$
 - r, s are relatively prime integers and they are in opposite parity
28. Which of the following integer is a sum of three squares?
- 7
 - 15
 - 60
 - 30
29. Which of the following statement is true?
- Every positive integer can be represented as a sum of four squares.
 - Every odd prime can be represented as a sum of two squares.
 - Every positive integer can be represented as a sum of three squares.
 - Every sufficiently large positive can be represented as a sum of four squares of positive integers.
30. Which of following Diophantine equation has positive integral solutions?
- $x^4 + y^4 = z^4$
 - $x^2 + y^2 = z^2$
 - $x^4 - y^4 = z^4$
 - $x^4 + y^4 = z^2$
31. For distinct odd primes p, q , which of the following is not the quadratic reciprocity law?
- $\left(\frac{p}{q}\right) \equiv p^{\frac{q-1}{2}} \pmod{q}$
 - $\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$
 - $\left(\frac{p}{q}\right) \equiv \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -\left(\frac{q}{p}\right) & \text{if } p \not\equiv 1 \pmod{4} \text{ and } q \not\equiv 1 \pmod{4} \end{cases}$
 - $\left(\frac{q}{p}\right) \equiv \begin{cases} \left(\frac{p}{q}\right) & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -\left(\frac{p}{q}\right) & \text{if } p \equiv 3 \pmod{4} \text{ and } q \equiv 3 \pmod{4} \end{cases}$

32. Which of the following Legendre symbol has the value -1?
- (a) $\left(\frac{100}{7}\right)$ b. $\left(\frac{2}{103}\right)$ c. $\left(\frac{-4}{101}\right)$ d. $\left(\frac{10}{7}\right)$
33. Which of following is a primitive pythagorean triple?
- (a) 16, 12, 20
(b) 36, 15, 39
(c) 209, 60, 241
(d) 168, 49, 175
34. Which of the following is not true?
- (a) $89|(2^{44} - 1)$
(b) $97|(2^{48} - 1)$
(c) $41|(2^{20} - 1)$
(d) $31|(2^{15} - 1)$
35. Which of the following statement is not true?
- (a) All the numbers $p, p + 2, p + 4$ are primes.
(b) The product of k -consecutive positive integers is divisible by $k!$.
(c) Every integer $n > 1$ has a prime divisor.
(d) An integer a is congruent modulo n to its remainder when it is divided by n .