

Important Concepts and Formulae :

- **Types of quantities :** Mainly there are two types of quantities - scalar quantities and vector quantities.
- **Scalar quantity :** The quantities having only magnitude, but no direction are called **scalar quantities**. e.g. speed, mass, density, volume, temperature etc.
- **Vector quantity :** The quantities having magnitude and direction both are called **vectors**. e.g. velocity, acceleration, displacement etc.

Representation of Vectors :

A vector is presented by a line segment whose initial end-point and terminal end-point are fixed. If P is the initial end-point and Q is the terminal end-point, then the vector is represented by $\overrightarrow{PQ}$. PQ is called its length or measure or magnitude.

Magnitude of $\overrightarrow{PQ}$ is $|\overrightarrow{PQ}| = PQ$.

- **Types of Vectors :**

- (1) **Zero vector :** If the magnitude of a vector is zero or if the initial and terminal end-points of the vector are the same, such a vector is called a **zero vector**. It is denoted by $\vec{0}$.
- (2) **Equal vectors :** If the magnitude and direction of two vectors are the same, the vectors are called **equal vectors**.
- (3) **Unit vector :** A vector whose magnitude is 1 is called a **unit vector**. A unit vector in the direction of $\vec{x}$ is denoted by $\hat{x}$. If $\vec{x}$ is a non-zero vector, then the unit vector in its direction is $\hat{x} = \frac{1}{|\vec{x}|} \vec{x}$.
- (4) **Vectors in the same direction :** If the directions of two non-zero vectors are same, they are called **vectors in the same direction**.
If there exists a scalar $k > 0$ such that $\vec{x} = k\vec{y}$, then $\vec{x}$ and $\vec{y}$ are vectors in the same direction.
- (5) **Vectors in opposite directions :** If the directions of two non-zero vectors are opposite to each other, they are called **vectors in opposite directions**.
For $\vec{x} \neq \vec{0}$, $\vec{y} \neq \vec{0}$, if $\vec{x} = k\vec{y}$, $k < 0$ then $\vec{x}$ and $\vec{y}$ are vectors in opposite directions. Thus, vectors in the same or in the opposite directions are called parallel vectors.
- (6) **Collinear vectors :** Two vectors whose directions are same or opposite are called **Collinear vectors**.
- (7) **Vectors in different directions :** If the directions of two vectors are neither the same nor opposite, then they are called **vectors in different directions**.
If $\vec{x} \neq \vec{0}$, $\vec{y} \neq \vec{0}$ and for any non-zero $k \in \mathbb{R}$, $\vec{x} \neq k\vec{y}$, then $\vec{x}$ and $\vec{y}$ are vectors in different directions.
- (8) **Negative of a vector :** A vector with the same magnitude as the given vector has and in the direction opposite to that of the given vector is called **negative of the given vector**.

(9) **Bound vector** : A vector with the initial end-point as origin is called a **bound vector**.

(10) **Free vector** : If the initial end-point of a vector is any point other than origin, it is called a **free vector**.

(11) **Position vector** : If O is the origin, the position vector of point P is $\vec{OP}$. If the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively, then $\vec{AB} = \vec{b} - \vec{a}$.

Position vector of origin is $\vec{0}$.

(12) **The sum of two vectors** : The sum of two vectors $\vec{x}$ and $\vec{y}$ is vector $\vec{x} + \vec{y}$. This is the resultant vector of $\vec{x}$ and $\vec{y}$.

Laws to find the sum of vectors :

(1) **Triangle law** : As shown in the figure, if $\vec{AB} = \vec{x}$ and $\vec{BC} = \vec{y}$ are the two sides of a triangle, then the third side of the triangle $\vec{AC}$ will be the vector $\vec{x} + \vec{y}$. This law is called the triangle law of addition of vectors.

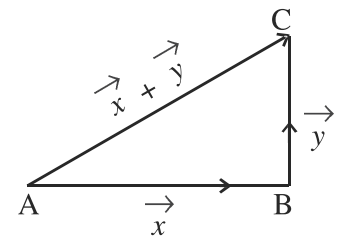


Figure 20.1

(2) **Parallelogram law** : In parallelogram OACB, if two adjacent sides are $\vec{OA} = \vec{x}$ and $\vec{OB} = \vec{y}$, then the diagonal $\vec{OC}$ through the common initial end-point O will represent the sum of $\vec{x}$ and $\vec{y}$. $\vec{OC} = \vec{OA} + \vec{OB} = \vec{x} + \vec{y}$

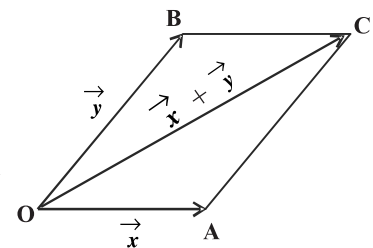


Figure 20.2

(3) **Polygonal law** : If $n - 1$ sides of a polygon represent the vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_{n-1}$ in order, then the n th side of the polygon will represent $\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \dots + \vec{x}_{n-1}$.

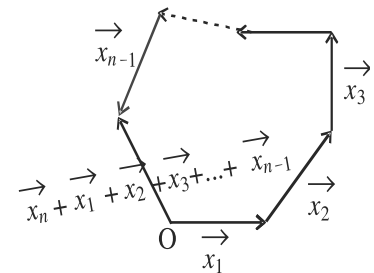


Figure 20.3

Components of a vector :

$\hat{i}, \hat{j}, \hat{k}$ represent unit vectors in the positive directions of X-axis, Y-axis and Z-axis respectively. The position vector of a point P(x_1, x_2, x_3) in R^3 is $\vec{x}$.

where $\vec{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$. Here, $x_1 \hat{i}, x_2 \hat{j}$ and $x_3 \hat{k}$ are the projection vectors of $\vec{x}$ on X-axis, Y-axis and Z-axis. x_1, x_2, x_3 are the scalar components of $\vec{x}$ and $x_1 \hat{i}, x_2 \hat{j}, x_3 \hat{k}$ are the vector components of $\vec{x}$.

Magnitude of $\vec{x}$ is $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Linear combination of vectors :

If $\vec{r} = a\vec{x} + b\vec{y} + c\vec{z}$, then $\vec{r}$ is a linear combination of vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$, where a , b , c are real constants.

Inner product (Scalar product or dot product) of vectors :

If $\vec{r} = x_1\hat{i} + x_2\hat{j}$ and $\vec{y} = y_1\hat{i} + y_2\hat{j}$ are vectors of R^2 , then the inner product is denoted by $\vec{x} \cdot \vec{y}$ and is defined as $x_1y_1 + x_2y_2$.

$$\therefore \vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2$$

Similarly, if $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ and $\vec{y} = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$ are vectors in R^3 , then $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3$.

Properties of inner product :

(1) Inner product is a scalar quantity and so it is also called scalar product.

$$(2) \quad \vec{x} \cdot \vec{x} \geq 0 \quad \text{and} \quad \vec{x} \cdot \vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$$

$$(3) \quad \vec{x} \cdot \vec{x} = |\vec{x}|^2$$

$$(4) \quad \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

$$(5) \quad \vec{x} \cdot (k\vec{y}) = (k\vec{x}) \cdot \vec{y} = k(\vec{x} \cdot \vec{y}) \quad \text{where } k \in R$$

$$(6) \quad \vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

$$(7) \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$(8) \quad \hat{i} \cdot \hat{j} = 0, \hat{j} \cdot \hat{k} = 0, \hat{i} \cdot \hat{k} = 0$$

$$(9) \quad \text{Cauchy-Schwartz inequality : } |\vec{x} \cdot \vec{y}| \leq |\vec{x}| |\vec{y}|$$

For non-zero vectors $\vec{x}$ and $\vec{y}$,

$$(i) \quad |\vec{x} \cdot \vec{y}| = |\vec{x}| |\vec{y}| \Leftrightarrow \vec{x} \text{ and } \vec{y} \text{ are in the same or opposite directions.}$$

$$(ii) \quad |\vec{x} \cdot \vec{y}| < |\vec{x}| |\vec{y}| \Leftrightarrow \vec{x} \text{ and } \vec{y} \text{ are in different directions.}$$

$$(10) \quad \text{Triangle inequality : } |\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$$

If $\vec{x}$ and $\vec{y}$ are non-zero vectors,

$$(i) \quad |\vec{x} + \vec{y}| = |\vec{x}| + |\vec{y}| \Leftrightarrow \vec{x} \text{ and } \vec{y} \text{ are in the same direction.}$$

$$(ii) \quad |\vec{x} + \vec{y}| < |\vec{x}| + |\vec{y}| \Leftrightarrow \vec{x} \text{ and } \vec{y} \text{ are in opposite direction or in different directions.}$$

Note : (1) $(\vec{x} + \vec{y}) \cdot (\vec{x} - \vec{y}) = |\vec{x}|^2 - |\vec{y}|^2$

(2) $|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 + 2(\vec{x} \cdot \vec{y})$

(3) $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2(\vec{x} \cdot \vec{y})$

Work : The work done by a force is a scalar quantity. Work done $w = \vec{F} \cdot \vec{d}$ where $\vec{d}$ is the displacement of the object in a particular direction under the force $\vec{F}$.

Outer product of vectors and its properties :

If $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ and $\vec{y} = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$ are vectors in R^3 , then their outer product (cross product) is $(x_2y_3 - x_3y_2)\hat{i} + (x_3y_1 - x_1y_3)\hat{j} + (x_1y_2 - x_2y_1)\hat{k}$ and is denoted by $\vec{x} \times \vec{y}$.

The outer product of two vectors can also be calculated using determinants as

$$\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Properties :

(1) Outer product is a vector quantity and so it is also known as cross product.

(2) $\vec{x} \times \vec{x} = \vec{0}$

(3) $\vec{x} \times \vec{0} = \vec{0} \times \vec{x} = \vec{0}$

(4) $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$

(5) $\vec{x} \times (k\vec{y}) = (k\vec{x}) \times \vec{y} = k(\vec{x} \times \vec{y})$ where, $k \in R$

(6) $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$

(7) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

(8) $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

● **Note :** (i) Outer product is not defined in R^2 .

(ii) If $\vec{x}$ and $\vec{y}$ are collinear vectors, then $\vec{x} \times \vec{y} = \vec{0}$ i.e. $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}$ where

$$\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \text{ and } \vec{y} = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}.$$

Relation between dot and cross products :

$$|\vec{x} \times \vec{y}|^2 + |\vec{x} \cdot \vec{y}|^2 = |\vec{x}|^2 |\vec{y}|^2. \text{ This relation is called Lagrange's identity.}$$

Inner product of four vectors : If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors, their inner product is

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}.$$

Scalar triple product (box product) and its properties :

If $\vec{x}, \vec{y}$ and $\vec{z}$ are vectors in R^3 , $\vec{x} \cdot (\vec{y} \times \vec{z})$ is called their scalar triple product or box

product and is denoted by $[\vec{x} \ \vec{y} \ \vec{z}]$. It is a scalar. $[\vec{x} \ \vec{y} \ \vec{z}] = D$, where $D = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

(1) If $\vec{x}$, $\vec{y}$ and $\vec{z}$ are the coterminus edges of a parallelopiped, then its volume is the modulus of $[\vec{x} \ \vec{y} \ \vec{z}]$ i.e. $V = |[\vec{x} \ \vec{y} \ \vec{z}]|$.

(2) If $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$, $\vec{y} = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$, $\vec{z} = z_1\hat{i} + z_2\hat{j} + z_3\hat{k}$

are the coterminus edges of a tetrahedron, then

$$\text{volume of the tetrahedron} = \frac{1}{6} |[\vec{x} \ \vec{y} \ \vec{z}]|$$

If A, B, C, D are the vertices of a tetrahedron, then its volume

$$V = \frac{1}{6} |[\vec{AB} \ \vec{AC} \ \vec{AD}]| = \frac{1}{6} |[\vec{BA} \ \vec{BC} \ \vec{BD}]|$$

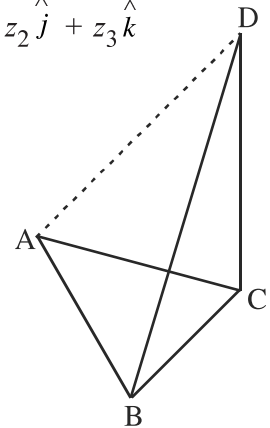


Figure 20.4

$$(3) \quad [\vec{x} \ \vec{y} \ \vec{z}] = [\vec{y} \ \vec{z} \ \vec{x}] = [\vec{z} \ \vec{x} \ \vec{y}]$$

$$(4) \quad [\vec{x} \ \vec{x} \ \vec{y}] = 0, [\vec{x} \ \vec{y} \ \vec{x}] = 0, [\vec{x} \ \vec{y} \ \vec{y}] = 0$$

$$(5) \quad [k\vec{x} \ \vec{y} \ \vec{z}] = [\vec{x} \ k\vec{y} \ \vec{z}] = [\vec{x} \ \vec{y} \ kz] = k[\vec{x} \ \vec{y} \ \vec{z}]; \text{ where } k \in \mathbb{R}$$

$$(6) \quad [\vec{x} \ \vec{y} \ \vec{0}] = 0$$

$$(7) \quad [\vec{x} \ \vec{y} \ \vec{z}] = -[\vec{y} \ \vec{x} \ \vec{z}]$$

(If $[\vec{x} \ \vec{y} \ \vec{z}] = k$, then interchanging any two vectors of $\vec{x}, \vec{y}, \vec{z}$ we get $-k$.)

$$(8) \quad [\vec{x} \times \vec{y} \ \vec{y} \times \vec{z} \ \vec{z} \times \vec{x}] = [\vec{x} \ \vec{y} \ \vec{z}]^2$$

$$(9) \quad [\vec{x} \ \vec{y} \ \vec{z}]^2 = \begin{vmatrix} \vec{x} \cdot \vec{x} & \vec{x} \cdot \vec{y} & \vec{x} \cdot \vec{z} \\ \vec{x} \cdot \vec{y} & \vec{y} \cdot \vec{y} & \vec{y} \cdot \vec{z} \\ \vec{x} \cdot \vec{z} & \vec{y} \cdot \vec{z} & \vec{z} \cdot \vec{z} \end{vmatrix}$$

(10) If $|[\vec{x} \ \vec{y} \ \vec{z}]| = |\vec{x}| |\vec{y}| |\vec{z}|$, then parallelopiped will become a cuboid.

Vector Triple Product :

If $\vec{x}$, $\vec{y}$, $\vec{z}$ are vectors in $\mathbb{R}^3$, then their vector triple product is,

$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z}) \vec{y} - (\vec{x} \cdot \vec{y}) \vec{z}.$$

If $\vec{r} = \vec{x} \times (\vec{y} \times \vec{z})$, then $\vec{r} \perp \vec{x}$ and $\vec{r} \perp (\vec{y} \times \vec{z})$ and $\vec{r}$ is a vector in the plane of $\vec{y}$ and $\vec{z}$.

$$\text{Note : } (\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x} \cdot \vec{z}) \vec{y} - (\vec{y} \cdot \vec{z}) \vec{x}$$

Collinear vectors :

Vectors which are parallel to the given line are called collinear vectors. Their magnitudes can be different.

For non-zero vectors $\vec{x}$ and $\vec{y}$, if $\vec{x} = k\vec{y}$, $k \in \mathbb{R} - \{0\}$, then $\vec{x}$ and $\vec{y}$ are collinear vectors.

$\vec{x} = x_1 \hat{i} + x_2 \hat{j}$ and $\vec{y} = y_1 \hat{i} + y_2 \hat{j}$ are collinear, if and only if $x_1 y_2 - x_2 y_1 = 0$.

Vectors $\vec{x}$ and $\vec{y}$ of $\mathbb{R}^3$ are collinear if and only if $\vec{x} \times \vec{y} = \vec{0}$.

$\vec{x}$ and $\vec{y}$ are collinear if and only if $|\vec{x} \cdot \vec{y}| = |\vec{x}| |\vec{y}|$.

Coplanar vectors :

Vectors lying in the same plane are called coplanar vectors.

If at least one of $\alpha, \beta, \gamma \in \mathbb{R}$ is non-zero and $\alpha \vec{x} + \beta \vec{y} + \gamma \vec{z} = \vec{0}$, then $\vec{x}, \vec{y}$, and $\vec{z}$ are coplanar or linearly dependent vectors.

If $\vec{x}, \vec{y}$ and $\vec{z}$ are non-coplanar (linearly independent) vectors, then

$$\alpha \vec{x} + \beta \vec{y} + \gamma \vec{z} = \vec{0} \Rightarrow \alpha = 0, \beta = 0 \text{ and } \gamma = 0.$$

If $\vec{r}$ is in the plane of $\vec{x}$ and $\vec{y}$, then $\vec{r} = \alpha \vec{x} + \beta \vec{y}$, $\alpha^2 + \beta^2 \neq 0$.

In $\mathbb{R}^3$; $\vec{x}, \vec{y}$ and $\vec{z}$ are coplanar if and only if $[\vec{x} \ \vec{y} \ \vec{z}] = 0$.

- **Angle between two non-zero vectors :** If the measure of the angle between two non-zero

vectors $(\vec{x}, \vec{y}) = \alpha$, then $\alpha = \cos^{-1} \frac{\vec{x} \cdot \vec{y}}{|\vec{x}| |\vec{y}|}$, $\alpha \in [0, \pi]$;

Results :

- If $\vec{x}$ and $\vec{y}$ are vectors in the same direction, then $\alpha = 0$

- If $\vec{x}$ and $\vec{y}$ are in opposite directions, then $\alpha = \pi$

- If $\vec{x} \cdot \vec{y} = 0$ if and only if $\vec{x} \perp \vec{y}$ i.e. $\alpha = \frac{\pi}{2}$

- $\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \alpha$

- $|\vec{x} \times \vec{y}| = |\vec{x}| |\vec{y}| \sin \alpha$

- Unit vector perpendicular to both $\vec{x}$ and $\vec{y}$ is $\pm \frac{\vec{x} \times \vec{y}}{|\vec{x} \times \vec{y}|}$.

- Area of ΔABC is $\Delta = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors representing the sides of ΔABC , then the area of ΔABC is

$$\Delta = \frac{1}{2} \sqrt{|\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \cdot \vec{b}|^2}$$

- If $\vec{AC}$ and $\vec{BD}$ are the diagonals of parallelogram $ABCD$, then the area of parallelogram

$$ABCD = \frac{1}{2} |\vec{AC} \times \vec{BD}|$$

Angle bisector :

If $\vec{r}$ is a bisector of the angle between $\vec{x}$ and $\vec{y}$, then $\vec{r} = \left(\frac{\vec{x}}{|\vec{x}|} + \frac{\vec{y}}{|\vec{y}|} \right)$.

Projection of a vector :

If $\vec{x}$ and $\vec{y}$ are neither zero vectors nor perpendicular vectors, then projection vector of

$\vec{x}$ on $\vec{y}$ is $\left(\frac{\vec{x} \cdot \vec{y}}{|\vec{y}|^2} \right) \vec{y}$. It is denoted by $\text{Proj}_{\vec{y}} \vec{x}$. So, $\text{Proj}_{\vec{y}} \vec{x} = \left(\frac{\vec{x} \cdot \vec{y}}{|\vec{y}|^2} \right) \vec{y}$

Magnitude of projection vector :

$\frac{|\vec{x} \cdot \vec{y}|}{|\vec{y}|}$ is the magnitude of projection vector of $\vec{x}$ on $\vec{y}$.

$\frac{\vec{x} \cdot \vec{y}}{|\vec{y}|}$ is called the component of projection vector of $\vec{x}$ on $\vec{y}$. It is denoted by $\text{Comp}_{\vec{y}} \vec{x}$.

Direction cosines, direction angles and direction ratios of a vector :

If α , β and γ are the measures of the angles made by the vector $\vec{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$ with the positive directions of X, Y and Z axes, then α , β and γ are called the direction angles of $\vec{x}$ and $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are called its direction cosines.

If $\vec{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$, $\vec{x} \neq \vec{0}$ and $m \neq 0$, then $m \vec{x} = mx_1 \hat{i} + mx_2 \hat{j} + mx_3 \hat{k}$. The numbers mx_1 , mx_2 , mx_3 are called the direction ratios (direction numbers) of $\vec{x}$.

- $\cos \alpha = \frac{x_1}{|\vec{x}|}$, $\cos \beta = \frac{x_2}{|\vec{x}|}$ and $\cos \gamma = \frac{x_3}{|\vec{x}|}$

- $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

- $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$

Division of a line-segment (Section formula) :

If $\vec{x}$ and $\vec{y}$ are the position vectors of points A and B respectively and if point P($\vec{r}$)

divides $\overline{AB}$ in the ratio λ from A, then $\vec{r} = \frac{\lambda \vec{y} + \vec{x}}{\lambda + 1}$. ($\lambda \neq -1$)

- If $\lambda = m : n$ then $\vec{r} = \frac{m \vec{y} + n \vec{x}}{m + n}$
- The position vector of midpoint of $\overline{AB}$ is $\frac{\vec{x} + \vec{y}}{2}$.
- If A($\vec{x}$), B($\vec{y}$) and C($\vec{z}$) are the vertices of ΔABC , then the position vector of centroid of ΔABC is $\frac{\vec{x} + \vec{y} + \vec{z}}{3}$.
- The position vector of incentre of ΔABC will be $\frac{a \vec{x} + b \vec{y} + c \vec{z}}{a + b + c}$,
where, $a = |\overrightarrow{BC}|$, $b = |\overrightarrow{CA}|$, $c = |\overrightarrow{AB}|$.

Inverse system of vectors :

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero, non collinear and non-coplanar vectors, then we can get vectors

$$\vec{a}', \vec{b}' \text{ and } \vec{c}' \text{ such that } \vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]}, \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a} \ \vec{b} \ \vec{c}]}, \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a} \ \vec{b} \ \vec{c}]}.$$

Vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are called inverse system of vectors of $\vec{a}$, $\vec{b}$, $\vec{c}$.

Properties :

- (1) $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$
- (2) $\vec{a} \cdot \vec{b}' = \vec{a} \cdot \vec{c}' = \vec{b} \cdot \vec{a}' = \vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = \vec{c} \cdot \vec{b}' = 0$
- (3) $[\vec{a}', \vec{b}', \vec{c}'] = \frac{1}{[\vec{a} \ \vec{b} \ \vec{c}]}$
- (4) $\vec{a}', \vec{b}'$ and $\vec{c}'$ are non-zero, non-collinear and non-coplanar vectors.
- (5) $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}' = \vec{0}$
- (6) $\hat{i} = \hat{i}', \hat{j} = \hat{j}', \hat{k} = \hat{k}'$

: Multiple Choice Questions :

- (1) If $\vec{a}$ is a unit vector, then $(\vec{a} \cdot \hat{i})^2 + (\vec{a} \cdot \hat{j})^2 + (\vec{a} \cdot \hat{k})^2 = \dots$
 (A) 0 (B) 1 (C) 2 (D) 3

Sol. : Taking $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$$\vec{a} \cdot \hat{i} = a_1, \vec{a} \cdot \hat{j} = a_2, \text{ and } \vec{a} \cdot \hat{k} = a_3$$

$$\therefore (\vec{a} \cdot \hat{i})^2 + (\vec{a} \cdot \hat{j})^2 + (\vec{a} \cdot \hat{k})^2 = a_1^2 + a_2^2 + a_3^2 = 1$$

Ans. : (B)

- (2) If $\vec{a}$ is a unit vector, then $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = \dots$
 (A) 0 (B) 1 (C) 2 (D) 3

Sol. : Taking $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$$\text{and } \vec{a} \times \hat{i} = a_3 \hat{j} - a_2 \hat{k}; \vec{a} \times \hat{j} = a_1 \hat{k} - a_3 \hat{i} \text{ and } \vec{a} \times \hat{k} = a_2 \hat{i} - a_1 \hat{j}$$

$$\begin{aligned} \therefore |\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 &= (a_3^2 + a_2^2) + (a_1^2 + a_3^2) + (a_2^2 + a_1^2) \\ &= 2(a_1^2 + a_2^2 + a_3^2) = 2(1) = 2 \end{aligned}$$

Ans. : (C)

- (3) If $\vec{a}$ is a unit vector making an angle of measure $\frac{\pi}{4}$ with $\vec{b} = \hat{i}$ and perpendicular to $\vec{c} = \hat{k}$, then $\vec{a} = \dots$

$$(A) \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \quad (B) -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} \quad (C) \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{\sqrt{2}} \hat{k} \quad (D) \frac{1}{\sqrt{2}} \hat{j} - \frac{1}{\sqrt{2}} \hat{k}$$

Sol. : Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$. $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$$\cos(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}. \text{ From this, } \cos \frac{\pi}{4} = \frac{a_1}{1}$$

$$\therefore \frac{1}{\sqrt{2}} = a_1$$

$$\vec{a} \perp \vec{c} \Rightarrow \vec{a} \cdot \vec{c} = 0 \Rightarrow a_3 = 0$$

$$\text{Also, } a_1^2 + a_2^2 + a_3^2 = 1$$

$$\therefore \frac{1}{2} + a_2^2 + 0 = 1$$

$$\therefore a_2 = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \vec{a} = \frac{1}{\sqrt{2}} \hat{i} \pm \frac{1}{\sqrt{2}} \hat{j}$$

Ans. : (A)

(4) If $\vec{a} \cdot \vec{x} = 0$, $\vec{a} \cdot \vec{y} = 0$, $\vec{a} \cdot \vec{z} = 0$, then $[\vec{x} \ \vec{y} \ \vec{z}] = \dots$ ($\vec{a} \neq \vec{0}$)

- (A) 1 (B) $|\vec{a}|$ (C) 0 (D) not possible

Sol. : Since $\vec{a}$ is perpendicular to three vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$, the vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ are coplanar.

$$\therefore [\vec{x} \ \vec{y} \ \vec{z}] = 0$$

Ans. : (C)

Second method : $\vec{a} \cdot \vec{x} = 0$, $\vec{a} \cdot \vec{y} = 0$, $\vec{a} \cdot \vec{z} = 0$

$\vec{x}$, $\vec{y}$, $\vec{z}$ may be coplanar or non-coplanar.

If $\vec{x}$, $\vec{y}$, $\vec{z}$ are coplanar, then $[\vec{x} \ \vec{y} \ \vec{z}] = 0$.

If $\vec{x}$, $\vec{y}$, $\vec{z}$ are non-coplanar, then each non-zero $\vec{a} \in \mathbb{R}^3$ can be written as $\vec{a} = p\vec{x} + q\vec{y} + r\vec{z}$.

$$\vec{a} \cdot \vec{a} = \vec{a} \cdot (p\vec{x} + q\vec{y} + r\vec{z}) = p(\vec{a} \cdot \vec{x}) + q(\vec{a} \cdot \vec{y}) + r(\vec{a} \cdot \vec{z}) = 0$$

$$\therefore |\vec{a}|^2 = 0$$

$$\therefore |\vec{a}| = 0. \text{ But } \vec{a} \neq \vec{0}. \text{ A contradiction.}$$

$$\therefore \vec{x}, \vec{y}, \vec{z} \text{ are not non-coplanar.}$$

$$\therefore \vec{x}, \vec{y}, \vec{z} \text{ are coplanar.}$$

$$\therefore [\vec{x}, \vec{y}, \vec{z}] = 0$$

Third method : If $\vec{x} \neq \vec{0}$, $\vec{y} \neq \vec{0}$, $\vec{z} \neq \vec{0}$,

$$\vec{a} \perp \vec{x}, \vec{a} \perp \vec{y}$$

$$\therefore \vec{a} = \pm(\vec{x} \times \vec{y})$$

$$\therefore \vec{a} \cdot \vec{z} = 0 \Rightarrow \pm(\vec{x} \times \vec{y}) \cdot \vec{z} = 0$$

$$\Rightarrow |\vec{x} \cdot \vec{y} \cdot \vec{z}| = 0$$

If any one of $\vec{x}$, $\vec{y}$ or $\vec{z}$ is $\vec{0}$, then

$$|\vec{x} \cdot \vec{y} \cdot \vec{z}| = 0$$

(5) If $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{0}$, then ($\vec{a} \neq k(\vec{b} \times \vec{c})$, $\vec{a} \neq \vec{0}$)

- (A) $\vec{b} \parallel \vec{c}$ (B) $\vec{a} \parallel \vec{b}$ (C) $\vec{a} \parallel \vec{c}$ (D) $\vec{b} \perp \vec{c}$

Sol. : $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{0} \Rightarrow \vec{a} \parallel (\vec{b} \times \vec{c})$ or $\vec{a} = \vec{0}$ or $\vec{b} \times \vec{c} = \vec{0}$

$\vec{a} \neq \vec{0}$. Since $\vec{a} \neq k(\vec{b} \times \vec{c})$, $\vec{a} \parallel (\vec{b} \times \vec{c})$ is not possible. So, $\vec{b} \times \vec{c} = \vec{0}$

$$\therefore \vec{b} \parallel \vec{c}$$

Ans. : (A)

(6) If for unit vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, if $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{c}}{2}$, then $(\vec{a}, \vec{b}) = \dots$

- (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{5\pi}{6}$ (D) $\frac{2\pi}{3}$

Sol. : $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{c}}{2}$

$$\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{\vec{c}}{2}$$

$$|\vec{c}| = 1 \Rightarrow \vec{c} \neq \vec{0}. \text{ So, } \vec{b} \times \vec{c} \neq \vec{0} \text{ as } \vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{c}}{2}$$

$$\therefore (\vec{a} \cdot \vec{c}) \vec{b} \times \vec{b} - (\vec{a} \cdot \vec{b})(\vec{b} \times \vec{c}) = \frac{\vec{b} \times \vec{c}}{2}$$

$$\therefore -(\vec{a} \cdot \vec{b})(\vec{b} \times \vec{c}) = \frac{\vec{b} \times \vec{c}}{2}$$

$$\therefore \vec{a} \cdot \vec{b} = -\frac{1}{2} \quad (\vec{b} \times \vec{c} \neq \vec{0})$$

$$\therefore |\vec{a}| |\vec{b}| \cos(\vec{a}, \vec{b}) = -\frac{1}{2}$$

$$\therefore (1)(1) \cos(\vec{a}, \vec{b}) = -\frac{1}{2}$$

$$\therefore (\vec{a}, \vec{b}) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Ans. : (D)

- (7) $\vec{x} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{y} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{z} = \hat{i} + 3\hat{k}$. If $\vec{x} + \lambda\vec{y}$ is perpendicular to $\vec{z}$, then $\lambda = \dots$

- (A) 2 (B) 5 (C) 0 (D) 6

Sol. : $\vec{x} + \lambda\vec{y} = (2 + 2\lambda)\hat{i} + (3 + \lambda)\hat{j} + (1 - \lambda)\hat{k}$ is perpendicular to $\vec{z} = \hat{i} + 3\hat{k}$.

$$\therefore (2 + 2\lambda)(1) + (3 + \lambda)(0) + (1 - \lambda)(3) = 0$$

$$\therefore 2 + 2\lambda + 3 - 3\lambda = 0$$

$$\therefore 5 = \lambda$$

Ans. : (B)

- (8) $\vec{a}$ and $\vec{b}$ are unit vectors and $(\vec{a}, \vec{b}) = \theta$, then $2\sin\frac{\theta}{2} = \dots$

- (A) $\frac{1}{2}|\vec{a} - \vec{b}|$ (B) $|\vec{a} - \vec{b}|$ (C) $\frac{1}{2}|\vec{a} + \vec{b}|$ (D) $|\vec{a} + \vec{b}|$

Sol. : $4\sin^2\frac{\theta}{2} = 2(1 - \cos\theta)$

$$= 1 + 1 - 2(1)(1) \cos\theta$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos\theta \quad (\vec{a} \text{ and } \vec{b} \text{ are unit vectors})$$

$$= |\vec{a} - \vec{b}|^2$$

$$\therefore 2\sin\frac{\theta}{2} = |\vec{a} - \vec{b}|$$

Ans. : (B)

- (9) If $\vec{x} \cdot (\vec{y} \times \vec{z}) = |\vec{x}| |\vec{y}| |\vec{z}|$, then

- (A) $\vec{x} \cdot \vec{y}$, $\vec{y} \cdot \vec{z}$, $\vec{z} \cdot \vec{x}$ are non-zero.

- (B) One of $\vec{x} \cdot \vec{y} \neq 0$, $\vec{y} \cdot \vec{z} \neq 0$, $\vec{z} \cdot \vec{x} \neq 0$ is true.

- (C) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{z} = \vec{z} \cdot \vec{x} = 1$

- (D) $\vec{x} \cdot \vec{y} = 0$, $\vec{y} \cdot \vec{z} = 0$, $\vec{z} \cdot \vec{x} = 0$

Sol. : $\vec{x} \cdot (\vec{y} \times \vec{z}) = [\vec{x} \ \vec{y} \ \vec{z}]$ is the volume of a parallelepiped with coterminal edges $\vec{x}$, $\vec{y}$ and $\vec{z}$.

Since, $[\vec{x} \ \vec{y} \ \vec{z}] = |\vec{x}| |\vec{y}| |\vec{z}|$, the parallelepiped becomes a cuboid.

$$\therefore \vec{x} \perp \vec{y}, \vec{y} \perp \vec{z} \text{ and } \vec{z} \perp \vec{x}$$

$$\therefore \vec{x} \cdot \vec{y} = 0, \vec{y} \cdot \vec{z} = 0 \text{ and } \vec{z} \cdot \vec{x} = 0$$

Ans. : (D)

Second method :

$$\therefore \vec{x} \cdot (\vec{y} \times \vec{z}) = |\vec{x}| |\vec{y} \times \vec{z}| \cos \alpha \quad \text{where } \alpha = (\vec{x} \wedge \vec{y} \times \vec{z})$$

$$= |\vec{x}| |\vec{y}| |\vec{z}| \sin \beta \cos \alpha \quad \text{where } \beta = (\vec{y} \wedge \vec{z})$$

$$= |\vec{x}| |\vec{y}| |\vec{z}| \quad (\text{given})$$

$$\therefore \sin \beta = \cos \alpha = 1$$

$$\therefore \beta = \frac{\pi}{2}$$

$$\therefore \vec{y} \perp \vec{z}$$

$$\therefore \vec{y} \cdot \vec{z} = 0$$

Similarly, from $\vec{y} \cdot (\vec{z} \times \vec{x})$, $\vec{z} \cdot \vec{x} = 0$ and from $\vec{z} \cdot (\vec{x} \times \vec{y})$, $\vec{x} \cdot \vec{y} = 0$

$$\therefore \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{z} = \vec{z} \cdot \vec{x} = 0$$

Ans. : (D)

- (10) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the vectors representing respectively the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of ΔABC , then [IIT : 2000, AIEEE : 2012]

$$(A) \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$$

$$(B) \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$(C) \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$$

$$(D) \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$$

Sol. : $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\therefore \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$$\therefore \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\therefore \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$$

$$\therefore \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

Similarly, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

$$\therefore \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

Ans. : (B)

- (11) If $\vec{x} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{y} = \hat{i} - \hat{j}$, then the unit vector perpendicular to $\vec{x} + \vec{y}$ and $\vec{x} - \vec{y}$ is

(A) $\frac{1}{6\sqrt{3}}(2\hat{i} - 2\hat{j} + 10\hat{k})$

(B) $\frac{1}{6\sqrt{3}}(2\hat{i} + 2\hat{j} - 10\hat{k})$

(C) $\frac{1}{3\sqrt{3}}(\hat{i} + \hat{j} + 5\hat{k})$

(D) $\frac{1}{3\sqrt{3}}(\hat{i} - \hat{j} + 5\hat{k})$

Sol. : $(\vec{x} + \vec{y}) \times (\vec{x} - \vec{y}) = -(\vec{x} \times \vec{y}) + (\vec{y} \times \vec{x})$

$$= -2(\vec{x} \times \vec{y}) = -2(-\hat{i} - \hat{j} - 5\hat{k}) = 2\hat{i} + 2\hat{j} + 10\hat{k}$$

Unit vector perpendicular to $(\vec{x} + \vec{y})$ and $(\vec{x} - \vec{y}) = \pm \frac{(\vec{x} + \vec{y}) \times (\vec{x} - \vec{y})}{|(\vec{x} + \vec{y}) \times (\vec{x} - \vec{y})|}$

$$= \pm \frac{1}{3\sqrt{3}}(\hat{i} + \hat{j} + 5\hat{k}) \quad \text{Ans. : (C)}$$

- (12) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar unit vectors, then $[2\vec{a} - \vec{b} \quad 2\vec{b} - \vec{c} \quad 2\vec{c} - \vec{a}] = \dots$ [IIT : 2000]

(A) 0

(B) 1

(C) $-\sqrt{3}$

(D) $\sqrt{3}$

Sol. : Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, $2\vec{a} - \vec{b}$, $2\vec{b} - \vec{c}$ and $2\vec{c} - \vec{a}$ will too be coplanar.

$$\therefore [2\vec{a} - \vec{b} \quad 2\vec{b} - \vec{c} \quad 2\vec{c} - \vec{a}] = 0$$

Note : $[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$

$$\therefore [2\vec{a} - \vec{b} \quad 2\vec{b} - \vec{c} \quad 2\vec{c} - \vec{a}] = 7[\vec{a} \quad \vec{b} \quad \vec{c}] = 0$$

Ans. : (A)

- (13) Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$ are given.

If $\vec{c}$ is a vector such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$, then $\vec{c} = \dots$

(A) $\frac{5}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$

(B) $-\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

(C) $\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

(D) $-\frac{5}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$

Sol. : Suppose $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Now, $\vec{a} \cdot \vec{c} = 3$. So, $c_1 + c_2 + c_3 = 3$ (1)

$$\vec{a} \times \vec{c} = \vec{b}$$

$$\therefore (c_3 - c_2)\hat{i} + (c_1 - c_3)\hat{j} + (c_2 - c_1)\hat{k} = 0\hat{i} + \hat{j} - \hat{k}$$

$$\therefore c_3 - c_2 = 0; \quad c_1 - c_3 = 1 \text{ and } c_2 - c_1 = -1$$

$$\therefore c_3 = c_2 \quad \text{(2)}$$

$$c_1 = 1 + c_3 \quad (3)$$

By solving (1), (2) and (3),

$$c_3 = \frac{2}{3} = c_2 \quad \text{and} \quad c_1 = \frac{5}{3}$$

$$\therefore \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Ans. : (C)

- (14) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors, then the value of $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2$ cannot be greater than [IIT : 2001]

- (A) 4 (B) 9 (C) 8 (D) 6

Sol. : $|\vec{a} + \vec{b} + \vec{c}|^2 \geq 0$

$$\therefore |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \geq 0$$

$$\therefore 1 + 1 + 1 \geq -2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\therefore 3 \geq -2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \quad (1)$$

$$\text{Now, } |\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2) - 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$= 2(1 + 1 + 1) - 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\leq 6 + 3 = 9$$

(From (1))

$$\left[\text{Value 9 is possible if we take } \vec{a} = \frac{\vec{i} - \vec{j}}{\sqrt{2}}, \vec{b} = \frac{\vec{j} - \vec{k}}{\sqrt{2}}, \vec{c} = \frac{\vec{k} - \vec{i}}{\sqrt{2}}. \right]$$

Ans. : (B)

- (15) If ABCDEF is a regular hexagon and $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = \lambda \vec{AD}$, then $\lambda = \dots$

- (A) 6 (B) 4 (C) 3 (D) 2

Sol. : By triangle law of addition of vectors,

$$\begin{aligned} \text{In } \triangle ACD, \vec{AD} &= \vec{AC} + \vec{CD} \\ &= \vec{AC} + \vec{AF} \end{aligned} \quad (1)$$

$$\begin{aligned} \text{and in } \triangle AED, \vec{AD} &= \vec{AE} + \vec{ED} \\ &= \vec{AE} + \vec{AB} \end{aligned} \quad (2)$$

Adding, (1) and (2),

$$2\vec{AD} = \vec{AB} + \vec{AC} + \vec{AE} + \vec{AF}$$

$$\therefore 3\vec{AD} = \vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}. \text{ So, } \lambda = 3.$$

Ans. : (C)

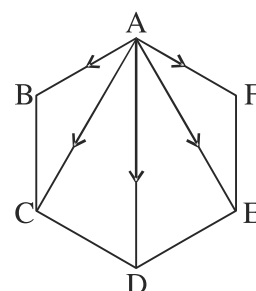


Figure 20.5

- (16) In a trapezium, $\overrightarrow{BC} = m\overrightarrow{AD}$. For vector $\overrightarrow{x}$, $\overrightarrow{x} = \overrightarrow{AC} + \overrightarrow{BD}$ and $\overrightarrow{x}$ is collinear with $\overrightarrow{AD}$.

If $\overrightarrow{x} = n\overrightarrow{AD}$ then $n = \dots$

- (A) $m + 1$ (B) $m - 1$ (C) $1 - m$ (D) $-m - 1$

Sol. : $\overrightarrow{x} = \overrightarrow{AC} + \overrightarrow{BD}$

$$\therefore n\overrightarrow{AD} = \overrightarrow{AC} + (\overrightarrow{BC} + \overrightarrow{CD})$$

$$\therefore n\overrightarrow{AD} = \overrightarrow{AC} + m\overrightarrow{AD} + \overrightarrow{CD}$$

$$\therefore n\overrightarrow{AD} = (\overrightarrow{AC} + \overrightarrow{CD}) + m\overrightarrow{AD}$$

$$\therefore n\overrightarrow{AD} = \overrightarrow{AD} + m\overrightarrow{AD}$$

$$\therefore n\overrightarrow{AD} = (1 + m)\overrightarrow{AD}$$

$$\therefore n = m + 1$$

Ans. : (A)

- (17) If $\overrightarrow{a} = \hat{i} - \hat{k}$, $\overrightarrow{b} = x\hat{i} + \hat{j} + (1 - x)\hat{k}$, $\overrightarrow{c} = y\hat{i} + x\hat{j} + (1 + x - y)\hat{k}$ then $[\overrightarrow{a} \ \overrightarrow{b} \ \overrightarrow{c}]$ is not dependent on [IIT : 2001, AIEEE : 2005]

- (A) only x (B) only y (C) any one of x and y (D) x and y

Sol. : $[\overrightarrow{a} \ \overrightarrow{b} \ \overrightarrow{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & (1-x) \\ y & x & 1+x-y \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & x & 1 \end{vmatrix}$$

$$= 1$$

$$(C_3 \rightarrow C_3 + C_1 - C_2)$$

$\therefore [\overrightarrow{a} \ \overrightarrow{b} \ \overrightarrow{c}]$ is not dependent on x and y .

Ans. : (D)

- (18) If two vectors $\overrightarrow{x} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\overrightarrow{y} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel (collinear), then ($\overrightarrow{x} \neq 0, \overrightarrow{y} \neq 0$)

(A) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(B) $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(C) $a_1 = b_1, a_2 = b_2, a_3 = b_3$

(D) no relation between coordinates $a_i, b_i; i = 1, 2, 3$

Sol. : Since non-zero vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ are parallel, $\overrightarrow{x} = k\overrightarrow{y}$, $k \neq 0$

$$\therefore a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = k(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = k$$

(If any $b_i = 0$, then $a_i = 0$.)

Ans. : (A)

- (19) If $\vec{x} = -\hat{i} + 3\hat{k}$ and $\vec{y} = \hat{i} - 3\hat{j} + 2\hat{k}$. If $\vec{y} = \vec{a} + \vec{b}$, where $\vec{a}$ is parallel to $\vec{x}$ and $\vec{b} \perp \vec{x}$, then $(\vec{a}, \vec{b}) = \dots$

- (A) $\left(\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}, \hat{i} - \frac{7}{2}\hat{j} + \frac{1}{2}\hat{k}\right)$ (B) $\left(-\frac{1}{2}\hat{i} + \frac{3}{2}\hat{k}, \frac{3}{2}\hat{i} - 3\hat{j} + \frac{1}{2}\hat{k}\right)$
 (C) $\left(-\frac{1}{2}\hat{j} - \frac{3}{2}\hat{k}, \frac{3}{2}\hat{i} - 3\hat{j} + \frac{7}{2}\hat{k}\right)$ (D) $\left(-\frac{1}{2}\hat{i} + \hat{j} + \frac{3}{2}\hat{k}, \frac{3}{2}\hat{i} - 4\hat{j} + \frac{1}{2}\hat{k}\right)$

Sol. : $\vec{a}$ and $\vec{x}$ are parallel vectors.

$$\therefore \vec{a} = \alpha(-\hat{i} + 3\hat{k}) \text{ and } \vec{b} = \vec{y} - \vec{a} = (1 + \alpha)\hat{i} - 3\hat{j} + (2 - 3\alpha)\hat{k}$$

Now, $\vec{b} \perp \vec{x}$. So, $\vec{b} \cdot \vec{x} = 0$

$$\therefore -(1 + \alpha) + 3(2 - 3\alpha) = 0$$

$$\therefore 5 = 10\alpha. \text{ So, } \alpha = \frac{1}{2}$$

$$\therefore \vec{a} = -\frac{1}{2}\hat{i} + \frac{3}{2}\hat{k} \text{ and } \vec{b} = \frac{3}{2}\hat{i} - 3\hat{j} + \frac{1}{2}\hat{k}$$

Ans. : (B)

- (20) If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{b} = \hat{j}$ and $\vec{c}$ is a vector such that $\vec{a}$, $\vec{c}$ and $\vec{b}$ follow the right hand thumb rule, then $\vec{c} = \dots$ [AIEEE : 2002]

- (A) $z\hat{i} - x\hat{k}$ (B) $\vec{0}$ (C) $y\hat{i}$ (D) $-z\hat{i} - x\hat{k}$

Sol. : Since $\vec{a}$, $\vec{c}$, $\vec{b}$ follow the right hand thumb rule, then $\vec{c} = \vec{b} \times \vec{a}$

$$\therefore \vec{c} = z\hat{i} - x\hat{k}$$

Ans. : (A)

- (21) $\vec{a} = 3\hat{i} - 5\hat{j}$, $\vec{b} = 6\hat{i} + 3\hat{j}$. If $\vec{c} = \vec{a} \times \vec{b}$, then $|\vec{a}| : |\vec{b}| : |\vec{c}| = \dots$ [AIEEE : 2002]

- (A) $\sqrt{34} : \sqrt{45} : \sqrt{39}$ (B) $\sqrt{34} : \sqrt{45} : 39$ (C) $34 : 39 : 45$ (D) $39 : 35 : 34$

Sol. : $|\vec{a}| = \sqrt{9+25} = \sqrt{34}$; $|\vec{b}| = \sqrt{36+9} = \sqrt{45}$

and $\vec{c} = \vec{a} \times \vec{b} = 39\hat{k}$. So, $|\vec{c}| = 39$

$$\therefore |\vec{a}| : |\vec{b}| : |\vec{c}| = \sqrt{34} : \sqrt{45} : 39$$

Ans. : (B)

- (22) If $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} - \hat{k}$, then the unit vector perpendicular to $\vec{a} + \vec{b}$ and $\vec{b} - \vec{c}$ is

- (A) $\pm\hat{i}$ (B) $\pm\hat{j}$ (C) $\pm\hat{k}$ (D) $\pm\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

Sol. : $\vec{a} + \vec{b} = 3\hat{i} + \hat{j}$ and $\vec{b} - \vec{c} = 3\hat{j}$

$$\therefore (\vec{a} + \vec{b}) \times (\vec{b} - \vec{c}) = 9\hat{k}$$

The unit vector perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{c})$ is $\pm \frac{[(\vec{a} + \vec{b}) \times (\vec{b} - \vec{c})]}{|(\vec{a} + \vec{b}) \times (\vec{b} - \vec{c})|} = \pm \frac{9\hat{k}}{|9\hat{k}|} = \pm \hat{k}$

Ans. : (C)

(23) If $\vec{a}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$ satisfy the right hand thumb rule, then $\hat{a} = \dots$

(A) $\frac{1}{5\sqrt{6}}(11\hat{i} - 2\hat{j} + 5\hat{k})$

(B) $\frac{1}{5\sqrt{6}}(11\hat{i} + 2\hat{j} - 5\hat{k})$

(C) $\frac{1}{5\sqrt{6}}(11\hat{i} + 2\hat{j} + 5\hat{k})$

(D) $\frac{1}{5}(11\hat{i} - 2\hat{j} - 5\hat{k})$

Sol. : $\vec{a}, \vec{b}, \vec{c}$ follow right hand thumb rule.

$\therefore \hat{a}, \vec{b}, \vec{c}$ will also follow right hand thumb rule.

$$\hat{a} = \frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|} = \frac{11\hat{i} + 2\hat{j} - 5\hat{k}}{5\sqrt{6}}$$

Ans. : (B)

(24) If $\vec{a}, \vec{b} = \hat{k} - \hat{j}$ and $\vec{c} = 2\hat{j} + \hat{k}$ are coplanar and $\vec{a} \perp \vec{b}$, then $\hat{a} = \dots$

(A) $\frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$

(B) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

(C) $\frac{1}{\sqrt{2}}(\hat{j} - \hat{k})$

(D) $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

Sol. : Suppose $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Now, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

$$\therefore [\vec{a} \vec{b} \vec{c}] = 0$$

$$\therefore \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0$$

$$\therefore -3a_1 = 0. \text{ So, } a_1 = 0$$

and $\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$. From this, $a_3 - a_2 = 0$. So, $a_3 = a_2$

$$\therefore \vec{a} = a_2\hat{j} + a_2\hat{k} \text{ and so } |\vec{a}| = \sqrt{2}|a_2|$$

$$\therefore \hat{a} = \pm \frac{\vec{a}}{|\vec{a}|} = \pm \frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$$

Ans. : (A)

Second method :

Vector coplanar with $\vec{b}$ and $\vec{c}$ and perpendicular to

$$\begin{aligned}\vec{b} & \text{ is } \vec{b} \times (\vec{b} \times \vec{c}) \\ & = (\vec{b} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{b}) \vec{c} = (-2 + 1)(\hat{k} - \hat{j}) - 2(2\hat{j} + \hat{k}) = -3\hat{j} - 3\hat{k}\end{aligned}$$

$$\therefore \text{ Required unit vector } = \pm \frac{(-3\hat{j} - 3\hat{k})}{3\sqrt{2}} = \pm \frac{1}{\sqrt{2}} (\hat{j} + \hat{k}) \quad \text{Ans. : (A)}$$

(25) $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{j} + \hat{k}$. If $\vec{c}$ is a unit vector, then the minimum value of $[\vec{a} \vec{b} \vec{c}]$ is

- (A) $\sqrt{89}$ (B) $-\sqrt{89}$ (C) 0 (D) $-\sqrt{15}$

Sol. : $[\vec{a} \vec{b} \vec{c}] = [\vec{c} \vec{a} \vec{b}]$

$$= \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$= |\vec{c}| |\vec{a} \times \vec{b}| \cos\theta \quad \text{where } \theta = (\vec{c}, \vec{a} \times \vec{b})$$

The minimum value of $\cos\theta = -1$ and so the minimum value of $[\vec{a} \vec{b} \vec{c}]$ is $-|\vec{a} \times \vec{b}|$.

$$\text{Now, } |\vec{a} \times \vec{b}| = |-7\hat{i} - 2\hat{j} + 6\hat{k}| = \sqrt{49 + 4 + 36} = \sqrt{89}$$

So, minimum value of $[\vec{a} \vec{b} \vec{c}]$ is $-\sqrt{89}$. Ans. : (B)

(26) A vector making angles of same measure with $\vec{a} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$, $\vec{b} = \frac{1}{5}(-3\hat{i} - 4\hat{k})$, $\vec{c} = \hat{j}$ and having magnitude $\sqrt{51}$ is

- (A) $5\hat{i} + \hat{j} + 5\hat{k}$ (B) $5\hat{i} - \hat{j} + 5\hat{k}$ (C) $5\hat{i} + \hat{j} - 5\hat{k}$ (D) $5\hat{i} - \hat{j} - 5\hat{k}$

Sol. : Suppose the required vector is $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$.

$$\text{Now, } |\vec{x}| = \sqrt{51}$$

$$\therefore x_1^2 + x_2^2 + x_3^2 = 51 \quad (1)$$

For vectors $\vec{a}, \vec{b}, \vec{c}$, $|\vec{a}| = 1, |\vec{b}| = 1, |\vec{c}| = 1$

$\vec{x}$ is making angles of equal measure with $\vec{a}, \vec{b}, \vec{c}$.

$$\therefore \frac{\vec{x} \cdot \vec{a}}{|\vec{x}| |\vec{a}|} = \frac{\vec{x} \cdot \vec{b}}{|\vec{x}| |\vec{b}|} = \frac{\vec{x} \cdot \vec{c}}{|\vec{x}| |\vec{c}|}$$

$$\therefore \vec{x} \cdot \vec{a} = \vec{x} \cdot \vec{b} = \vec{x} \cdot \vec{c}$$

$$\therefore \frac{1}{3}(2x_1 - 2x_2 + x_3) = \frac{1}{5}(-3x_1 - 4x_3) = x_2$$

$$\therefore 2x_1 - 2x_2 + x_3 = 3x_2 \text{ and } -3x_1 - 4x_3 = 5x_2$$

$$\therefore 2x_1 - 5x_2 + x_3 = 0 \text{ and } 3x_1 + 5x_2 + 4x_3 = 0$$

$$\therefore \frac{x_1}{-25} = \frac{x_2}{-5} = \frac{x_3}{25}$$

(By Cramer's rule)

$$\therefore \frac{x_1}{5} = \frac{x_2}{1} = \frac{x_3}{-5} = k \text{ (Assume)}$$

Substituting $x_1 = 5k$, $x_2 = k$, $x_3 = -5k$ in (1)

$k^2 = 1$. It follows that $k = \pm 1$

$$\therefore \vec{x} = \pm(5\hat{i} + \hat{j} - 5\hat{k})$$

Ans. : (C)

(27) If $\vec{a} + 2\vec{b}$ and $5\vec{a} - 4\vec{b}$ are perpendicular to each other for unit vectors $\vec{a}$ and $\vec{b}$, then

$$(\vec{a}, \vec{b}) = \dots\dots$$

[IIT : 2002]

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\cos^{-1} \frac{1}{3}$

(D) $\cos^{-1} \frac{2}{7}$

Sol. : $(\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 4\vec{b}) = 0$

$$\therefore 5|\vec{a}|^2 + 6(\vec{a} \cdot \vec{b}) - 8|\vec{b}|^2 = 0$$

$$\therefore 5 + 6(\vec{a} \cdot \vec{b}) - 8 = 0$$

$$\left(|\vec{a}| = |\vec{b}| = 1 \right)$$

$$\therefore \vec{a} \cdot \vec{b} = \frac{1}{2}$$

$$\text{Now, } (\vec{a}, \vec{b}) = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$$

Ans. : (B)

(28) If the inner product of the unit vector parallel to the sum of $4\hat{i} - 5\hat{j} + 2\hat{k}$ and

$2\hat{i} + 3\hat{j} + a\hat{k}$ with $\hat{i} + \hat{j} + \hat{k}$ is 1, then $a = \dots\dots$

(A) 1

(B) -1

(C) 0

(D) 2

Sol. : The sum of the given vectors is $\vec{x} = 6\hat{i} - 2\hat{j} + (2+a)\hat{k}$.

$\therefore$ A vector parallel to $\vec{x}$ is $\vec{y} = m\vec{x}$, $m \neq 0$

$$= m(6\hat{i} - 2\hat{j} + (2+a)\hat{k})$$

$$\therefore \hat{y} = \pm \frac{(6\hat{i} - 2\hat{j} + (2+a)\hat{k})}{\sqrt{44 + 4a + a^2}}$$

Now, $\hat{y} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$

$$\therefore \pm(6 - 2 + 2 + a) = \sqrt{a^2 + 4a + 44}$$

$$\therefore a^2 + 12a + 36 = a^2 + 4a + 44$$

$$\therefore 8a = 8. \text{ So, } a = 1$$

Ans. : (A)

(29) If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $\vec{x} = (1, a, a^2)$, $\vec{y} = (1, b, b^2)$ and $\vec{z} = (1, c, c^2)$ are

non-coplanar, then $abc = \dots$

(A) $(a-b)(b-c)(c-a)$

(B) 3

(C) 0

(D) -1

Sol. : $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\therefore (1 + abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

Since, $\vec{x}, \vec{y}, \vec{z}$ are non-coplanar, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$

$$\therefore 1 + abc = 0$$

$$\therefore abc = -1$$

Ans. : (D)

(30) $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between $3\hat{i} + 4\hat{j}$ and $\vec{c}$. The unit vector in the direction of $\vec{c}$ is

(A) $\frac{1}{15} (11\hat{i} + 10\hat{j} + 2\hat{k})$

(B) $\frac{1}{15} (11\hat{i} - 10\hat{j} + 2\hat{k})$

(C) $\frac{-1}{15} (11\hat{i} + 10\hat{j} + 2\hat{k})$

(D) $\frac{-1}{15} (11\hat{i} - 10\hat{j} + 2\hat{k})$

Sol. : Suppose $\vec{OP} = \hat{b}$ and $\vec{OQ} = \hat{c}$, where $\vec{b} = 3\hat{i} + 4\hat{j}$

$$\therefore \vec{OP} = \hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{5}(3\hat{i} + 4\hat{j})$$

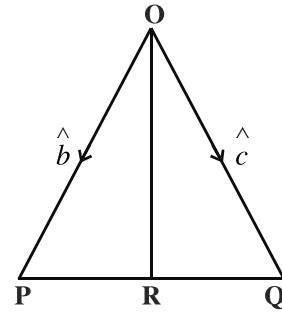


Figure 20.6

Suppose the midpoint of $\overline{PQ}$ is R.

$\therefore \overline{OR} \perp \overline{PQ}$ ($OP = OQ = 1$, so $\triangle OPQ$ is isosceles and $\overline{OR}$ is the median.)

Here $\triangle OPR \cong \triangle OQR$.

$$\therefore m\angle POR = m\angle QOR$$

$\therefore \vec{OR}$ is the angle bisector of $\angle POQ$.

$\therefore \vec{OR}$ is the angle bisector of $\hat{b}$ and $\hat{c}$ and so of $\vec{b}$ and $\vec{c}$.

But, $-\hat{i} + \hat{j} - \hat{k}$ is the angle bisector $\vec{b}$ and $\vec{c}$.

$$\therefore \vec{OR} = \lambda(-\hat{i} + \hat{j} - \hat{k}); \lambda \neq 0$$

$$\text{Now, } \vec{OP} + \vec{OQ} = \vec{OR} + \vec{RP} + \vec{OR} + \vec{RQ}$$

$$\begin{aligned} &= 2\vec{OR} + \vec{RP} - \vec{RP} \\ &= 2\vec{OR} \end{aligned}$$

$$\therefore \frac{1}{2}(\hat{b} + \hat{c}) = \lambda(-\hat{i} + \hat{j} - \hat{k})$$

$$\therefore \hat{c} = 2\lambda(-\hat{i} + \hat{j} - \hat{k}) - \hat{b}$$

$$= -2\lambda\hat{i} + 2\lambda\hat{j} - 2\lambda\hat{k} - \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\therefore \hat{c} = \left(-2\lambda - \frac{3}{5}\right)\hat{i} + \left(2\lambda - \frac{4}{5}\right)\hat{j} - 2\lambda\hat{k} \quad (1)$$

$$\text{Also, } \sqrt{\left(-2\lambda - \frac{3}{5}\right)^2 + \left(2\lambda - \frac{4}{5}\right)^2 + 4\lambda^2} = 1$$

$$\therefore 12\lambda^2 - \frac{4}{5}\lambda + 1 = 1$$

$$\therefore \lambda\left(12\lambda - \frac{4}{5}\right) = 0$$

But, $\lambda \neq 0$. So, $\lambda = \frac{1}{15}$

Placing value of λ in (1),

$$\hat{c} = \frac{-1}{15}(11\hat{i} + 10\hat{j} + 2\hat{k})$$

Ans. : (C)

- (31) If $A(\vec{x})$, $B(\vec{y})$ and $C(\vec{z})$ are vertices of an equilateral triangle ABC and orthocentre of ΔABC is origin, then is true.

(A) $\vec{x} + \vec{y} = \vec{z}$ (B) $|\vec{x}|^2 + |\vec{y}|^2 = |\vec{z}|^2$ (C) $\vec{x} + \vec{y} + \vec{z} = \vec{0}$ (D) $|\vec{x}| + |\vec{y}| = |\vec{z}|$

Sol. : Orthocentre of ΔABC is origin.

$\therefore$ Position vector of origin is $\vec{0}$.

Since, ΔABC is an equilateral triangle,

position vector of its orthocentre = position vector of centroid

$$\therefore \vec{0} = \frac{\vec{x} + \vec{y} + \vec{z}}{3}$$

$$\therefore \vec{x} + \vec{y} + \vec{z} = \vec{0}$$

Ans. : (C)

- (32) If $\vec{v} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{w} = \hat{i} + 3\hat{k}$ and $\vec{u}$ is a unit vector, then the maximum value of $[\vec{u} \vec{v} \vec{w}]$ is [IIT : 2002]

(A) -1 (B) $\sqrt{10} + \sqrt{16}$ (C) $\sqrt{59}$ (D) $\sqrt{6}$

Sol. : $[\vec{u} \vec{v} \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = |\vec{u}| |\vec{v} \times \vec{w}| \cos(\vec{u}, \vec{v} \times \vec{w})$

Maximum value of $\cos(\vec{u}, \vec{v} \times \vec{w})$ is 1 and $|\vec{u}| = 1$

$$\therefore \text{Maximum value of } [\vec{u} \vec{v} \vec{w}] = |\vec{v} \times \vec{w}| = |3\hat{i} - 7\hat{j} - \hat{k}| = \sqrt{59} \quad \text{Ans. : (C)}$$

- (33) $\vec{x} = 3\hat{i} + 2\hat{j} + 4\hat{k}$ and $\vec{y} = 4\hat{i} + 7\hat{j} + 4\hat{k}$. If $\vec{z} = \vec{x} \times \vec{y}$, then $|\vec{x}|^2 : |\vec{y}|^2 : |\vec{z}|^2 = \dots\dots$

(A) $\sqrt{29} : 9 : \sqrt{585}$ (B) $29 : 81 : 585$ (C) $9 : 15 : 3$ (D) $29 : 81 : 487$

Sol. : $|\vec{x}|^2 = 29$, $|\vec{y}|^2 = 81$

$$\vec{z} = \vec{x} \times \vec{y}$$

$$= -20\hat{i} + 4\hat{j} + 13\hat{k}$$

$$\therefore |\vec{z}|^2 = 400 + 16 + 169 = 585$$

$$\therefore |\vec{x}|^2 : |\vec{y}|^2 : |\vec{z}|^2 = 29 : 81 : 585$$

Ans. : (B)

(34) $\vec{x} = \hat{i} - \hat{j}$, $\vec{y} = \hat{j} - \hat{k}$ and $\vec{z} = \hat{k} - \hat{i}$. If $\vec{w}$ is a vector with magnitude $\sqrt{6}$, perpendicular to $\vec{x}$ and coplanar with $\vec{y}$ and $\vec{z}$, then $\vec{w} = \dots$

- (A) $\pm(\hat{i} - \hat{j} + \hat{k})$ (B) $\pm(\hat{i} + \hat{j} - \hat{k})$ (C) $\pm(\hat{i} - \hat{j} + 2\hat{k})$ (D) $\pm(\hat{i} + \hat{j} - 2\hat{k})$

Sol. : Suppose $\vec{w} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\vec{x} \perp \vec{w} \Rightarrow \vec{x} \cdot \vec{w} = 0$$

$$\therefore a_1 - a_2 = 0. \text{ So, } a_1 = a_2 \quad (1)$$

$\vec{w}$, $\vec{y}$ and $\vec{z}$ are coplanar.

$$\therefore [\vec{w} \ \vec{y} \ \vec{z}] = 0$$

$$\therefore \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\therefore a_1 + a_2 + a_3 = 0 \quad (2)$$

From (1) and (2),

$$2a_1 + a_3 = 0. \text{ So, } a_3 = -2a_1$$

Now, $|\vec{w}| = \sqrt{6}$

$$\therefore a_1^2 + a_2^2 + a_3^2 = 6$$

$$\therefore a_1^2 + a_1^2 + 4a_1^2 = 6$$

$$\therefore a_1^2 = 1. \text{ So, } a_1 = \pm 1 = a_2 \text{ and } a_3 = \mp 2$$

$$\therefore \vec{w} = \pm(\hat{i} + \hat{j} - 2\hat{k})$$

Ans. : (D)

(35) $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} + \hat{j}$. If $\vec{a} \cdot \vec{c} = |\vec{c}|$, $(\vec{a} \times \vec{b}, \vec{c}) = \frac{\pi}{6}$ and $|\vec{c} - \vec{a}| = 2\sqrt{2}$ for vector $\vec{c}$, then $|(\vec{a} \times \vec{b}) \times \vec{c}| = \dots$

- (A) 1 (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) 2

Sol. : $|\vec{c} - \vec{a}| = 2\sqrt{2}$

$$\therefore |\vec{c} - \vec{a}|^2 = 8$$

$$\therefore |\vec{c}|^2 - 2(\vec{c} \cdot \vec{a}) + |\vec{a}|^2 = 8$$

$$\therefore |\vec{c}|^2 - 2|\vec{c}| + 9 = 8. \text{ So, } |\vec{c}|^2 - 2|\vec{c}| + 1 = 0$$

$$\therefore (|\vec{c}| - 1)^2 = 0. \text{ So, } |\vec{c}| = 1$$

$$\text{Now, } |(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}| |\vec{c}| \sin((\vec{a} \times \vec{b}), \vec{c})$$

$$= |2\hat{i} - 2\hat{j} + \hat{k}| (1) \left(\frac{1}{2}\right) = \frac{3}{2}$$

Ans. : (B)

(36) If the position vectors of vertices A, B, C of right angled ΔABC are

$$2\hat{i} - \hat{j} + \hat{k}, \lambda\hat{i} - 3\hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}, \quad m\angle B = \frac{\pi}{2}, \text{ then } \lambda = \dots$$

(A) 1 or 2

(B) -1 or 2

(C) 1 or -2

(D) -1 or -2

Sol. : $m\angle B = \frac{\pi}{2} \Rightarrow \vec{AB} \perp \vec{BC}$

$$\Rightarrow \vec{AB} \cdot \vec{BC} = 0$$

$$\Rightarrow ((\lambda - 2)\hat{i} - 2\hat{j}) \cdot ((1 - \lambda)\hat{i} - 6\hat{k}) = 0$$

$$\Rightarrow (\lambda - 2)(1 - \lambda) = 0$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 1$$

Ans. : (A)

(37) The vector with magnitude $\sqrt{2}$, coplanar with $\hat{j} + \hat{k}$ and $\hat{i} + \hat{k}$ and perpendicular to $-2\hat{i} - 4\hat{j} + 2\hat{k}$ is

(A) $\hat{i} + \hat{k}$

(B) $\hat{i} + 2\hat{j} - \hat{k}$

(C) $\hat{i} + 2\hat{j} + 2\hat{k}$

(D) $3\hat{i} - 6\hat{j} - 3\hat{k}$

Sol. : Let the required vector be $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$.

$a\hat{i} + b\hat{j} + c\hat{k}$, $\hat{j} + \hat{k}$ and $\hat{i} + \hat{k}$ are coplanar.

$$\therefore \begin{vmatrix} a & b & c \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\therefore a + b - c = 0$$

Also, $a\hat{i} + b\hat{j} + c\hat{k}$ and $-2\hat{i} - 4\hat{j} + 2\hat{k}$ are perpendicular vectors.

$$\therefore -2a - 4b + 2c = 0$$

$$\therefore a + 2b - c = 0$$

Also, $a + b - c = 0$

$$\therefore \text{Let } \frac{a}{-1} = \frac{b}{0} = \frac{c}{-1} = \lambda$$

$$\therefore \vec{r} = \lambda(-\hat{i} - \hat{k}), \text{ and } |\vec{r}| = \sqrt{2} \Rightarrow \lambda = \pm 1$$

$$\therefore \vec{r} = \hat{i} + \hat{k} \text{ or } -\hat{i} - \hat{k}$$

Ans. : (A)

(38) If $\vec{a}$ and $\vec{b}$ are unit vectors, then $|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$ has maximum value

(A) 1

(B) 2

(C) $2\sqrt{2}$

(D) $\sqrt{2}$

Sol. : $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$

$$= 1 + 1 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$= 2 + 2\cos\theta = 2(1 + \cos\theta) = 2 \times 2\cos^2\frac{\theta}{2}$$

$$\therefore |\vec{a} + \vec{b}| = 2\cos\frac{\theta}{2} \quad 0 \leq \frac{\theta}{2} \leq \frac{\pi}{2}$$

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta \\ &= 2 - 2\cos\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2} \end{aligned}$$

$$\therefore |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$$

$$|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}| = 2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \leq 2\sqrt{2} \quad \left(\text{Maximum value of } \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \text{ is } \sqrt{2}\right)$$

$$\therefore \text{Maximum value of } |\vec{a} + \vec{b}| + |\vec{a} - \vec{b}| \text{ is } 2\sqrt{2}. \quad \text{Ans. : (C)}$$

(39) If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors, then $(\vec{u} + \vec{v} - \vec{w}) \cdot [(\vec{u} - \vec{v}) \times (\vec{v} - \vec{w})] = \dots$ [AIEEE : 2003]

(A) $\vec{u} \cdot (\vec{v} \times \vec{w})$

(B) $\vec{u} \cdot (\vec{w} \times \vec{v})$

(C) $3\vec{u} \cdot (\vec{v} \times \vec{w})$

(D) 0

$$\begin{aligned} \text{Sol. : } (\vec{u} + \vec{v} - \vec{w}) \cdot [(\vec{u} - \vec{v}) \times (\vec{v} - \vec{w})] &= (\vec{u} + \vec{v} - \vec{w}) \cdot [(\vec{u} \times \vec{v}) - (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})] \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) - \vec{v} \cdot (\vec{u} \times \vec{w}) - \vec{w} \cdot (\vec{u} \times \vec{v}) \\ &= [\vec{u} \quad \vec{v} \quad \vec{w}] - [\vec{v} \quad \vec{u} \quad \vec{w}] - [\vec{w} \quad \vec{u} \quad \vec{v}] \\ &= [\vec{u} \quad \vec{v} \quad \vec{w}] + [\vec{u} \quad \vec{v} \quad \vec{w}] - [\vec{u} \quad \vec{v} \quad \vec{w}] \\ &= [\vec{u} \quad \vec{v} \quad \vec{w}] \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) \end{aligned}$$

Ans. : (A)

Second method :

Let, $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$

$$\therefore D = \begin{vmatrix} u_1 + v_1 - w_1 & u_2 + v_2 - w_2 & u_3 + v_3 - w_3 \\ u_1 - v_1 & u_2 - v_2 & u_3 - v_3 \\ v_1 - w_1 & v_2 - w_2 & v_3 - w_3 \end{vmatrix}$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 - v_1 & u_2 - v_2 & u_3 - v_3 \\ v_1 - w_1 & v_2 - w_2 & v_3 - w_3 \end{vmatrix} \quad (R_1 \rightarrow R_1 - R_2 - R_3)$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ -w_1 & -w_2 & -w_3 \end{vmatrix} \quad (R_2 \rightarrow R_2 + R_1 \text{ and } R_3 \rightarrow R_3 - R_1)$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad (R_1 \rightarrow R_2)$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

Ans. : (A)

(40) $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i} + 2\hat{j} + 3\hat{k}$. If for $\hat{n}$, $\vec{u} \cdot \hat{n} = 0$, $\vec{v} \cdot \hat{n} = 0$, then $|\vec{w} \cdot \hat{n}| = \dots$ [AIEEE : 2003]

- (A) 1 (B) 2 (C) 3 (D) 0

Sol. : $\vec{u} \perp \hat{n}$ and $\vec{v} \perp \hat{n}$. For this, $\hat{n} = \pm \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \pm \hat{k}$

$\therefore |\vec{w} \cdot \hat{n}| = 3$

Ans. : (C)

(41) If the vectors $\vec{x} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{y} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{a}$ is a vector such that $\vec{a} \perp \vec{x}$ and $\vec{a} \perp \vec{y}$, then $\vec{a} = \dots$

- (A) $18\hat{i} + 3\hat{j} - 8\hat{k}$ (B) $22\hat{i} + \hat{j} - 8\hat{k}$
(C) $-22\hat{i} + 2\hat{j} + 6\hat{k}$ (D) $-22\hat{i} - 4\hat{j} + 10\hat{k}$

Sol. : $\vec{a} \perp \vec{x}$ and $\vec{a} \perp \vec{y}$

A vector perpendicular to both $\vec{x}$ and $\vec{y}$ is $\pm(\vec{x} \times \vec{y})$.

$\therefore \vec{a}$ and $\pm(\vec{x} \times \vec{y})$ are collinear vectors.

$\therefore \vec{a} = \lambda(\vec{x} \times \vec{y}) = \lambda(11\hat{i} + 2\hat{j} - 5\hat{k})$ ($\lambda \neq 0$)

If $\lambda = -2$, $\vec{a} = -22\hat{i} - 4\hat{j} + 10\hat{k}$

Ans. : (D)

(42) If $\vec{AB} = 3\hat{i} + 4\hat{k}$ and $\vec{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$, then the length of the median of ΔABC from A to $\overline{BC}$ is [AIEEE : 2003]

- (A) $\sqrt{72}$ (B) $\sqrt{33}$ (C) $\sqrt{288}$ (D) $\sqrt{18}$

Sol. : $\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2} = 4\hat{i} - \hat{j} + 4\hat{k}$

$\therefore |\vec{AD}| = \sqrt{16 + 1 + 16} = \sqrt{33}$

Ans. : (B)

(43) If for vectors $\vec{a}, \vec{b}, \vec{c}$, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 1, |\vec{b}| = 2, |\vec{c}| = 3$, then $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \dots$ [AIEEE : 2003]

- (A) -7 (B) 7 (C) 1 (D) 0

Sol. : $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

$\therefore 0 = 1 + 4 + 9 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

$\therefore \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -7$

Ans. : (A)

- (44) If the volume of a parallelopiped with edges $\hat{i} + a\hat{j} + \hat{k}$, $\hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ is minimum, then $a = \dots\dots$, $0 \leq a \leq 1$.

[IIT : 2003]

- (A) $\sqrt{3}$ (B) 2 (C) $\frac{1}{\sqrt{3}}$ (D) 3

Sol. : Volume of parallelopiped, $V = \text{Modulus of } \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = \text{Modulus of } 1 - a + a^3 = |1 - a + a^3|$

$$|1 - a + a^3| = |1 - a(1 - a^2)|$$

$$0 \leq a \leq 1 \Rightarrow 0 \leq a^2 \leq 1 \Rightarrow 0 \leq 1 - a^2 \leq 1$$

$$\Rightarrow 0 \leq a(1 - a^2) \leq a \leq 1$$

$$\therefore 1 - a + a^3 \leq 0$$

$$\therefore V = |1 - a + a^3| = 1 - a + a^3$$

$$\frac{dV}{da} = 0 \Rightarrow 3a^2 - 1 = 0 \Rightarrow a = \frac{1}{\sqrt{3}}, 0 < a < 1$$

$$\frac{d^2V}{da^2} = 6a = \frac{6}{\sqrt{3}} > 0$$

$$\therefore V \text{ is minimum when } a = \frac{1}{\sqrt{3}}.$$

$$V(0) = 1, V(1) = 1, V\left(\frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} = 1 - \frac{2}{3\sqrt{3}} \text{ is global minimum for } a = \frac{1}{\sqrt{3}}.$$

Ans. : (C)

- (45) For the volume of the parallelopiped with edges $\hat{i} + x\hat{j}$, $4\hat{i} - \hat{j} + x\hat{k}$ and $-x\hat{i} + \hat{j} - \hat{k}$ to be maximum, $x = \dots\dots$ where $x \in [0, \sqrt{3}]$

- (A) -3 (B) 1 (C) -1 (D) 3

Sol. : Volume of parallelopiped, $V = \text{Modulus of } \begin{vmatrix} 1 & x & 0 \\ 4 & -1 & x \\ -x & 1 & -1 \end{vmatrix} = \text{Modulus of } 1 + 3x - x^3$

$$V = |1 + 3x - x^3|$$

$$0 \leq x \leq \sqrt{3} \Rightarrow 0 \leq x^2 \leq 3 \Rightarrow 0 \leq (3 - x^2)$$

$$\therefore V = 1 + 3x - x^3$$

$$\frac{dV}{dx} = 0 \Rightarrow 3 - 3x^2 = 0 \Rightarrow x = 1$$

$$\frac{d^2V}{dx^2} = -6x < 0$$

$$\therefore V \text{ is maximum for } 0 \leq x \leq \sqrt{3} \text{ at } x = 1.$$

$$V(0) = 1, V(1) = 3, V(\sqrt{3}) = 1$$

$$\therefore V \text{ is global maximum for } x = 1.$$

Ans. : (B)

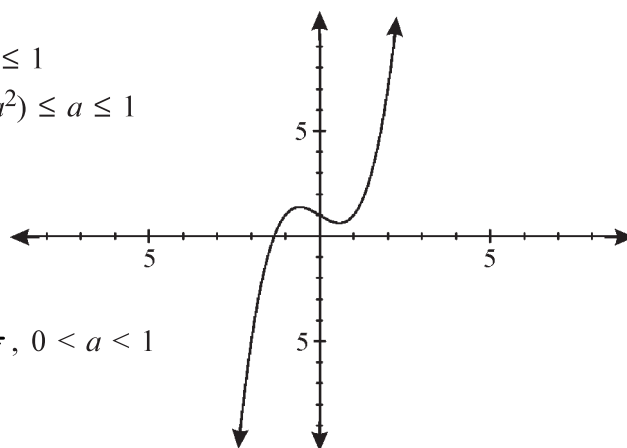


Figure 20.7

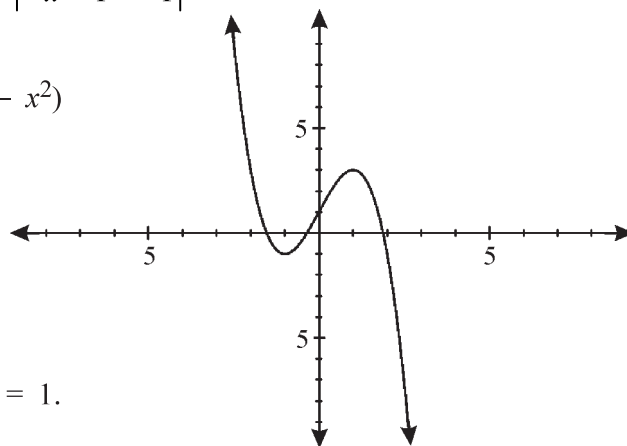


Figure 20.8

Note : $V = |1 + 3x - x^3|$

$\frac{dV}{dx}$ changes sign from +ve to -ve at $x = 1$.

$\therefore$ For $0 < x < 1$, V increases; for $x > 1$, V decreases

$\therefore V$ is maximum for $x = 1$.

- (46) Two forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ are acting on a particle and hence the particle moves from the point (1, 2, 3) to (5, 4, 1). The work done on the particle is [AIEEE : 2003, 2004]

(A) 30 units (B) 40 units (C) 50 units (D) 20 units

Sol. : Resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 = 7\hat{i} + 2\hat{j} - 4\hat{k}$

Displacement $\vec{d} = 4\hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore$ Work done, $w = \vec{F} \cdot \vec{d} = 28 + 4 + 8 = 40$ units

Ans. : (B)

- (47) On applying two forces $\vec{F}_1 = 2\hat{i} + a\hat{j} - \hat{k}$ and $\vec{F}_2 = 3\hat{i} + 4\hat{j} + \hat{k}$ on a particle, its displacement is $\vec{d} = 2\hat{i} - 3\hat{j} + 5\hat{k}$. If the work done is 34 units, then $a = \dots$

(A) -12 (B) 12 (C) -1 (D) 1

Sol. : $\vec{F} = \vec{F}_1 + \vec{F}_2 = 5\hat{i} + (a + 4)\hat{j}$

Work, $w = \vec{F} \cdot \vec{d}$

$\therefore 34 = 10 - 3a - 12$. So, $36 = -3a$. So, $a = -12$

Ans. : (A)

- (48) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero vectors and no two of them are collinear. If $\vec{a} + 2\vec{b}$ and $\vec{c}$ are collinear and $\vec{b} + 3\vec{c}$ and $\vec{a}$ are collinear, then $\vec{a} + 2\vec{b} + 6\vec{c} = \dots$ (λ is a non-zero constant)

[AIEEE : 2004]

(A) $\lambda \vec{a}$ (B) $\lambda \vec{b}$ (C) $\lambda \vec{c}$ (D) $\vec{0}$

Sol. : $\vec{a} + 2\vec{b}$ and $\vec{c}$ are collinear.

$\therefore \vec{a} + 2\vec{b} = k_1 \vec{c}$, $k_1 \in \mathbb{R} - \{0\}$

$\vec{b} + 3\vec{c}$ and $\vec{a}$ are collinear.

$\therefore \vec{b} + 3\vec{c} = k_2 \vec{a}$, $k_2 \in \mathbb{R} - \{0\}$

$\therefore \vec{a} + 2\vec{b} - 2(\vec{b} + 3\vec{c}) = k_1 \vec{c} - 2k_2 \vec{a}$

$\therefore \vec{a} - 6\vec{c} = k_1 \vec{c} - 2k_2 \vec{a}$

$\vec{a}$ and $\vec{c}$ are non-collinear.

Comparing coefficients of $\vec{a}$ and $\vec{c}$,

$1 = -2k_2$ and $-6 = k_1$

$\therefore k_2 = -\frac{1}{2}$ and $k_1 = -6$

Now, $\vec{a} + 2\vec{b} = k_1 \vec{c}$

$\therefore \vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$

Ans. : (D)

- (49) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and λ is a real number, then $\vec{a} + 2\vec{b} + 3\vec{c}, \lambda\vec{b} + \mu\vec{c}$ and $(2\lambda - 1)\vec{c}$ are non-coplanar for values of λ .

(A) all (B) all except one (C) all except two (D) exactly two

Sol. : If the vectors $\vec{a} + 2\vec{b} + 3\vec{c}, \lambda\vec{b} + \mu\vec{c}$ and $(2\lambda - 1)\vec{c}$ are coplanar,

then $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

$\therefore \lambda(2\lambda - 1) = 0$

$\therefore \lambda = \frac{1}{2}$ or $\lambda = 0$

$\therefore$ The given vectors are non-coplanar for all value of λ except two.

Ans. : (C)

- (50) $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$ for non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$. If θ is the measure of the angle between $\vec{b}$ and $\vec{c}$, then $\cos\theta = \dots\dots$

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{2}}{3}$ (C) $-\frac{1}{3}$ (D) $-\frac{\sqrt{2}}{3}$

Sol. : $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$

$\therefore (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$

(If $\vec{a}$ and $\vec{b}$ are collinear, then $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$. From this, $\vec{a} = \vec{0}$. But $\vec{a}$ is a non-zero vector.)

Comparing coefficients of $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{c} = 0$ and $-\vec{b} \cdot \vec{c} = \frac{1}{3} |\vec{b}| |\vec{c}|$

$\therefore |\vec{b}| |\vec{c}| \cos\theta = \frac{-1}{3} |\vec{b}| |\vec{c}|$

$\therefore \cos\theta = -\frac{1}{3}$

Ans. : (C)

- (51) The unit vector which is coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is

(A) $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$ (B) $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$ (C) $\frac{\hat{j} - \hat{k}}{\sqrt{2}}$ (D) $\frac{\hat{j} + \hat{k}}{\sqrt{2}}$

Sol. : The vector perpendicular to $\vec{x} = \hat{i} + \hat{j} + \hat{k}$ and coplanar with $\vec{y} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{z} = \hat{i} + 2\hat{j} + \hat{k}$ is $\pm(\vec{x} \times (\vec{y} \times \vec{z}))$.

Now, $\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z}) \vec{y} - (\vec{x} \cdot \vec{y}) \vec{z} = 4(\hat{i} + \hat{j} + 2\hat{k}) - 4(\hat{i} + 2\hat{j} + \hat{k}) = -4\hat{j} + 4\hat{k}$

$\therefore$ Required unit vector = $\pm \frac{(-\hat{j} + \hat{k})}{\sqrt{2}}$. So, required vector = $\frac{\hat{j} - \hat{k}}{\sqrt{2}}$

Ans. : (C)

- (52) If a unit vector $\vec{x}$ makes an angle of measure $\frac{\pi}{4}$ with X-axis, an angle of measure $\frac{\pi}{3}$ with Y-axis and an acute angle θ with Z-axis, then θ and $\vec{x}$ are respectively

(A) $\frac{\pi}{3}, \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ (B) $\frac{\pi}{3}, \frac{\sqrt{2}\hat{i} + \hat{j} + \hat{k}}{2}$ (C) $\frac{\pi}{3}, \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$ (D) $\frac{\pi}{3}, \frac{\sqrt{2}\hat{i} - \hat{j} + \hat{k}}{2}$

Sol. : If α, β, γ are the measures of the angles which $\vec{x}$ makes with axes, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \text{ and } \hat{x} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}.$$

$$\therefore \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\theta = 1$$

$$\therefore \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1. \text{ So, } \cos^2\theta = \frac{1}{4}$$

$$\therefore \cos\theta = \frac{1}{2} \quad (\theta \text{ is acute})$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\text{and } \vec{x} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} = \frac{1}{2}(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$$

Ans. : (B)

- (53) If $[\vec{a} \ \vec{b} \ \vec{a} \times \vec{b}] = \frac{1}{4}$ for unit vectors $\vec{a}$ and $\vec{b}$, then $(\vec{a}, \vec{b}) = \dots$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Sol. : $[\vec{a} \ \vec{b} \ \vec{a} \times \vec{b}] = \frac{1}{4}$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = \frac{1}{4}$$

$$\therefore |\vec{a} \times \vec{b}|^2 = \frac{1}{4}. \text{ So, } |\vec{a} \times \vec{b}| = \frac{1}{2}$$

$$\therefore |\vec{a}| |\vec{b}| \sin(\vec{a}, \vec{b}) = \frac{1}{2} \Rightarrow \sin(\vec{a}, \vec{b}) = \frac{1}{2}$$

$$\therefore (\vec{a}, \vec{b}) = \frac{\pi}{6}$$

Ans. : (D)

- (54) If the adjacent sides of a parallelogram are $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = 4\hat{i} + 3\hat{j}$, then the measure of the angle between its diagonals is

(A) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (B) $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ (C) $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ (D) $\frac{\pi}{2}$

Sol. : Diagonal vectors are $\vec{a} + \vec{b} = 7\hat{i} + 7\hat{j}$ and $\vec{a} - \vec{b} = -\hat{i} + \hat{j}$.

$$\text{Now, } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 25 - 25 = 0$$

$$\therefore (\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b})$$

$\therefore$ The measure of the angle between the diagonals is $\frac{\pi}{2}$.

Ans. : (D)

(55) If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$, then $\vec{b} = \dots$

[IIT : 2004]

(A) $\hat{i} - \hat{j} + \hat{k}$ (B) $2\hat{j} - \hat{k}$ (C) $\hat{i}$ (D) $2\hat{i}$

Sol. : $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = 3$

$$\vec{a} \times (\vec{a} \times \vec{b}) = (\hat{i} + \hat{j} + \hat{k}) \times (\hat{j} - \hat{k})$$

$$\therefore (\vec{a} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{a}) \vec{b} = \hat{i} \times \hat{j} - \hat{i} \times \hat{k} - \hat{j} \times \hat{k} + \hat{k} \times \hat{j}$$

$$\therefore 1(\hat{i} + \hat{j} + \hat{k}) - 3\vec{b} = \hat{k} + \hat{j} - \hat{i} - \hat{i}$$

$$\therefore 3\hat{i} = 3\vec{b}$$

$$\therefore \vec{b} = \hat{i}$$

Ans. : (C)

(56) The unit vector which is perpendicular to $\vec{a} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and coplanar with $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ is

[IIT : 2004]

(A) $\frac{2\hat{i} - 6\hat{j} + \hat{k}}{\sqrt{41}}$ (B) $\frac{2\hat{i} - 3\hat{j}}{\sqrt{13}}$ (C) $\frac{3\hat{j} - \hat{k}}{\sqrt{10}}$ (D) $\frac{4\hat{i} - 3\hat{j} - 3\hat{k}}{\sqrt{34}}$

Sol. : The vector which is perpendicular to $\vec{a}$ and coplanar with $\vec{b}$ and $\vec{c}$ is $\vec{a} \times (\vec{b} \times \vec{c})$.

$$\begin{aligned} \text{Now, } \vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \\ &= 7(2\hat{i} + \hat{j} + \hat{k}) - 14(\hat{i} - \hat{j} + \hat{k}) \\ &= 21\hat{j} - 7\hat{k} \end{aligned}$$

$$\therefore \text{Required unit vector} = \frac{3\hat{j} - \hat{k}}{\sqrt{10}}$$

Ans. : (C)

Second method :

Suppose $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$$\therefore [\vec{x} \ \vec{b} \ \vec{c}] = 0 \quad (1)$$

$$\text{and } \vec{x} \perp \vec{a} \Rightarrow \vec{x} \cdot \vec{a} = 0 \quad (2)$$

Now, required vector can be obtained from (1), (2) and $|\vec{x}| = 1$.

Ans. : (C)

- (57) If the sum of two unit vectors is a unit vector, then the magnitude of the difference between the vectors is

(A) 1 (B) $1 + \sqrt{3}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Sol. : Let $\vec{a} + \vec{b} = \vec{c}$, $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$$\text{Now, } |\vec{a} + \vec{b}|^2 = |\vec{c}|^2$$

$$\therefore |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 1$$

$$\therefore 1 + 1 + 2|\vec{a}||\vec{b}|\cos\theta = 1 \quad (\text{Taking } (\vec{a}, \vec{b}) = \theta)$$

$$\therefore \cos\theta = -\frac{1}{2}$$

$$\text{Now, } |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta = 1 + 1 - 2\left(-\frac{1}{2}\right) = 3$$

$$\therefore |\vec{a} - \vec{b}| = \sqrt{3}$$

Ans. : (D)

Second method :

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$$

$$\therefore |\vec{a} - \vec{b}|^2 = 2 + 2 - 1 = 3$$

$$\therefore |\vec{a} - \vec{b}| = \sqrt{3}$$

Ans. : (D)

- (58) $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors, $\vec{b}$ and $\vec{c}$ are not parallel. If $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$, then

$$(\vec{a}, \vec{b}) = \dots\dots$$

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Sol. : $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$

$$\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{1}{\sqrt{2}}\vec{b} + \frac{1}{\sqrt{2}}\vec{c}$$

Now, $\vec{b}$ and $\vec{c}$ are not parallel. So, $\vec{a} \cdot \vec{b} = \frac{-1}{\sqrt{2}}$

$$(\vec{a}, \vec{b}) = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \cos^{-1} \left(\frac{-1}{\sqrt{2}} \right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Ans. : (B)

- (59) If C is the midpoint of $\overline{AB}$ and $P \notin \overleftrightarrow{AB}$, then [AIEEE : 2005]

(A) $\vec{PA} + \vec{PB} = \vec{PC}$

(B) $\vec{PA} + \vec{PB} = 2\vec{PC}$

(C) $\vec{PA} + \vec{PB} + \vec{PC} = \vec{0}$

(D) $\vec{PA} + \vec{PB} + 2\vec{PC} = \vec{0}$

Sol. : From the triangle law of vectors,

For, ΔPAC , $\vec{PA} + \vec{AC} = \vec{PC}$ (1)

and for ΔPCB , $\vec{PB} + \vec{BC} = \vec{PC}$

$\therefore \vec{PB} - \vec{AC} = \vec{PC}$

($\vec{AC} = \vec{CB}$) (2)

From (1) and (2),

$\vec{PA} + \vec{PB} = 2\vec{PC}$

Ans. : (B)

Second method :

If P is taken as origin, the position vectors of A, B, C are $\vec{a}$, $\vec{b}$, $\frac{\vec{a} + \vec{b}}{2}$ respectively.

$\therefore \vec{PC} = \frac{\vec{a} + \vec{b}}{2}$

$\therefore 2\vec{PC} = \vec{a} + \vec{b} = \vec{PA} + \vec{PB}$

Ans. : (B)

- (60) If $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar, then $c = \dots\dots$

(a, b, c are different positive numbers.)

[AIEEE : 2005]

(A) Arithmetic mean of a and b

(B) Geometric mean of a and b

(C) Harmonic mean of a and b

(D) 0

Sol. : Vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar and so, $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\therefore a(-c) - a(b - c) + c(c) = 0$

$\therefore -ac - ab + ac + c^2 = 0$

$\therefore c^2 = ab$

$\therefore c$ is the G.M. of a and b .

Ans. : (B)

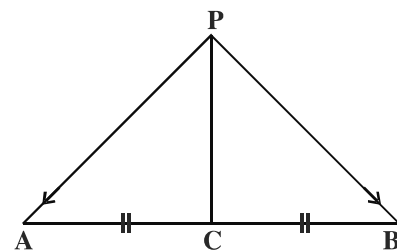


Figure 20.9

(61) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non coplanar vectors and λ is a real number, then

$$[\lambda(\vec{a} + \vec{b}) \quad \lambda^2 \vec{b} \quad \lambda \vec{c}] = [\vec{a} \quad \vec{b} + \vec{c} \quad \vec{b}] \text{ is } \dots \quad [\text{AIEEE : 2005}]$$

- (A) not possible for any λ . (B) possible for only one value of λ .
 (C) possible for only two values of λ . (D) possible for three values of λ .

Sol. : $[\lambda(\vec{a} + \vec{b}) \quad \lambda^2 \vec{b} \quad \lambda \vec{c}] = [\vec{a} \quad \vec{b} + \vec{c} \quad \vec{b}]$

$$\therefore \lambda(\vec{a} + \vec{b}) \cdot (\lambda^2 \vec{b} \times \lambda \vec{c}) = \vec{a} \cdot ((\vec{b} + \vec{c}) \times \vec{b})$$

$$\therefore \lambda^4 \{ \vec{a} \cdot (\vec{b} \times \vec{c}) \} = \vec{a} \cdot (\vec{c} \times \vec{b})$$

$$\therefore \lambda^4 \{ \vec{a} \cdot (\vec{b} \times \vec{c}) \} = -\vec{a} \cdot (\vec{b} \times \vec{c})$$

But $\vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$

$$\therefore \lambda^4 = -1 \text{ is not possible.}$$

$$\therefore \text{The given statement is not possible for any } \lambda.$$

Ans. : (A)

(62) If $2\vec{a} + \vec{b} = \hat{i} + \hat{j}$ and $\vec{a} + 2\vec{b} = \hat{i} - \hat{j}$ for vectors $\vec{a}$ and $\vec{b}$, then $\cos(\vec{a}, \vec{b}) = \dots$

- (A) $\frac{4}{5}$ (B) $-\frac{4}{5}$ (C) $\frac{4}{9}$ (D) $-\frac{4}{9}$

Sol. : $2\vec{a} + \vec{b} = \hat{i} + \hat{j}$ and $\vec{a} + 2\vec{b} = \hat{i} - \hat{j}$

$$\therefore 2\vec{a} + \vec{b} = \hat{i} + \hat{j} \text{ and } 2\vec{a} + 4\vec{b} = 2\hat{i} - 2\hat{j}$$

$$\therefore 3\vec{b} = \hat{i} - 3\hat{j}$$

$$\therefore \vec{b} = \frac{\hat{i} - 3\hat{j}}{3} \text{ and } \vec{a} = \hat{i} - \hat{j} - 2\left(\frac{\hat{i} - 3\hat{j}}{3}\right) = \frac{\hat{i} + 3\hat{j}}{3}$$

$$\therefore |\vec{b}| = \frac{\sqrt{10}}{3} \text{ and } |\vec{a}| = \frac{\sqrt{10}}{3}$$

$$\text{Now, } \cos(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{-8}{\frac{10}{9}} = -\frac{4}{5}$$

Ans. : (B)

(63) If $|\vec{F}_1| = |\vec{F}_2| = 5$ and the work done due to these forces is H and displacement is $\vec{d}$, then maximum value of H =

- (A) $|\vec{d}|$ (B) $2|\vec{d}|$ (C) $5|\vec{d}|$ (D) $10|\vec{d}|$

Sol. : $|\vec{F}_1| = |\vec{F}_2| = 5$

$$|\vec{F}| = |\vec{F}_1 + \vec{F}_2| \leq |\vec{F}_1| + |\vec{F}_2| \leq 10$$

$$\therefore \text{Maximum H} = 10|\vec{d}|$$

Ans. : (D)

(64) $|\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a} \perp \vec{b}$, $\vec{b} \perp \vec{c}$, $\vec{c} \perp \vec{a}$. Then $\vec{a} + \vec{b} + \vec{c}$ makes an angle of measure with any one of the given vectors.

- (A) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (C) $\cos^{-1}\left(\frac{1}{3}\right)$ (D) $\cos^{-1}\left(\frac{2}{3}\right)$

Sol. : Suppose $|\vec{a}| = |\vec{b}| = |\vec{c}| = m$

$$\text{Now, } |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 3m^2$$

$$(\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0)$$

$$\therefore |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}m$$

$$\begin{aligned} \text{Now, } \cos(\vec{a} + \vec{b} + \vec{c}, \vec{a}) &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} \\ &= \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{(\sqrt{3}m)m} = \frac{m^2}{\sqrt{3}m^2} = \frac{1}{\sqrt{3}} \end{aligned}$$

$$\therefore (\vec{a} + \vec{b} + \vec{c}, \vec{a}) = \cos^{-1} \frac{1}{\sqrt{3}}$$

Ans. : (A)

(65) Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$. If the magnitude of the projection of a vector in the plane of $\vec{a}$ and $\vec{b}$ on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, then the vector is [IIT : 2006]

- (A) $4\hat{i} + \hat{j} + 4\hat{k}$ (B) $3\hat{i} + \hat{j} - 3\hat{k}$ (C) $2\hat{i} + \hat{j} - 2\hat{k}$ (D) $4\hat{i} + \hat{j} - 4\hat{k}$

Sol. : A vector in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{r} = m\vec{a} + n\vec{b}$

$$= (m + n)\hat{i} + (2m - n)\hat{j} + (m + n)\hat{k}$$

Magnitude of projection of $\vec{r}$ on $\vec{c}$ is $\frac{1}{\sqrt{3}}$.

$$\therefore \frac{|\vec{r} \cdot \vec{c}|}{|\vec{c}|} = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{|(m + n)(1) + (2m - n)(1) + (m + n)(-1)|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore |2m - n| = 1$$

$$\therefore 2m - n = 1 \quad \text{or} \quad 2m - n = -1$$

$$\therefore 2m - 1 = n \quad 2m + 1 = n$$

$$\therefore \vec{r} = (3m - 1)\hat{i} + \hat{j} + (3m - 1)\hat{k} \quad \text{or} \quad \vec{r} = (3m + 1)\hat{i} - \hat{j} + (3m + 1)\hat{k}$$

Out of the given options, the only option with the coefficients of $\hat{i}$ and $\hat{k}$ to be same is $4\hat{i} + \hat{j} + 4\hat{k}$.

Ans. : (A)

- (66) The position vectors of the vertices of a triangle are $\vec{x}$, $\vec{y}$, $\vec{z}$. If the circumcentre of the triangle is origin, its orthocentre is

(A) $\vec{x} + \vec{y} + \vec{z}$ (B) $\frac{\vec{x} + \vec{y} + \vec{z}}{2}$ (C) $\frac{\vec{x} + \vec{y} + \vec{z}}{3}$ (D) origin

Sol. : The circumcentre of the triangle is origin ($\vec{0}$) and the position vector of centroid is $\frac{\vec{x} + \vec{y} + \vec{z}}{3}$. Centroid divides the line-segment joining orthocentre ($\vec{h}$) and circumcentre in the ratio 2:1 from orthocentre.

$$\therefore \frac{\vec{x} + \vec{y} + \vec{z}}{3} = \frac{2(\vec{0}) + 1(\vec{h})}{2 + 1}$$

$$\therefore \vec{h} = \vec{x} + \vec{y} + \vec{z}$$

Ans. : (A)

(67)
$$\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} = \dots$$

(A) $[\vec{a} \ \vec{b} \ \vec{c}]$ (B) $[\vec{a} \ \vec{b} \ \vec{c}]^2$ (C) $[\vec{a} \ \vec{b} \ \vec{c}]^3$ (D) 0

Sol. : Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, $\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$,

$$\begin{aligned} \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} &= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1 b_1 + a_2 b_2 + a_3 b_3 & a_1 c_1 + a_2 c_2 + a_3 c_3 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 & b_1^2 + b_2^2 + b_3^2 & b_1 c_1 + b_2 c_2 + b_3 c_3 \\ a_1 c_1 + a_2 c_2 + a_3 c_3 & b_1 c_1 + b_2 c_2 + b_3 c_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(Using properties of determinants)} \\ &= [\vec{a} \ \vec{b} \ \vec{c}]^2 \end{aligned}$$

Ans. : (B)

- (68) Vectors $-\lambda^2 \hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \lambda^2 \hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - \lambda^2 \hat{k}$, will be coplanar for values of λ .
 (A) zero (B) one (C) two (D) three

Sol. : If the vectors are coplanar,
$$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\therefore -\lambda^2(\lambda^4 - 1) - 1(-\lambda^2 - 1) + 1(1 + \lambda^2) = 0$$

$$\therefore (\lambda^2 + 1)(\lambda^2 + 1)(2 - \lambda^2) = 0$$

$$\text{Since, } \lambda^2 \geq 0, \lambda^2 = 2$$

$$\therefore \lambda = \pm\sqrt{2}$$

$\therefore$ Two values of λ satisfy the condition.

Ans. : (C)

(69) If for vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$, $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, then $x = \dots$ [AIEEE : 2007]

(A) 1 (B) -4 (C) -2 (D) 0

Sol. : $\vec{c}$ is in the plane of $\vec{a}$ and $\vec{b}$. So, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$$\therefore [\vec{a} \ \vec{b} \ \vec{c}] = 0$$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & (x-2) & -1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 0 & 0 & 1 \\ 2 & -3 & 2 \\ 2 & x-1 & -1 \end{vmatrix} = 0 \quad \left(\begin{array}{l} C_1 \rightarrow C_1 - C_2 \\ C_2 \rightarrow C_2 - C_3 \end{array} \right)$$

$$\therefore 2(x-1) + 6 = 0. \text{ From this, } x = -2.$$

Ans. : (C)

(70) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero, non coplanar vectors and $\vec{b}_1 = \vec{b} - \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a}$, $\vec{b}_2 = \vec{b} + \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a}$,

$$\vec{c}_1 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2} \vec{b}_1, \vec{c}_2 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2} \vec{b}_1, \vec{c}_3 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{c}|^2} \vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2} \vec{b}_1, \text{ and}$$

$$\vec{c}_4 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{c}|^2} \vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{b}|^2} \vec{b}_1, \text{ then } \dots \text{ are mutually perpendicular. [IIT : 2005]}$$

(A) $(\vec{a}, \vec{b}_1, \vec{c}_3)$ (B) $(\vec{a}, \vec{b}_1, \vec{c}_2)$ (C) $(\vec{a}, \vec{b}_1, \vec{c}_1)$ (D) $(\vec{a}, \vec{b}_2, \vec{c}_2)$

$$\text{Sol. : } \vec{a} \cdot \vec{b}_1 = \vec{a} \cdot \left(\vec{b} - \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} \right) = \vec{a} \cdot \vec{b} - \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} |\vec{a}|^2 = 0$$

$$\therefore \vec{a} \perp \vec{b}_1$$

$$\begin{aligned}\text{Now, } \vec{b}_1 \cdot \vec{c}_3 &= \vec{b}_1 \cdot \left(\vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2} \vec{b}_1 \right) \\ &= \vec{b}_1 \cdot \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} \cdot \vec{b}_1 + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2} |\vec{b}_1|^2 \neq 0\end{aligned}$$

∴ Option (A) is not the answer.

$$\begin{aligned}\text{Now, } \vec{b}_1 \cdot \vec{c}_2 &= \vec{b}_1 \cdot \left(\vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2} \vec{b}_1 \right) \\ &= \vec{b}_1 \cdot \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} \cdot \vec{b}_1 - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2} |\vec{b}_1|^2 = \vec{b}_1 \cdot \vec{c} - \vec{b}_1 \cdot \vec{c} = \vec{0} \quad (\vec{a} \cdot \vec{b}_1 = 0)\end{aligned}$$

$$\therefore \vec{b}_1 \perp \vec{c}_2$$

$$\begin{aligned}\text{and } \vec{a} \cdot \vec{c}_2 &= \vec{a} \cdot \left(\vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2} \vec{b}_1 \right) \\ &= \vec{a} \cdot \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2} |\vec{a}|^2 - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2} \vec{a} \cdot \vec{b}_1 = \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{c} = \vec{0} \quad (\vec{a} \cdot \vec{b}_1 = 0)\end{aligned}$$

$$\therefore \vec{a} \perp \vec{c}_2$$

∴ Vectors in $(\vec{a}, \vec{b}_1, \vec{c}_2)$ are mutually perpendicular.

(Note : This is called Gram - Schmidt orthogonalization process.)

Ans. : (B)

(71) $\vec{a}, \vec{b}, \vec{c}$ are non-zero, mutually perpendicular vectors with equal magnitudes and if vector $\vec{x}$

satisfies $\vec{a} \times \{(\vec{x} - \vec{b}) \times \vec{a}\} + \vec{b} \times \{(\vec{x} - \vec{c}) \times \vec{b}\} + \vec{c} \times \{(\vec{x} - \vec{a}) \times \vec{c}\} = \vec{0}$, then $\vec{x} = \dots$

$$(A) \frac{1}{2}(\vec{a} + \vec{b} - \vec{c}) \quad (B) \frac{1}{3}(\vec{a} + \vec{b} - \vec{c}) \quad (C) \frac{1}{2}(\vec{a} + \vec{b} + \vec{c}) \quad (D) \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$$

Sol. : Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = m$ and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

$$\vec{a} \times \{(\vec{x} - \vec{b}) \times \vec{a}\} + \vec{b} \times \{(\vec{x} - \vec{c}) \times \vec{b}\} + \vec{c} \times \{(\vec{x} - \vec{a}) \times \vec{c}\} = \vec{0}$$

$$\begin{aligned}\therefore \vec{a} \cdot \vec{a}(\vec{x} - \vec{b}) - (\vec{a} \cdot (\vec{x} - \vec{b}))\vec{a} + \vec{b} \cdot \vec{b}(\vec{x} - \vec{c}) - (\vec{b} \cdot (\vec{x} - \vec{c}))\vec{b} + \vec{c} \cdot \vec{c}(\vec{x} - \vec{a}) - (\vec{c} \cdot (\vec{x} - \vec{a}))\vec{c} \\ = \vec{0}\end{aligned}$$

$$\therefore m^2(\vec{x} - \vec{b}) + m^2(\vec{x} - \vec{c}) + m^2(\vec{x} - \vec{a}) - (\vec{a} \cdot \vec{x})\vec{a} - (\vec{b} \cdot \vec{x})\vec{b} - (\vec{c} \cdot \vec{x})\vec{c} = \vec{0}$$

$$\therefore 3m^2 \vec{x} - m^2(\vec{a} + \vec{b} + \vec{c}) - (\vec{a} \cdot \vec{x})\vec{a} - (\vec{b} \cdot \vec{x})\vec{b} - (\vec{c} \cdot \vec{x})\vec{c} = \vec{0}$$

Taking inner product with $\vec{a}$,

$$3m^2(\vec{x} \cdot \vec{a}) - m^2(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{x})(\vec{a} \cdot \vec{a}) = 0$$

$$\therefore 3m^2(\vec{x} \cdot \vec{a}) - m^2(m^2) - (\vec{a} \cdot \vec{x})m^2 = 0$$

$$\therefore 2(\vec{x} \cdot \vec{a}) - m^2 = 0 \quad (m \neq 0)$$

$$\therefore \vec{x} \cdot \vec{a} = \frac{m^2}{2}. \text{ Similarly, } \vec{x} \cdot \vec{b} = \frac{m^2}{2} \text{ and } \vec{x} \cdot \vec{c} = \frac{m^2}{2}$$

$$\therefore 3m^2 \vec{x} - m^2(\vec{a} + \vec{b} + \vec{c}) - \frac{m^2}{2}(\vec{a} + \vec{b} + \vec{c}) = \vec{0}$$

$$\therefore 3\vec{x} = \frac{3}{2}(\vec{a} + \vec{b} + \vec{c})$$

$$\therefore \vec{x} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c}) \quad \text{Ans. : (C)}$$

$$(72) \text{ If } \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \times (\vec{c} \times \vec{a}) \text{ and } [\vec{a} \ \vec{b} \ \vec{c}] \neq 0, \text{ then } \vec{c} \times (\vec{a} \times \vec{b}) = \dots..$$

$$(A) \vec{a} \times \vec{b} \quad (B) \vec{b} \times \vec{c} \quad (C) \vec{c} \times \vec{a} \quad (D) \vec{0}$$

$$\text{Sol. : } \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \times (\vec{c} \times \vec{a})$$

$$\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}$$

$$\therefore (\vec{b} \cdot \vec{c})\vec{a} + (\vec{a} \cdot \vec{c})\vec{b} - 2(\vec{a} \cdot \vec{b})\vec{c} = \vec{0}$$

Since, $[\vec{a} \ \vec{b} \ \vec{c}] \neq 0$, $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar.

$$\therefore l\vec{a} + m\vec{b} + n\vec{c} = \vec{0} \Rightarrow l = m = n = 0$$

$$\therefore \vec{b} \cdot \vec{c} = 0, \vec{a} \cdot \vec{c} = 0 \text{ and } \vec{a} \cdot \vec{b} = 0$$

$$\text{Now, } \vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b} = \vec{0} \quad \text{Ans. : (D)}$$

$$(73) \text{ If } \vec{a} = \alpha \hat{i} + 2\hat{j} + \beta \hat{k} \text{ is in the plane of } \vec{b} = \hat{i} + \hat{j} \text{ and } \vec{c} = \hat{j} + \hat{k} \text{ and } \vec{a} \text{ bisects the angle between } \vec{b} \text{ and } \vec{c}, \text{ then the values of } \alpha \text{ and } \beta \text{ are respectively [AIEEE : 2008]}$$

$$(A) 1 \text{ and } 1 \quad (B) 2 \text{ and } 2 \quad (C) 1 \text{ and } 2 \quad (D) 2 \text{ and } 1$$

$$\text{Sol. : The vector bisecting angle between } \vec{b} \text{ and } \vec{c} \text{ is } \vec{a} = \lambda \left(\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|} \right).$$

$$\therefore \alpha \hat{i} + 2\hat{j} + \beta \hat{k} = \lambda \left(\frac{\hat{i} + \hat{j}}{\sqrt{2}} + \frac{\hat{j} + \hat{k}}{\sqrt{2}} \right)$$

$$\therefore \alpha \hat{i} + 2\hat{j} + \beta \hat{k} = \frac{\lambda}{\sqrt{2}} \hat{i} + \sqrt{2}\lambda \hat{j} + \frac{\lambda}{\sqrt{2}} \hat{k}$$

$$\therefore \alpha = \frac{\lambda}{\sqrt{2}}, \quad 2 = \sqrt{2}\lambda, \quad \beta = \frac{\lambda}{\sqrt{2}}$$

$$\therefore \lambda = \sqrt{2}$$

$$\therefore \alpha = 1 \text{ and } \beta = 1$$

Ans. : (A)

(74) If $\vec{a} = 8\vec{b}$, $\vec{c} = -7\vec{b}$ for non-zero vectors, $\vec{a}$, $\vec{b}$, $\vec{c}$, then $(\vec{a}, \vec{c}) = \dots$

(A) π (B) 0 (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Sol. : $\cos(\vec{a}, \vec{c}) = \frac{\vec{a} \cdot \vec{c}}{|\vec{a}| |\vec{c}|} = \frac{(8\vec{b}) \cdot (-7\vec{b})}{|8\vec{b}| |-7\vec{b}|} = -\frac{56|\vec{b}|^2}{56|\vec{b}|^2} = -1$

$$\therefore (\vec{a}, \vec{c}) = \pi$$

Ans. : (A)

Note : $\vec{a}$ is in the direction of $\vec{b}$ and $\vec{c}$ is in the direction opposite to that of $\vec{b}$.

$$\therefore (\vec{a}, \vec{c}) = \pi$$

(75) The edges of a parallelopiped are of length 1 unit and they are parallel to non-coplanar vectors

$\hat{a}$, $\hat{b}$, $\hat{c}$. If $\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{c} = \hat{c} \cdot \hat{a} = \frac{1}{2}$, then the volume of the parallelopiped is [IIT : 2008]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{3}}$

Sol. : The volume of the parallelopiped will be equal to the magnitude of $[\hat{a} \hat{b} \hat{c}]$.

$$\begin{aligned} \text{Now, } [\hat{a} \hat{b} \hat{c}]^2 &= \begin{vmatrix} \hat{a} \cdot \hat{a} & \hat{a} \cdot \hat{b} & \hat{a} \cdot \hat{c} \\ \hat{a} \cdot \hat{b} & \hat{b} \cdot \hat{b} & \hat{b} \cdot \hat{c} \\ \hat{a} \cdot \hat{c} & \hat{b} \cdot \hat{c} & \hat{c} \cdot \hat{c} \end{vmatrix} \\ &= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = 1\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{3}{4} - \frac{1}{8} - \frac{1}{8} = \frac{1}{2} \end{aligned}$$

$$\therefore [\hat{a} \hat{b} \hat{c}] = \frac{1}{\sqrt{2}}$$

$$\therefore \text{The volume of parallelopiped} = \frac{1}{\sqrt{2}}$$

Ans. : (A)

- (76) Let the angle between $\vec{a}$ and $\vec{b}$ be acute. At time t , set of points P is $\vec{OP}$ which is $\hat{a} \cos t + \hat{b} \sin t$. When P is at a maximum distance from origin, the length of $\vec{OP}$ is m . If the unit vector in the direction of $\vec{OP}$ is $\hat{u}$, then [IIT : 2008]

$$(A) \hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|} \text{ and } m = (1 + \hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$(B) \hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|} \text{ and } m = (1 + \hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$(C) \hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|} \text{ and } m = (1 + 2\hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$(D) \hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|} \text{ and } m = (1 + 2\hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

Sol. : $|\vec{OP}| = |\hat{a} \cos t + \hat{b} \sin t|$

$$\begin{aligned} \therefore m &= (|\hat{a}|^2 \cos^2 t + |\hat{b}|^2 \sin^2 t + 2\hat{a} \cdot \hat{b} \cos t \sin t)^{\frac{1}{2}} \\ &= (\cos^2 t + \sin^2 t + 2\hat{a} \cdot \hat{b} \sin 2t)^{\frac{1}{2}} \\ &= (1 + \hat{a} \cdot \hat{b} \sin 2t)^{\frac{1}{2}} \end{aligned}$$

For maximum $|\vec{OP}|$, $\sin 2t = 1$. From this, $t = \frac{\pi}{4}$

$$\therefore m = (1 + \hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$\therefore \vec{OP} = \hat{a} \cos \frac{\pi}{4} + \hat{b} \sin \frac{\pi}{4} = \frac{\hat{a} + \hat{b}}{\sqrt{2}}$$

$$\therefore \text{The unit vector in the direction of } \vec{OP} \text{ is } \hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|} \quad \text{Ans. : (A)}$$

- (77) On applying a force of magnitude 5 in the direction of $\hat{i} + 2\hat{j} - 2\hat{k}$ on a particle, the particle is displaced from (3, 1, 2) to (7, 5, 3), then the work done is

$$(A) \frac{70}{3}$$

$$(B) \frac{50}{3}$$

$$(C) \frac{25}{3}$$

$$(D) \frac{35}{3}$$

Sol. : Force, $\vec{F} = \frac{5(\hat{i} + 2\hat{j} - 2\hat{k})}{|\hat{i} + 2\hat{j} - 2\hat{k}|} = \frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$

Displacement $\vec{d} = (7, 5, 3) - (3, 1, 2) = 4\hat{i} + 4\hat{j} + \hat{k}$

$$\therefore \text{Work} = \vec{F} \cdot \vec{d}$$

$$= \frac{5}{3}(4 + 8 - 2) = \frac{50}{3}$$

Ans. : (B)

(78) If $(\vec{a}, \vec{b}) = \frac{5\pi}{6}$ and $\text{Comp}_{\vec{b}} \vec{a} = -2\sqrt{3}$, then $|\vec{a}| = \dots$

(A) 4

(B) 6

(C) 12

(D) $4|\vec{b}|$

Sol. : $\text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$\therefore -2\sqrt{3} = \frac{|\vec{a}| |\vec{b}| \cos \frac{5\pi}{6}}{|\vec{b}|}$$

$$\therefore -2\sqrt{3} = |\vec{a}| \left(-\frac{\sqrt{3}}{2} \right)$$

$$\therefore |\vec{a}| = 4$$

Ans. : (A)

(79) Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are of same magnitude and the angle between any two of them has same measure. If $\vec{a} = \hat{j} + \hat{k}$ and $\vec{b} = \hat{k} + \hat{i}$, then $\vec{c} = \dots$

(A) $\hat{i} + \hat{j} + 2\hat{k}$

(B) $\hat{i} - \hat{j}$

(C) $\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{4}{3}\hat{k}$

(D) $-\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{4}{3}\hat{k}$

Sol. : $|\vec{a}| = |\vec{b}| = \sqrt{2} = |\vec{c}|$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$. From this $x^2 + y^2 + z^2 = 2$ (1)

The angle between any two of the vectors has the same measure.

$$\therefore \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}| |\vec{c}|} = \frac{\vec{c} \cdot \vec{a}}{|\vec{c}| |\vec{a}|}$$

$$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$$

$$\therefore 1 = y + z \text{ and } 1 = x + z$$

$$\therefore 1 - z = y \text{ and } 1 - z = x$$

Substituting values of x and y in (1),

$$(1 - z)^2 + (1 - z)^2 + z^2 = 2$$

$$\therefore 3z^2 - 4z = 0$$

$$\therefore z = 0 \text{ or } z = \frac{4}{3}$$

$$\therefore x = y = 1 \text{ or } x = y = 1 - \frac{4}{3} = -\frac{1}{3}$$

$$\therefore \vec{c} = \hat{i} + \hat{j} \text{ or } \vec{c} = -\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{4}{3}\hat{k}$$

Ans. : (D)

- (80) If the points with position vectors $3\hat{i} + 60\hat{j}$, $-52\hat{i} + a\hat{j}$ and $-8\hat{i} + 40\hat{j}$ are collinear, then $a = \dots$
- (A) 20 (B) -20 (C) 40 (D) -40

Sol. : $A(3\hat{i} + 60\hat{j})$, $B(-52\hat{i} + a\hat{j})$ and $C(-8\hat{i} + 40\hat{j})$ are collinear.

$$\therefore \vec{AB} = \lambda \vec{BC}$$

$$\therefore (-52\hat{i} + a\hat{j}) - (3\hat{i} + 60\hat{j}) = \lambda \{(-8\hat{i} + 40\hat{j}) - (-52\hat{i} + a\hat{j})\}$$

$$\therefore -55\hat{i} + (a - 60)\hat{j} = \lambda(44\hat{i} + (40 - a)\hat{j})$$

$$\therefore -55 = 44\lambda \text{ and } a - 60 = \lambda(40 - a)$$

$$\therefore -\frac{5}{4} = \lambda \text{ and } a - 60 = -\frac{5}{4}(40 - a)$$

$$\therefore 4a - 240 = -200 + 5a$$

$$\therefore -40 = a$$

Ans. : (D)

- (81) If ABCDEF is a regular hexagon and $\vec{AB} = x$, $\vec{BC} = y$, $\vec{CD} = z$, then $\vec{AE} = \dots$
- (A) $x + y$ (B) $y + z$ (C) $z + x$ (D) $x + y + z$

Sol. : $\vec{AE} = \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE}$

$$= x + y + z + \vec{BA}$$

$$= x + y + z - \vec{AB}$$

$$= x + y + z - x$$

$$= y + z$$

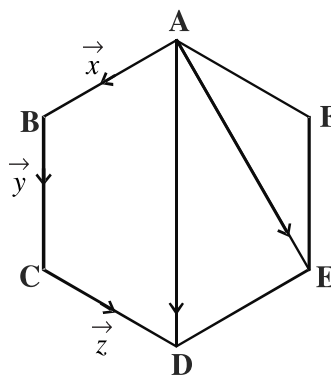


Figure 20.10

Ans. : (B)

Second method :

$$\vec{AE} = \vec{AF} + \vec{FE}$$

$$= \vec{BC} + \vec{CD} = y + z$$

Ans. : (B)

- (82) If ABCDEF is a regular hexagon, then $\vec{AD} + \vec{EB} + \vec{FC} = \dots$

- (A) $\vec{AB}$ (B) $2\vec{AB}$ (C) $3\vec{AB}$ (D) $4\vec{AB}$

Sol. : For a regular hexagon ABCDEF,

$$\vec{AD} \parallel \vec{BC} \text{ and } \vec{AD} = 2\vec{BC}$$

$$\vec{EB} \parallel \vec{FA} \text{ and } \vec{EB} = 2\vec{FA}$$

$$\vec{FC} \parallel \vec{AB} \text{ and } \vec{FC} = 2\vec{AB}$$

$$\text{Now, } \vec{AD} + \vec{EB} + \vec{FC} = 2(\vec{BC} + \vec{FA} + \vec{AB})$$

$$= 2(\vec{BC} + \vec{FB})$$

$$= 2\vec{FC}$$

$$= 2(2\vec{AB})$$

$$= 4\vec{AB}$$

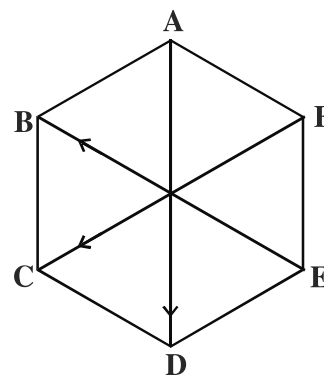


Figure 20.11

Ans. : (D)

(83) If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are non-coplanar vectors and

$$[3\vec{u} \quad p\vec{v} \quad p\vec{w}] - [p\vec{v} \quad \vec{w} \quad q\vec{u}] - [2\vec{w} \quad q\vec{v} \quad q\vec{u}] = 0, \text{ then [AIEEE : 2009]}$$

(A) only one value for (p, q)

(B) two values for (p, q)

(C) more than two values are possible, but not all the values

(D) all values are possible for (p, q)

$$\text{Sol. : } [3\vec{u} \quad p\vec{v} \quad p\vec{w}] - [p\vec{v} \quad \vec{w} \quad q\vec{u}] - [2\vec{w} \quad q\vec{v} \quad q\vec{u}] = 0$$

$$\therefore 3p^2[\vec{u} \quad \vec{v} \quad \vec{w}] - pq[\vec{u} \quad \vec{v} \quad \vec{w}] + 2q^2[\vec{u} \quad \vec{v} \quad \vec{w}] = 0$$

$$\therefore 3p^2 - pq + 2q^2 = 0 \quad (\text{Since } \vec{u}, \vec{v}, \vec{w} \text{ are non-coplanar, } [\vec{u} \quad \vec{v} \quad \vec{w}] \neq 0)$$

$$\therefore q^2 - 24q^2 \geq 0 \quad (\text{Taking it as a quadratic equation in } p)$$

$$\therefore -23q^2 \geq 0$$

$$\therefore q^2 \leq 0. \text{ But since } q^2 \geq 0, q = 0 \text{ and from that } p = 0$$

$$\therefore (p, q) = (0, 0) \text{ is the only possible value.}$$

Ans. : (A)

(84) If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are unit vectors such that $\vec{a} \cdot \vec{c} = \frac{1}{2}$ and $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$, then [IIT : 2009]

(A) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar

(B) $\vec{a}$, $\vec{b}$, $\vec{d}$ are non-coplanar

(C) $\vec{b}$ and $\vec{d}$ are not parallel

(D) $\vec{a}$ and $\vec{d}$ are parallel and $\vec{b}$ and $\vec{c}$ are parallel

Sol. : Suppose $(\vec{a}, \vec{b}) = \alpha, (\vec{c}, \vec{d}) = \beta$ and $(\vec{a} \times \vec{b}, \vec{c} \times \vec{d}) = \gamma$

Now, $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$

$$\therefore |\vec{a} \times \vec{b}| |\vec{c} \times \vec{d}| \cos \gamma = 1$$

$$\therefore (|\vec{a}| |\vec{b}| \sin \alpha) (|\vec{c}| |\vec{d}| \sin \beta) \cos \gamma = 1$$

$$\therefore \sin \alpha \sin \beta \cos \gamma = 1$$

This is possible only if $\sin \alpha = 1, \sin \beta = 1$ and $\cos \gamma = 1$. ($\sin \alpha \geq 0, \sin \beta \geq 0$)

$$\therefore \alpha = \frac{\pi}{2}, \quad \beta = \frac{\pi}{2}, \quad \gamma = 0$$

$$\therefore \vec{a} \perp \vec{b}, \vec{c} \perp \vec{d} \text{ and } (\vec{a} \times \vec{b}) \parallel (\vec{c} \times \vec{d}). \text{ So, } \vec{a} \times \vec{b} = k(\vec{c} \times \vec{d})$$

Taking dot product with $\vec{c}$ and $\vec{d}$ respectively,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = k(\vec{c} \times \vec{d}) \cdot \vec{c} = 0 \text{ and } (\vec{a} \times \vec{b}) \cdot \vec{d} = k(\vec{c} \times \vec{d}) \cdot \vec{d} = 0$$

$$\therefore \vec{a}, \vec{b}, \vec{c} \text{ are coplanar and } \vec{a}, \vec{b}, \vec{d} \text{ are coplanar.}$$

$\therefore$ Option (A) and (B) are not possible.

$$\text{Let, } \vec{b} \parallel \vec{d} \text{ if possible. Then } \vec{b} = \pm \vec{d} \quad (|\vec{b}| = |\vec{d}| \text{ and } \vec{b} \parallel \vec{d})$$

$$\text{Also, } (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1. \text{ So, } (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{b}) = \pm 1 \quad (\text{Taking } \vec{d} = \pm \vec{b})$$

$$\therefore [\vec{a} \times \vec{b} \quad \vec{c} \quad \vec{b}] = \pm 1$$

$$\therefore [\vec{c} \quad \vec{b} \quad \vec{a} \times \vec{b}] = \pm 1$$

$$\therefore \vec{c} \cdot (\vec{b} \times (\vec{a} \times \vec{b})) = \pm 1$$

$$\therefore \vec{c} \cdot ((\vec{b} \cdot \vec{b}) \vec{a} - (\vec{b} \cdot \vec{a}) \vec{b}) = \pm 1$$

$$\therefore \vec{c} \cdot \vec{a} = \pm 1. \text{ But } \vec{a} \cdot \vec{c} = \frac{1}{2} \quad (\vec{b} \cdot \vec{b} = |\vec{b}|^2 = 1 \text{ and } \vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0)$$

$\therefore \vec{b}$ and $\vec{d}$ are not parallel vectors.

$$\text{Taking } \vec{a} \parallel \vec{d} \text{ and } \vec{b} \parallel \vec{c}, \text{ then } \vec{d} = \pm \vec{a} \text{ and } \vec{c} = \pm \vec{b}$$

$$\text{But then } \vec{a} \cdot \vec{c} = 0. \text{ But } \vec{a} \cdot \vec{c} = \frac{1}{2}$$

$\therefore$ Option (C) is true.

Ans. : (C)

(85) If $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k})$, then $\vec{a} = \dots$

- (A) $\hat{i} + \hat{j} + \hat{k}$ (B) $\hat{i}$ (C) $\hat{j}$ (D) $\hat{k}$

Sol. : Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\text{Now, } \vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k})$$

$$\therefore a_1 = a_1 + a_2 = a_1 + a_2 + a_3$$

$$\therefore a_1 = a_1 + a_2 \text{ and } a_1 + a_2 = a_1 + a_2 + a_3$$

$$\therefore a_2 = 0 \text{ and } a_3 = 0$$

$$\therefore \vec{a} = a_1 \hat{i}. \text{ Taking } a_1 = 1, \vec{a} = \hat{i}$$

Ans. : (B)

(86) If $x\hat{i} + 8\hat{j}$ is a vector in the direction of $-3\hat{i} + 4\hat{j}$ and has magnitude 10, then $x = \dots$

- (A) 6 (B) 2 (C) -6 (D) -2

Sol. : A unit vector in the direction of $-3\hat{i} + 4\hat{j}$ is $\frac{1}{5}(-3\hat{i} + 4\hat{j})$

$$\therefore \text{The vector in the direction of } -3\hat{i} + 4\hat{j} \text{ and with magnitude 10 is } = \frac{10}{5}(-3\hat{i} + 4\hat{j})$$

$$\therefore x\hat{i} + 8\hat{j} = -6\hat{i} + 8\hat{j}$$

$$\therefore x = -6$$

Ans. : (C)

(87) If $\vec{a} = (2, 1, -1)$, $\vec{b} = (5, -1, 1)$ and $\vec{c} = (1, -1, 0)$, then a vector with magnitude 3 and parallel to $\vec{a} - \vec{b} + \vec{c}$, but in the opposite direction, is $\dots$

- (A) $\frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$ (B) $2\hat{i} - \hat{j} + 2\hat{k}$ (C) $\frac{1}{3}(-2\hat{i} + \hat{j} - 2\hat{k})$ (D) $-2\hat{i} + \hat{j} - 2\hat{k}$

Sol. : $\vec{a} - \vec{b} + \vec{c} = (2, 1, -1) - (5, -1, 1) + (1, -1, 0)$
 $= (-2, 1, -2)$

$$\therefore |\vec{a} - \vec{b} + \vec{c}| = \sqrt{4 + 1 + 4} = 3$$

$\therefore$ The vector with magnitude 3 and in the direction opposite to that of $\vec{a} - \vec{b} + \vec{c}$,

$$= -3 \left(\frac{\vec{a} - \vec{b} + \vec{c}}{|\vec{a} - \vec{b} + \vec{c}|} \right)$$

$$= 2\hat{i} - \hat{j} + 2\hat{k}$$

Ans. : (B)

(88) If the angle between any two of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ has measure $\frac{\pi}{3}$ and

$|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 2$, then $|\vec{a} + \vec{b} - \vec{c}| = \dots$

- (A) $3\sqrt{3}$ (B) $\sqrt{55}$ (C) 27 (D) 55

Sol. : Here, $(\vec{a}, \vec{b}) = (\vec{b}, \vec{c}) = (\vec{c}, \vec{a}) = \frac{\pi}{3}$

$$\begin{aligned} \text{Now, } |\vec{a} + \vec{b} - \vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{c} - \vec{c} \cdot \vec{a}) \\ &= 9 + 16 + 4 + 2 \left(|\vec{a}||\vec{b}|\cos\frac{\pi}{3} - |\vec{b}||\vec{c}|\cos\frac{\pi}{3} - |\vec{c}||\vec{a}|\cos\frac{\pi}{3} \right) \\ &= 29 + 2 \left\{ (3)(4)\left(\frac{1}{2}\right) - (4)(2)\frac{1}{2} - (2)(3)\frac{1}{2} \right\} \\ &= 29 + 2\{6 - 4 - 3\} = 29 - 2 = 27 \end{aligned}$$

$$\therefore |\vec{a} + \vec{b} - \vec{c}| = 3\sqrt{3} \quad \text{Ans. : (A)}$$

(89) Let $\vec{a} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. If $(\vec{a} \times \vec{b}) + \vec{c} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 3$, then $\vec{b} = \dots$

[AIEEE : 2010]

- (A) $-\hat{i} + \hat{j} - 2\hat{k}$ (B) $2\hat{i} - \hat{j} + 2\hat{k}$ (C) $\hat{i} - \hat{j} - 2\hat{k}$ (D) $\hat{i} + \hat{j} - 2\hat{k}$

Sol. : $(\vec{a} \times \vec{b}) + \vec{c} = \vec{0}$

$$\therefore \vec{a} \times [(\vec{a} \times \vec{b}) + \vec{c}] = \vec{a} \times \vec{0}$$

$$\therefore \vec{a} \times (\vec{a} \times \vec{b}) + \vec{a} \times \vec{c} = \vec{0}$$

$$\therefore (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\therefore 3(\hat{j} - \hat{k}) - 2\vec{b} + (-2\hat{i} - \hat{j} - \hat{k}) = \vec{0}$$

$$\therefore -2\hat{i} + 2\hat{j} - 4\hat{k} = 2\vec{b}$$

$$\therefore -\hat{i} + \hat{j} - 2\hat{k} = \vec{b} \quad \text{Ans. : (A)}$$

(90) If $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually perpendicular vectors, then $(\lambda, \mu) = \dots$

[AIEEE : 2010]

- (A) $(-3, 2)$ (B) $(2, -3)$ (C) $(-2, 3)$ (D) $(3, -2)$

Sol. : Given vectors are mutually perpendicular.

$$\begin{array}{l|l} \therefore \vec{a} \perp \vec{c} \Rightarrow \vec{a} \cdot \vec{c} = 0 & \vec{b} \perp \vec{c} \Rightarrow \vec{b} \cdot \vec{c} = 0 \\ \therefore \lambda - 1 + 2\mu = 0 & \therefore 2\lambda + 4 + \mu = 0 \\ \therefore \lambda + 2\mu = 1 & \therefore 2\lambda + \mu = -4 \end{array}$$

Solving the equations $(\lambda, \mu) = (-3, 2)$

Ans. : (A)

- (91) If the position vectors of points P, Q, R and S in a plane are $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively, then quadrilateral PQRS is

[IIT : 2010]

- (A) a parallelogram which is not a rectangle or rhombus
 (B) square
 (C) rectangle, but not a square
 (D) rhombus

Sol. : Sides are $\vec{PQ} = 6\hat{i} + \hat{j}$, $\vec{QR} = -\hat{i} + 3\hat{j}$, $\vec{RS} = -6\hat{i} - \hat{j}$, $\vec{SP} = \hat{i} - 3\hat{j}$ and diagonals are $\vec{PR} = 5\hat{i} + 4\hat{j}$ and $\vec{QS} = -7\hat{i} + 2\hat{j}$.

Here, since $\vec{PQ} \parallel \vec{RS}$ and $\vec{QR} \parallel \vec{SP}$, quadrilateral PQRS is a parallelogram.

$\vec{PQ} \cdot \vec{QR} \neq 0$. From this it follows that, it is not a rectangle or square.

$\vec{PR} \cdot \vec{QS} \neq 0$. From this it follows that, it is not a rhombus.

$\therefore$ The quadrilateral PQRS is a parallelogram which is not rectangle or rhombus. **Ans. : (A)**

- (92) The adjacent sides of a parallelogram ABCD are represented by vectors $\vec{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\vec{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$. Side $\vec{AD}$ of the parallelogram ABCD rotates in the plane of ABCD through an angle with measure α and it becomes $\vec{AD'}$. If $\vec{AD'}$ makes right angle with $\vec{AB}$, then $\cos\alpha = \dots$

[ITT : 2010]

- (A) $\frac{8}{9}$ (B) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

Sol. : Suppose $m\angle BAD = \beta$

$$\vec{AB} \cdot \vec{AD} = -2 + 20 + 22 = 40 > 0$$

$$\therefore \cos\beta > 0$$

$$\therefore 0 < \beta < \frac{\pi}{2}$$

i.e. β is an acute angle.

$$\cos\beta = \frac{\vec{AB} \cdot \vec{AD}}{|\vec{AB}||\vec{AD}|} = \frac{40}{(15)(3)} = \frac{8}{9}$$

$$\therefore \sin\beta = \frac{\sqrt{17}}{9}$$

Now, $(\vec{AD}, \vec{AD'}) = \alpha$ and $\vec{AD'} \perp \vec{AB}$

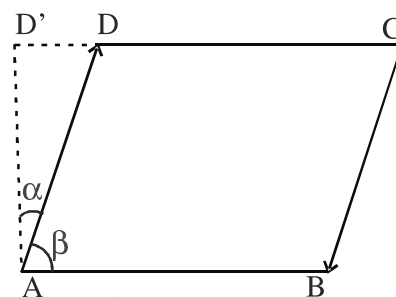


Figure 20.12

$$\therefore \alpha + \beta = \frac{\pi}{2}$$

$$\therefore \cos(\alpha + \beta) = 0$$

$$\therefore \cos \alpha \cos \beta - \sin \alpha \sin \beta = 0$$

$$\therefore \frac{8}{9} \cos \alpha - \frac{\sqrt{17}}{9} \sin \alpha = 0$$

$$\therefore 8 \cos \alpha = \sqrt{17} \sin \alpha$$

$$\therefore 64 \cos^2 \alpha = 17(1 - \cos^2 \alpha)$$

$$\therefore 81 \cos^2 \alpha = 17. \text{ So, } \cos \alpha = \frac{\sqrt{17}}{9}$$

Ans. : (B)

(93) If vectors $p\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ are coplanar, then

$$pqr - (p + q + r) = \dots\dots$$

[AIEEE : 2011]

(A) 2

(B) 0

(C) -1

(D) -2

Sol. : Since the vectors are coplanar,

$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$$\therefore p(qr - 1) - 1(r - 1) + 1(1 - q) = 0$$

$$\therefore pqr - p - r + 1 + 1 - q = 0$$

$$\therefore pqr - (p + q + r) = -2$$

Ans. : (D)

(94) $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. The magnitude of projection of $\vec{v}$, lying in the plane of $\vec{a}$ and $\vec{b}$ on $\vec{c}$ is $\frac{1}{\sqrt{3}}$. Then $\vec{v} = \dots\dots$ [IIT : 2011]

(A) $\hat{i} - 3\hat{j} + 3\hat{k}$

(B) $-3\hat{i} - 3\hat{j} - \hat{k}$

(C) $3\hat{i} - \hat{j} + 3\hat{k}$

(D) $\hat{i} + 3\hat{j} - 3\hat{k}$

Sol. : $\vec{v}$ is a vector in the plane of $\vec{a}$ and $\vec{b}$.

$$\therefore \vec{v} = m\vec{a} + n\vec{b}, m, n \in \mathbb{R}$$

$$= (m + n)\hat{i} + (m - n)\hat{j} + (m + n)\hat{k}$$

$$\left| \text{Proj}_{\vec{c}} \vec{v} \right| = \frac{|\vec{v} \cdot \vec{c}|}{|\vec{c}|}$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{|(m + n)(1) + (m - n)(-1) + (m + n)(-1)|}{\sqrt{3}}$$

$$\therefore \pm 1 = (m - n)$$

$$\therefore m - n = \pm 1$$

$\therefore$ In $\vec{v}$, the coefficients of $\hat{i}$ and $\hat{k}$ are the same. Also by taking $m - n = -1$,

Out of the given options, $\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$.

Ans. : (C)

- (95) The bisector of the angle between vectors $\vec{a} = 4\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$ is $\vec{c}$.

If $|\vec{c}| = 3\sqrt{6}$, then $\vec{c} = \dots$

- (A) $7\hat{i} + \hat{j} - \hat{k}$ (B) $\frac{7\hat{i} + 2\hat{j} - \hat{k}}{9}$ (C) $7\hat{i} - 2\hat{j} - \hat{k}$ (D) $\frac{7\hat{i} - 2\hat{j} - \hat{k}}{9}$

Sol. : Angle bisector $\vec{c} = m \left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} \right)$

$$= m \left(\frac{4\hat{i} + 4\hat{j} - 7\hat{k}}{9} + \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} \right)$$

$$= \frac{m}{9} (7\hat{i} - 2\hat{j} - \hat{k})$$

$$\therefore |\vec{c}|^2 = \frac{m^2}{81} (54) = 54 \quad \text{(Given)}$$

$$\therefore m = \pm 9$$

$$\therefore \vec{c} = 7\hat{i} - 2\hat{j} - \hat{k} \text{ or } \vec{c} = -7\hat{i} + 2\hat{j} + \hat{k} \quad \text{Ans. : (C)}$$

- (96) For a regular hexagon ABCDEF with centre at origin $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = \dots$

- (A) $6\vec{AO}$ (B) $5\vec{AO}$ (C) $4\vec{AO}$ (D) $3\vec{AO}$

Sol. : Suppose $\vec{AB} = \vec{a}$, $\vec{BC} = \vec{b} = \vec{AO}$

Now, $\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$

$$\vec{AD} = 2\vec{BC} = 2\vec{b}$$

$$\vec{AE} = \vec{AD} + \vec{DE}$$

$$= \vec{AD} - \vec{AB}$$

$$= 2\vec{b} - \vec{a}$$

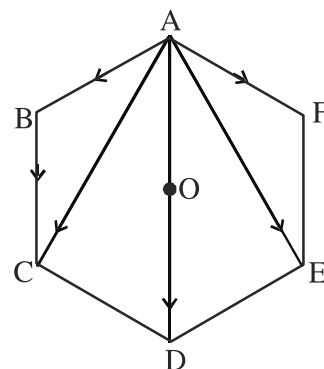


Figure 20.13

and $\vec{AF} = \vec{AE} + \vec{EF}$

$$= 2\vec{b} - \vec{a} - \vec{BC} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$$

$$\therefore \vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = \vec{a} + (\vec{a} + \vec{b}) + (2\vec{b}) + (2\vec{b} - \vec{a}) + (\vec{b} - \vec{a})$$

$$= 6\vec{b} = 6\vec{AO}$$

Ans. : (A)

(97) If $\vec{a} \perp \vec{b}$, then $\vec{a} \times [\vec{a} \times \{\vec{a} \times (\vec{a} \times \vec{b})\}] = \dots$

(A) $-|\vec{a}|^2 \vec{b}$ (B) $|\vec{a}|^4 \vec{b}$ (C) $-|\vec{a}|^2 (\vec{a} \times \vec{b})$ (D) $|\vec{a}|^4 (\vec{a} \times \vec{b})$

Sol. : $\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} = -|\vec{a}|^2 \vec{b}$ ($\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$)

$$\therefore \vec{a} \times [\vec{a} \times \{\vec{a} \times (\vec{a} \times \vec{b})\}] = \vec{a} \times [\vec{a} \times (-|\vec{a}|^2 \vec{b})]$$

$$= -|\vec{a}|^2 \{\vec{a} \times (\vec{a} \times \vec{b})\} = -|\vec{a}|^2 (-|\vec{a}|^2 \vec{b}) = |\vec{a}|^4 \vec{b}$$

Ans. : (B)

(98) If vectors $\lambda \hat{i} + \hat{j} + \hat{k}$, $(\lambda + 1)\hat{i} + \hat{j} + \hat{k}$ and $\lambda \hat{i} + \hat{j} - \hat{k}$ are coplanar, then the set of values of λ is

(A) R (B) $R - \{2\}$ (C) $R - \{1\}$ (D) $\emptyset$

Sol. : $\begin{vmatrix} \lambda & 1 & 1 \\ \lambda + 1 & 1 & 1 \\ \lambda & 1 & -1 \end{vmatrix} = \begin{vmatrix} \lambda & 1 & 0 \\ \lambda + 1 & 1 & 0 \\ \lambda & 1 & -2 \end{vmatrix}$ ($C_3 \rightarrow C_3 - C_2$)

$$= -2(\lambda - \lambda - 1) \neq 0, \forall \lambda \in R$$

$\therefore$ The given vectors can not be coplanar for any value of λ .

$\therefore$ The set of values of λ is $\emptyset$.

Ans. : (D)

(99) $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors. If $\vec{a}', \vec{b}'$ and $\vec{c}'$ are the inverses of $\vec{a}, \vec{b}$

and $\vec{c}$ respectively, then $\vec{a}' \cdot (\vec{a} + \vec{b}) + \vec{b}' \cdot (\vec{b} + \vec{c}) + \vec{c}' \cdot (\vec{c} + \vec{a}) = \dots$

(A) 0 (B) 1 (C) 2 (D) 3

Sol. : $\vec{a}'$ is the inverse vector of $\vec{a}$.

$$\therefore \vec{a}' \cdot \vec{a} = 1 \text{ and } \vec{a}' \cdot \vec{b} = 0 = \vec{a}' \cdot \vec{c} \text{ etc.}$$

$$\therefore \vec{a}' \cdot (\vec{a} + \vec{b}) + \vec{b}' \cdot (\vec{b} + \vec{c}) + \vec{c}' \cdot (\vec{c} + \vec{a}) = 3$$

Ans. : (D)

(100) Out of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, vectors $\vec{a}$ and $\vec{b}$ are not perpendicular to each other. If $\vec{a} \cdot \vec{d} = 0$ and $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$, then $\vec{d} = \dots$ ($\vec{b} \neq \vec{0}$, $\vec{c} \neq \vec{0}$, $\vec{c} \neq \vec{d}$) [AIEEE : 2011]

(A) $\vec{b} - \left(\frac{\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{c}$ (B) $\vec{c} + \left(\frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{b}$ (C) $\vec{b} + \left(\frac{\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{c}$ (D) $\vec{c} - \left(\frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{b}$

Sol. : $\vec{a}$ and $\vec{b}$ are not perpendicular to each other and so $\vec{a} \cdot \vec{b} \neq 0$ and $\vec{a} \cdot \vec{d} = 0$

Now $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$. So, $\vec{b} \times (\vec{c} - \vec{d}) = \vec{0}$

$\therefore \vec{b} \parallel (\vec{c} - \vec{d})$. From this, $\vec{c} - \vec{d} = m\vec{b}$, $m \in \mathbb{R} - \{0\}$ (1)

Taking dot product with $\vec{a}$,

$\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{d} = m(\vec{a} \cdot \vec{b})$. From this it follows that, $\vec{a} \cdot \vec{c} - 0 = m(\vec{a} \cdot \vec{b})$

$\therefore m = \frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}}$

Substituting value of m in (1), $\vec{c} - \vec{d} = \left(\frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{b}$

$\therefore \vec{c} - \left(\frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \right) \vec{b} = \vec{d}$

Ans. : (D)

(101) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero vectors and no two of them are collinear. If $\vec{a} + 3\vec{b}$ and $\vec{c}$ are collinear and $\vec{b} + 2\vec{c}$ and $\vec{a}$ are collinear, then $\vec{a} + 3\vec{b} + 6\vec{c} = \dots$ [AIEEE : 2011]

(A) $\vec{a}$ (B) $\vec{b}$ (C) $\vec{0}$ (D) $\vec{a} + \vec{c}$

Sol. : $\vec{a} + 3\vec{b}$ and $\vec{c}$ are collinear. So, $\vec{a} + 3\vec{b} = m\vec{c}$, $m \in \mathbb{R} - \{0\}$ (1)

$\vec{b} + 2\vec{c}$ and $\vec{a}$ are collinear. So, $\vec{b} + 2\vec{c} = n\vec{a}$, $n \in \mathbb{R} - \{0\}$

$\therefore 3\vec{b} + 6\vec{c} = 3n\vec{a}$ (2)

On subtracting (2) from (1),

$\vec{a} - 6\vec{c} = m\vec{c} - 3n\vec{a}$

$\therefore (1 + 3n)\vec{a} - (6 + m)\vec{c} = \vec{0}$

$\vec{a}$ and $\vec{c}$ are non-zero and non-collinear.

$\therefore 1 + 3n = 0$ and $6 + m = 0$. So, $n = -\frac{1}{3}$ and $m = -6$

Substituting the value of m in (1) (or n in (2)),

$$\vec{a} + 3\vec{b} + 6\vec{c} = \vec{0}$$

Ans. : (C)

(102) If $|\vec{a} + \vec{b}| = \sqrt{29}$ and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$ for vectors $\vec{a}$ and $\vec{b}$,

then a possible value of $(\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

[IIT : 2012]

(A) 0

(B) 3

(C) 4

(D) 8

Sol. : $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$

$$\therefore \vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b} = \vec{0}$$

$$\therefore \vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) + \vec{b} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = \vec{0}$$

$$\therefore (\vec{a} + \vec{b}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = \vec{0}$$

$$\therefore \vec{a} + \vec{b} = m(2\hat{i} + 3\hat{j} + 4\hat{k}), m \in \mathbb{R} - \{0\}$$

$$\therefore |\vec{a} + \vec{b}|^2 = m^2(4 + 9 + 16)$$

$$\therefore 29 = m^2(29). \quad \text{So, } 1 = m^2. \quad \text{So, } m = \pm 1$$

$$\therefore \vec{a} + \vec{b} = \pm(2\hat{i} + 3\hat{j} + 4\hat{k})$$

$$\begin{aligned} \text{Now, } (\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) &= \pm(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) \\ &= \pm(-14 + 6 + 12) = \pm 4 \end{aligned}$$

$\therefore$ A possible value of $(\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k})$ is 4.

Ans. : (C)

(103) $\vec{a}$ and $\vec{b}$ are unit vectors. If $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are mutually perpendicular,

then the measure of the angle between $\vec{a}$ and $\vec{b}$ is

[AIEEE : 2012]

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{4}$

Sol. : $\vec{c} \perp \vec{d}$. So, $\vec{c} \cdot \vec{d} = 0$

$$\therefore (\hat{a} + 2\hat{b}) \cdot (5\hat{a} - 4\hat{b}) = 0$$

$$\therefore 5|\hat{a}|^2 + 6\hat{a} \cdot \hat{b} - 8|\hat{b}|^2 = 0$$

$$\therefore 5(1) + 6|\hat{a}||\hat{b}|\cos\theta - 8(1) = 0, \text{ where } (\hat{a}, \hat{b}) = \theta$$

$$\therefore 6\cos\theta - 3 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Ans. : (C)

- (104) In parallelogram ABCD, $\vec{AB} = \vec{q}$ and $\vec{AD} = \vec{p}$ and $\angle BAD$ is an acute angle. If $\vec{r}$ represents the perpendicular from B to $\overline{AD}$, then $\vec{r} = \dots$ [AIEEE : 2012]

(A) $3\vec{q} - \frac{3(\vec{p} \cdot \vec{q})}{(\vec{p} \cdot \vec{p})} \vec{p}$ (B) $-\vec{q} + \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}} \right) \vec{p}$ (C) $\vec{q} - \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}} \right) \vec{p}$ (D) $-3\vec{q} + \frac{3(\vec{p} \cdot \vec{q})}{(\vec{p} \cdot \vec{p})} \vec{p}$

Sol. : Let $\overline{BE} \perp \overline{AD}$

$\therefore \vec{AE}$ is the projection of $\vec{q}$ on $\vec{p}$.

$$\therefore \vec{AE} = \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}} \right) \vec{p}$$

Now, in $\triangle ABE$, $\vec{AB} + \vec{BE} = \vec{AE}$

$$\therefore \vec{BE} = \vec{AE} - \vec{AB}$$

$$\therefore \vec{r} = \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}} \right) \vec{p} - \vec{q}$$

Ans. : (B)

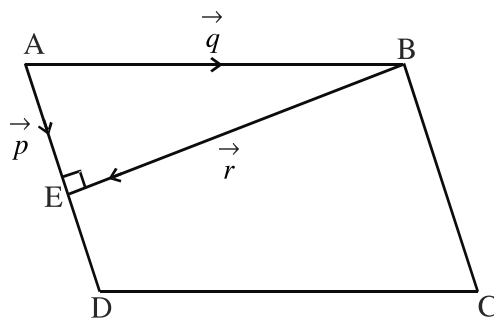


Figure 20.14

- (105) If $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$ for unit vectors $\vec{a}, \vec{b}, \vec{c}$, then $|4\vec{a} + 3\vec{b} + 3\vec{c}| = \dots$
- (A) 7 (B) 3 (C) 1 (D) 0

Sol. : $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$

$$\therefore 2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2) - 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 9$$

$$\therefore 2(1 + 1 + 1) - 9 = 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\therefore -3 = 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \quad (1)$$

$$\text{Now, } |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 3 - 3 = 0$$

$$\therefore |\vec{a} + \vec{b} + \vec{c}| = 0$$

$$\therefore \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\therefore \vec{b} + \vec{c} = -\vec{a}$$

$$\therefore |4\vec{a} + 3\vec{b} + 3\vec{c}| = |4\vec{a} + 3(-\vec{a})| = |\vec{a}| = 1$$

Ans. : (C)

(106) If $(\vec{a}, \vec{b}) = \frac{\pi}{4}$, then $\frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} = \dots$

(A) 1

(B) -1

(C) $\frac{1}{\sqrt{2}}$

(D) $-\frac{1}{\sqrt{2}}$

Sol. : $\frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} = \frac{|\vec{a}||\vec{b}|\sin(\vec{a}, \vec{b})}{|\vec{a}||\vec{b}|\cos(\vec{a}, \vec{b})} = \tan \frac{\pi}{4} = 1$

Ans. : (A)

(107) If the projections of a vector on axes are 3, -2, 6, respectively, then the direction *cosines* of the vector are

(A) $\frac{3}{5}, \frac{-2}{5}, \frac{6}{5}$

(B) $\frac{3}{25}, \frac{-2}{25}, \frac{6}{25}$

(C) $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$

(D) $\frac{3}{49}, \frac{-2}{49}, \frac{6}{49}$

Sol. : Suppose the projections of $\vec{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$ in the direction of $\hat{i}$ is 3.

$\therefore \frac{\vec{x} \cdot \hat{i}}{|\hat{i}|} = 3$. So, $x_1 = 3$

Similarly, $x_2 = -2$ and $x_3 = 6$

$\therefore \vec{x} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore |\vec{x}| = \sqrt{9 + 4 + 36} = 7$

$\therefore$ Direction *cosines* of $\vec{x}$ are $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$.

Ans. : (C)

Note : There is a unit vector in option (C) only and so it has to be the answer.

(108) $\vec{a}, \vec{b}, \vec{c}$ are the edges of a parallelepiped. If the dot product of any two of them is 3 and the magnitude of each of them is 3, then the volume of parallelepiped is

(A) 9

(B) $12\sqrt{15}$

(C) $6\sqrt{15}$

(D) 18

Sol. : According to the given conditions, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 3$ and $|\vec{a}| = |\vec{b}| = |\vec{c}| = 3$

$$[\vec{a} \ \vec{b} \ \vec{c}]^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 3 & 3 \\ 3 & 9 & 3 \\ 3 & 3 & 9 \end{vmatrix} = 27 \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 27(24 - 2 - 2) = 540$$

$\therefore$ Volume = $|\vec{a} \ \vec{b} \ \vec{c}|$

$= 6\sqrt{15}$

Ans. : (C)

(109) Inverse vectors of $\vec{a}$, $\vec{b}$, $\vec{c}$ are respectively $\vec{a'}$, $\vec{b'}$ and $\vec{c'}$. Then $\vec{a} \times \vec{a'} + \vec{b} \times \vec{b'} + \vec{c} \times \vec{c'}$,
=

- (A) $\vec{a} + \vec{b} + \vec{c}$ (B) $\vec{a'} + \vec{b'} + \vec{c'}$ (C) $[\vec{a} \ \vec{b} \ \vec{c}]$ (D) $\vec{0}$

$$\begin{aligned} \text{Sol. : } \vec{a} \times \vec{a'} + \vec{b} \times \vec{b'} + \vec{c} \times \vec{c'} &= \frac{\vec{a} \times (\vec{b} \times \vec{c})}{[\vec{a} \ \vec{b} \ \vec{c}]} + \frac{\vec{b} \times (\vec{c} \times \vec{a})}{[\vec{a} \ \vec{b} \ \vec{c}]} + \frac{\vec{c} \times (\vec{a} \times \vec{b})}{[\vec{a} \ \vec{b} \ \vec{c}]} \\ &= \frac{1}{[\vec{a} \ \vec{b} \ \vec{c}]} \{ (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} + (\vec{b} \cdot \vec{a}) \vec{c} - (\vec{b} \cdot \vec{c}) \vec{a} + (\vec{c} \cdot \vec{b}) \vec{a} - (\vec{c} \cdot \vec{a}) \vec{b} \} \\ &= \vec{0} \end{aligned}$$

Ans. : (D)

(110) $\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = \dots\dots$

- (A) $\vec{a} \cdot (\vec{b} \times \vec{a}) - \vec{b} \cdot (\vec{a} \times \vec{b})$ (B) $(\vec{b} \times \vec{a}) \cdot (\vec{a} \times \vec{a})$
(C) $(\vec{a} \cdot \vec{a})(\vec{b} \times \vec{a})$ (D) $[(\vec{a} \times \vec{b}) \cdot \vec{a}] \vec{a}$

$$\begin{aligned} \text{Sol. : } \vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] &= \vec{a} \times [(\vec{a} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{a}) \vec{b}] \\ &= (\vec{a} \cdot \vec{b}) (\vec{a} \times \vec{a}) - (\vec{a} \cdot \vec{a}) (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{b}) \vec{0} + (\vec{a} \cdot \vec{a}) (\vec{b} \times \vec{a}) \\ &= (\vec{a} \cdot \vec{a}) (\vec{b} \times \vec{a}) \end{aligned}$$

Ans. : (C)

(111) $|\vec{a} \times \vec{b}|^2 = \dots\dots$

- (A) $\begin{vmatrix} \vec{a} & \vec{b} \\ \vec{b} & \vec{a} \end{vmatrix}$ (B) $\begin{vmatrix} \vec{a} & \vec{a} \\ \vec{b} & \vec{b} \end{vmatrix}$ (C) $\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$ (D) $\begin{vmatrix} \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$

Sol. : According to Lagrange's identity.

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

$$\therefore |\vec{a} \times \vec{b}|^2 = (\vec{a} \cdot \vec{a}) (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b}) (\vec{a} \cdot \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$$

Ans. : (C)

(112) If the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{a}$ and $\vec{c}$ is obtuse for vectors $\vec{a} = x^2 \hat{i} + x \hat{j} + 2 \hat{k}$, $\vec{b} = \hat{i} - 3 \hat{j} + \hat{k}$ and $\vec{c} = 2 \hat{i} - 8 \hat{j} + 3 \hat{k}$, then $x \in \dots\dots$

- (A) (1, 3) (B) (2, 3) (C) $(-10, 11) \cup (2, 3)$ (D) (-1, 3)

Sol. : The angle between $\vec{a}$ and $\vec{b}$ is an acute angle.

$$\therefore \vec{a} \cdot \vec{b} > 0$$

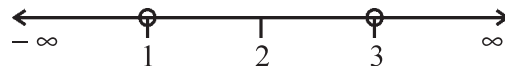
$$\therefore x^2 - 3x + 2 > 0$$

$$\therefore (x - 1)(x - 2) > 0$$

$$\therefore x \in (2, \infty) \text{ or } x \in (-\infty, 1)$$

$$\therefore x \in (-\infty, 1) \cup (2, \infty)$$

The angle between $\vec{a}$ and $\vec{c}$ is obtuse.



$$\therefore \vec{a} \cdot \vec{c} < 0$$

$$\therefore 2x^2 - 8x + 6 < 0$$

$$\therefore (x - 1)(x - 3) < 0$$

$$\therefore x \in (1, 3), \text{ But for } x \in (1, 2], \vec{a} \cdot \vec{b} \not> 0.$$

$$\therefore x \in (2, 3)$$

Ans. : (B)

(113) $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ and $\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$ are non-zero vectors

such that $\vec{c}$ is a unit vector and $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$. If $(\vec{a} \wedge \vec{b}) = \frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \dots$

(A) 0

(B) 1

(C) $\frac{1}{4} |\vec{a}|^2 |\vec{b}|^2$

(D) $\frac{3}{4} |\vec{a}|^2 |\vec{b}|^2 |\vec{c}|^2$

Sol. : Using property of determinants $|AB| = |A| |B|$, where A and B are square matrices,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(By using the property of converting rows into columns)

$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1 b_1 + a_2 b_2 + a_3 b_3 & a_1 c_1 + a_2 c_2 + a_3 c_3 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 & b_1^2 + b_2^2 + b_3^2 & b_1 c_1 + b_2 c_2 + b_3 c_3 \\ a_1 c_1 + a_2 c_2 + a_3 c_3 & b_1 c_1 + b_2 c_2 + b_3 c_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1 b_1 + a_2 b_2 + a_3 b_3 & 0 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 & b_1^2 + b_2^2 + b_3^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(Matrix multiplication)}$$

$$(\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \text{ and } |\vec{c}| = 1)$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a} \cdot \vec{b}|^2 = |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \frac{\pi}{6} = \frac{1}{4} |\vec{a}|^2 |\vec{b}|^2$$

Ans. : (C)

Second method :

$$\vec{c} \perp \vec{a} \text{ and } \vec{c} \perp \vec{b} \text{ and } |\vec{c}| = 1$$

$$\therefore \vec{c} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\begin{aligned} \text{Now, } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 &= [(\vec{a} \times \vec{b}) \cdot \vec{c}]^2 \\ &= \left(\frac{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} \right)^2 \\ &= |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \frac{\pi}{6} = \frac{1}{4} |\vec{a}|^2 |\vec{b}|^2 \end{aligned} \quad \text{Ans. : (C)}$$

(114) Non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other. If $\vec{r} \times \vec{a} = \vec{b}$, then $\vec{r} = \dots$
(α is a scalar.)

$$(A) \alpha(\vec{a} \times \vec{b}) \quad (B) \alpha \vec{b} - \frac{1}{|\vec{b}|^2}(\vec{a} \times \vec{b}) \quad (C) \alpha(\vec{b} \times \vec{a}) \quad (D) \alpha \vec{a} + \frac{1}{|\vec{a}|^2}(\vec{a} \times \vec{b})$$

Sol. : $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ are non coplanar.

$\therefore$ Any vector $\vec{r}$ can be written as $\vec{r} = \alpha \vec{a} + \beta \vec{b} + \gamma(\vec{a} \times \vec{b})$, where α, β, γ are scalars.

$$\text{Now, } \vec{r} \times \vec{a} = \alpha(\vec{a} \times \vec{a}) + \beta(\vec{b} \times \vec{a}) + \gamma((\vec{a} \times \vec{b}) \times \vec{a})$$

$$\therefore \vec{b} = \vec{0} - \beta(\vec{a} \times \vec{b}) + \gamma(\vec{a} \cdot \vec{a})\vec{b} - \gamma(\vec{a} \cdot \vec{b})\vec{a}$$

$$\therefore \vec{b} + \beta(\vec{a} \times \vec{b}) - \gamma|\vec{a}|^2 \vec{b} + \gamma(0)\vec{a} = \vec{0}$$

$$\therefore (1 - \gamma|\vec{a}|^2)\vec{b} + \beta(\vec{a} \times \vec{b}) = \vec{0}$$

$$\therefore 1 - \gamma|\vec{a}|^2 = 0 \text{ and } \beta = 0 \quad (\vec{a}, \vec{b}, \vec{a} \times \vec{b} \text{ are non-coplanar})$$

$$\therefore \gamma = \frac{1}{|\vec{a}|^2}$$

$$\therefore \vec{r} = \alpha \vec{a} + 0 \vec{b} + \frac{1}{|\vec{a}|^2}(\vec{a} \times \vec{b})$$

$$\therefore \vec{r} = \alpha \vec{a} + \frac{1}{|\vec{a}|^2}(\vec{a} \times \vec{b}) \quad \text{Ans. : (D)}$$

: Questions Asked in JEE Main and Advanced :

- (115) If vectors $\vec{AB} = 3\hat{i} + 4\hat{k}$ and $\vec{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ represent two sides of $\triangle ABC$, then the length of the median drawn from A is

[JEE-Main : 2013]

- (A) $\sqrt{18}$ (B) $\sqrt{72}$ (C) $\sqrt{33}$ (D) $\sqrt{45}$

Sol. : The median drawn from A is $\frac{\vec{AB} + \vec{AC}}{2} = \frac{8\hat{i} - 2\hat{j} + 8\hat{k}}{2} = 4\hat{i} - \hat{j} + 4\hat{k}$

$$\therefore \text{Its length} = \sqrt{16 + 1 + 16} = \sqrt{33}$$

Ans. : (C)

- (116) $\vec{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ represent the diagonals of a parallelogram and one vector with initial point P is $\vec{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$. The volume of the parallelopiped with edges as $\vec{PT}$, $\vec{PQ}$ and $\vec{PS}$ is

[JEE-Adv. : 2013]

- (A) 5 (B) 20 (C) 10 (D) 30

Sol. : $\vec{PQ} + \vec{QR} = \vec{PR}$

$$\therefore \vec{a} + \vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$$

and $\vec{SR} + \vec{RQ} = \vec{SQ}$

$$\therefore \vec{a} - \vec{b} = \hat{i} - 3\hat{j} - 4\hat{k}$$

Solving (1) and (2),

$$\vec{a} = 2\hat{i} - \hat{j} - 3\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\therefore \vec{PQ} = 2\hat{i} - \hat{j} - 3\hat{k} \text{ and } \vec{PS} = \hat{i} + 2\hat{j} + \hat{k} \text{ and } \vec{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$$

The volume of the parallelopiped with edges $\vec{PT}$, $\vec{PQ}$, $\vec{PS}$ = $|\begin{bmatrix} \vec{PT} & \vec{PQ} & \vec{PS} \end{bmatrix}|$

$$\text{Now, } \begin{bmatrix} \vec{PT} & \vec{PQ} & \vec{PS} \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(5) - 2(5) + 3(5) = 10$$

$\therefore$ The volume of the parallelopiped = 10

Ans. : (C)

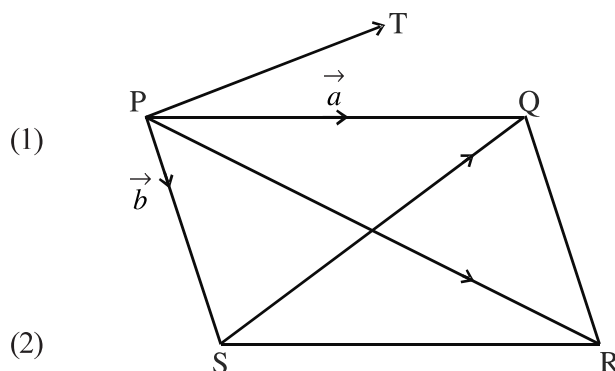


Figure 20.15

(117) Match the questions given in table I with the answers given in table II. [JEE Adv. : 2013]

	Table I	Table II	
(P)	If the volume of a parallelopiped with coterminus edges $\vec{a}, \vec{b}, \vec{c}$ is 2, then the volume of the parallelopiped with edges $2(\vec{a} \times \vec{b}), 3(\vec{b} \times \vec{c}), (\vec{c} \times \vec{a})$ is	1	100
(Q)	If the volume of a parallelopiped with edges $\vec{a}, \vec{b}, \vec{c}$ is 5, then the volume of the parallelopiped with edges $3(\vec{a} + \vec{b}), (\vec{b} + \vec{c})$ and $2(\vec{c} + \vec{a})$ is	2	30
(R)	If the area of a triangle with sides $\vec{a}$ and $\vec{b}$ is 20, then the area of triangle with sides $(2\vec{a} + 3\vec{b})$ and $(\vec{a} - \vec{b})$ is	3	24
(S)	If the area of a parallelogram with adjacent sides as $\vec{a}$ and $\vec{b}$ is 30, then the area of the parallelogram with sides as $(\vec{a} + \vec{b})$ and $\vec{a}$ is	4	60

	(P)	(Q)	(R)	(S)
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

Sol. : (P) : The volume of the parallelopiped with edges $\vec{a}, \vec{b}, \vec{c}$ is $|\vec{a} \vec{b} \vec{c}| = 2$

Now, the volume of the parallelopiped with edges $2(\vec{a} \times \vec{b}), 3(\vec{b} \times \vec{c})$ and $(\vec{c} \times \vec{a})$

$$= |[2(\vec{a} \times \vec{b}) \ 3(\vec{b} \times \vec{c}) \ (\vec{c} \times \vec{a})]|$$

$$= 6|[\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}]|$$

$$= 6[\vec{a} \ \vec{b} \ \vec{c}]^2 = 6(2)^2 = 24$$

$\therefore$ (P) $\rightarrow$ 3

[**Note:** Actually, no need to proceed now. (P) $\rightarrow$ 3 appears only in option (C). So, ans. is (C).]

(Q) : The volume of the parallelopiped with edges $\vec{a}, \vec{b}, \vec{c}$ is $|\vec{a} \vec{b} \vec{c}| = 5$

The volume of the parallelopiped with edges $3(\vec{a} + \vec{b}), (\vec{b} + \vec{c})$ and $2(\vec{c} + \vec{a})$ is

$$|[3(\vec{a} + \vec{b}) \ \vec{b} + \vec{c} \ 2(\vec{c} + \vec{a})]|$$

$$= 6|\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}| = 6|2[\vec{a} \quad \vec{b} \quad \vec{c}]| = 6(2)(5) = 60$$

$$\therefore (Q) \rightarrow 4$$

$$(R) : \text{The area of a triangle} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

$$\therefore 20 = \frac{1}{2}|\vec{a} \times \vec{b}|$$

$$\therefore 40 = |\vec{a} \times \vec{b}|$$

The area of the triangle with sides $2\vec{a} + 3\vec{b}$ and $\vec{a} - \vec{b}$,

$$= \frac{1}{2}|(2\vec{a} + 3\vec{b}) \times (\vec{a} - \vec{b})|$$

$$= \frac{1}{2}|-2(\vec{a} \times \vec{b}) + 3(\vec{b} \times \vec{a})| = \frac{1}{2}|-5(\vec{a} \times \vec{b})| = \frac{5}{2}|\vec{a} \times \vec{b}| = \frac{5}{2}(40) = 100$$

$$(R) \rightarrow 1$$

(S) : The area of the parallelogram with adjacent sides as $\vec{a}$ and $\vec{b} = |\vec{a} \times \vec{b}|$

$$\therefore |\vec{a} \times \vec{b}| = 30$$

So, the area of the parallelogram with adjacent sides $\vec{a} + \vec{b}$ and $\vec{a}$ is

$$= |(\vec{a} + \vec{b}) \times \vec{a}| = |\vec{b} \times \vec{a}| = |\vec{a} \times \vec{b}| = 30$$

$$\therefore (S) \rightarrow 2$$

So, (P) $\rightarrow$ 3, (Q) $\rightarrow$ 4, (R) $\rightarrow$ 1, (S) $\rightarrow$ 2

Ans. : (C)

(118) For unit vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, if $\vec{a} - \sqrt{3}\vec{b} + \vec{c} = \vec{0}$, then the angle between $\vec{a}$ and $\vec{c}$ has measure [JEE : 2013]

$$(A) \frac{\pi}{4}$$

$$(B) \frac{\pi}{3}$$

$$(C) \frac{\pi}{6}$$

$$(D) \frac{\pi}{2}$$

$$\text{Sol. : } \vec{a} - \sqrt{3}\vec{b} + \vec{c} = \vec{0}$$

$$\therefore \vec{a} + \vec{c} = \sqrt{3}\vec{b}$$

$$\therefore |\vec{a} + \vec{c}|^2 = 3|\vec{b}|^2$$

$$\therefore |\vec{a}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{c}|\cos(\vec{a}, \vec{c}) = 3|\vec{b}|^2$$

$$\therefore 1 + 1 + 2(1)(1)\cos(\vec{a}, \vec{c}) = 3(1)$$

$$\therefore \cos(\vec{a}, \vec{c}) = \frac{1}{2}$$

$$\therefore (\vec{a}, \vec{c}) = \frac{\pi}{3}$$

Ans. : (B)

- (119) $\vec{a}$ and $\vec{b}$ are non-collinear. If vectors $\vec{u} = (\alpha - 2)\vec{a} + \vec{b}$ and $\vec{v} = (2 + 3\alpha)\vec{a} - 3\vec{b}$ are collinear, then $\alpha = \dots$ [JEE : 2013]

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) $-\frac{2}{3}$

Sol. : $\vec{v}$ and $\vec{u}$ are collinear. So, $\vec{u} \times \vec{v} = \vec{0}$

$$\therefore ((\alpha - 2)\vec{a} + \vec{b}) \times ((2 + 3\alpha)\vec{a} - 3\vec{b}) = \vec{0}$$

$$\therefore -3(\alpha - 2)(\vec{a} \times \vec{b}) + (2 + 3\alpha)(\vec{b} \times \vec{a}) = 0$$

$$\therefore ((6 - 3\alpha) - (2 + 3\alpha))(\vec{a} \times \vec{b}) = \vec{0}$$

$$\therefore 4 - 6\alpha = 0 \quad (\text{Since } \vec{a}, \vec{b} \text{ are non-collinear } \vec{a} \times \vec{b} \neq \vec{0})$$

$$\therefore \frac{2}{3} = \alpha \quad \text{Ans. : (B)}$$

- (120) If projections of a vector in R^3 on X, Y, Z axes are 2, 3 and 6 respectively, then magnitude of the vector is [JEE : 2013]

(A) 12 (B) 7 (C) 9 (D) 6

Sol. : Suppose the position vector of A is $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and that of B is $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\therefore \vec{AB} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$$

$$\text{Projection of } \vec{AB} \text{ on X-axis is } \frac{\vec{AB} \cdot \hat{i}}{|\hat{i}|} = 2$$

$$\therefore (b_1 - a_1) = 2$$

Similarly, $(b_2 - a_2) = 3$ and $(b_3 - a_3) = 6$

$$|\vec{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2} = \sqrt{4 + 9 + 36} = 7 \quad \text{Ans. : (B)}$$

Second method : The vector is $\vec{r} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$\therefore |\vec{r}| = \sqrt{4 + 9 + 36} = 7$$

- (121) Suppose $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ are three vectors. If for some $\lambda \in R$, the magnitude of projection of $\vec{b} + \lambda\vec{c}$ on $\vec{a}$ is $\sqrt{\frac{2}{3}}$, then $\vec{b} + \lambda\vec{c} = \dots$ [JEE : 2013]

(A) $2\hat{i} + \hat{j} + 5\hat{k}$ (B) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (C) $2\hat{i} - \hat{j} - 5\hat{k}$ (D) $2\hat{i} + 3\hat{j} + 3\hat{k}$

Sol. : Magnitude of projection of $\vec{b} + \lambda\vec{c}$ on $\vec{a}$ is $\sqrt{\frac{2}{3}}$.

$$\vec{b} + \lambda\vec{c} = (1 + \lambda)\hat{i} + (2 + \lambda)\hat{j} - (1 + 2\lambda)\hat{k}$$

$$\therefore \frac{|(\vec{b} + \lambda\vec{c}) \cdot \vec{a}|}{|\vec{a}|} = \sqrt{\frac{2}{3}}$$

$$\therefore \frac{|2 + 2\lambda - 2 - \lambda - 1 - 2\lambda|}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

$$\therefore |-\lambda - 1| = 2$$

$$\therefore \lambda + 1 = 2 \quad \text{or} \quad \lambda + 1 = -2$$

$$\therefore \lambda = 1 \quad \lambda = -3$$

$$\therefore \vec{b} + \lambda \vec{c} = 2\hat{i} + 3\hat{j} - 3\hat{k} \quad \text{or} \quad \vec{b} + \lambda \vec{c} = -2\hat{i} - \hat{j} + 5\hat{k}$$

Ans. : (B)

$$(122) \quad \text{Vector } ((\hat{i} \times \vec{a}) \cdot \vec{b})\hat{i} + ((\hat{j} \times \vec{a}) \cdot \vec{b})\hat{j} + ((\hat{k} \times \vec{a}) \cdot \vec{b})\hat{k} = \dots$$

[JEE : 2013]

$$(A) \vec{b} \times \vec{a}$$

$$(B) \vec{a}$$

$$(C) \vec{a} \times \vec{b}$$

$$(D) \vec{b}$$

Sol. : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\text{Now } (\hat{i} \times \vec{a}) \cdot \vec{b} = [\hat{i} \quad \vec{a} \quad \vec{b}]$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

$$\text{Similarly } (\hat{j} \times \vec{a}) \cdot \vec{b} = a_3b_1 - a_1b_3 \quad \text{and} \quad (\hat{k} \times \vec{a}) \cdot \vec{b} = a_1b_2 - a_2b_1$$

$$\therefore ((\hat{i} \times \vec{a}) \cdot \vec{b})\hat{i} + ((\hat{j} \times \vec{a}) \cdot \vec{b})\hat{j} + ((\hat{k} \times \vec{a}) \cdot \vec{b})\hat{k}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \vec{a} \times \vec{b}$$

Ans. : (C)

Second method :

$$\left((\hat{i} \times \vec{a}) \cdot \vec{b} \right) \hat{i} + \left((\hat{j} \times \vec{a}) \cdot \vec{b} \right) \hat{j} + \left((\hat{k} \times \vec{a}) \cdot \vec{b} \right) \hat{k}$$

$$= \left((\hat{i} \cdot (\vec{a} \times \vec{b})) \right) \hat{i} + \left((\hat{j} \cdot (\vec{a} \times \vec{b})) \right) \hat{j} + \left((\hat{k} \cdot (\vec{a} \times \vec{b})) \right) \hat{k} = \vec{a} \times \vec{b}$$

Ans. : (C)

(123) The vertices of ΔABC in R^3 are $A(2, 3, 5)$, $B(-1, 3, 2)$, $C(\lambda, 5, \mu)$. If the median drawn from A makes angles of same measure with axes, then $(\lambda, \mu) = \dots$ [JEE : 2013]

$$(A) (10, 7)$$

$$(B) (7, 5)$$

$$(C) (7, 10)$$

$$(D) (5, 7)$$

Sol. : In ΔABC , midpoint of $\overline{BC}$ is $D\left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$.

$$\therefore \vec{AD} = \left(\frac{\lambda-5}{2}\right)\hat{i} + \hat{j} + \left(\frac{\mu-8}{2}\right)\hat{k}$$

$\vec{AD}$, makes equal angles with all the three axes.

$$\therefore \vec{AD} \cdot \hat{i} = \vec{AD} \cdot \hat{j} = \vec{AD} \cdot \hat{k}$$

$$\therefore \frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2}. \text{ So, } \lambda = 7, \mu = 10$$

$$\therefore (\lambda, \mu) = (7, 10)$$

Ans. : (C)

(124) $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c}$ are vectors such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ has measure $\frac{\pi}{6}$, then $|(\vec{a} \times \vec{b}) \times \vec{c}| = \dots$

(A) $\frac{1}{2}$

(B) $\frac{3\sqrt{3}}{2}$

(C) 3

(D) $\frac{3}{2}$

Sol. : $|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}| |\vec{c}| \sin \frac{\pi}{6}$

$$= |2\hat{i} - 2\hat{j} + \hat{k}| \cdot |\vec{c}| \left(\frac{1}{2}\right) = \frac{3}{2} |\vec{c}|$$

Now, $|\vec{c} - \vec{a}|^2 = 8$

$$\therefore |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a} \cdot \vec{c} = 8$$

$$\therefore |\vec{c}|^2 + 9 - 2|\vec{c}| - 8 = 0$$

$$(\vec{a} \cdot \vec{c} = |\vec{c}|)$$

$$\therefore |\vec{c}|^2 - 2|\vec{c}| + 1 = 0$$

$$\therefore (|\vec{c}| - 1)^2 = 0. \text{ So, } |\vec{c}| = 1$$

$$\therefore |(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{3}{2}$$

Ans. : (D)

(125) If $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = \lambda [\vec{a} \quad \vec{b} \quad \vec{c}]^2$, then $\lambda = \dots$

[JEE Main : 2014]

(A) 2

(B) 3

(C) 0

(D) 1

Sol. : We know that $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$

$$\therefore \lambda = 1$$

Ans. : (D)

Note : $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = (\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})]$

$$= (\vec{a} \times \vec{b}) \cdot \{((\vec{b} \times \vec{c}) \cdot \vec{a}) \vec{c} - ((\vec{b} \times \vec{c}) \cdot \vec{c}) \vec{a}\}$$

$$= (\vec{a} \times \vec{b}) \cdot \{[\vec{b} \quad \vec{c} \quad \vec{a}] \vec{c} - [\vec{b} \quad \vec{c} \quad \vec{c}] \vec{a}\}$$

$$= \{(\vec{a} \times \vec{b}) \cdot \vec{c}\} \{[\vec{b} \quad \vec{c} \quad \vec{a}] - 0\}$$

$$= [\vec{a} \quad \vec{b} \quad \vec{c}] [\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$= [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$

(126) For vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$, $|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt{2}$ and the angle between any two of them has measure $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then [JEE Adv. : 2014]

$$(A) \vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$$

$$(B) \vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$$

$$(C) \vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$$

$$(D) \vec{a} = -(\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$$

Sol. : $\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos(\vec{x} \wedge \vec{y}) = (\sqrt{2})(\sqrt{2}) \cos \frac{\pi}{3} = 2 \left(\frac{1}{2} \right) = 1$

Similarly, $\vec{y} \cdot \vec{z} = \vec{z} \cdot \vec{x} = 1$

$\vec{a}$ is perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$.

$$\therefore \vec{a} = \{m \vec{x} \times (\vec{y} \times \vec{z})\} = m\{(\vec{x} \cdot \vec{z}) \vec{y} - (\vec{x} \cdot \vec{y}) \vec{z}\} = m(\vec{y} - \vec{z})$$

Now, $\vec{a} \cdot \vec{y} = m(\vec{y} - \vec{z}) \cdot \vec{y} = m(|\vec{y}|^2 - \vec{z} \cdot \vec{y}) = m(2 - 1) = m$

$$\therefore \vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$$

$\therefore$ Option (B) is true.

$\vec{b}$ is perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$.

$$\therefore \vec{b} = n\{\vec{y} \times (\vec{z} \times \vec{x})\} = n\{(\vec{y} \cdot \vec{x}) \vec{z} - (\vec{y} \cdot \vec{z}) \vec{x}\} = n(\vec{z} - \vec{x})$$

Now, $\vec{b} \cdot \vec{z} = n(\vec{z} - \vec{x}) \cdot \vec{z} = n(|\vec{z}|^2 - \vec{x} \cdot \vec{z}) = n(2 - 1) = n$

$$\therefore \vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$$

$\therefore$ Option (A) is true.

Now $\vec{a} \cdot \vec{b} = ((\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})) \cdot ((\vec{b} \cdot \vec{z})(\vec{z} - \vec{x}))$

$$= (\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})(\vec{y} - \vec{z}) \cdot (\vec{z} - \vec{x})$$

$$= (\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})(\vec{y} \cdot \vec{z} - \vec{y} \cdot \vec{x} - |\vec{z}|^2 + \vec{z} \cdot \vec{x})$$

$$= (\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})(1 - 1 - 2 + 1)$$

$$= -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$$

$\therefore$ Option (C) is true.

Since (B) is true, (D) is also true.

In this way, options (A), (B), (C) and (D) are true.

Ans. : (A), (B), (C), (D)

(127) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors. The angle between any two of them has measure $\frac{\pi}{3}$.

If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$ for real constants p, q, r , then $\frac{p^2 + 2q^2 + r^2}{q^2} = \dots$

[JEE Adv. : 2014]

(A) 1

(B) 2

(C) 3

(D) 4

Sol. : $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \frac{\pi}{3}$

$$= (1)(1)\left(\frac{1}{2}\right)$$

$$\therefore \vec{a} \cdot \vec{b} = \frac{1}{2}$$

Similarly, $\vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = \frac{1}{2}$

Taking dot product of $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$ with $\vec{a}$,

$$\therefore \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{b} \times \vec{c}) = p|\vec{a}|^2 + q(\vec{a} \cdot \vec{b}) + r(\vec{a} \cdot \vec{c})$$

$$\therefore [\vec{a} \quad \vec{b} \quad \vec{c}] = p + \frac{q}{2} + \frac{r}{2} \quad (1)$$

Similarly, taking dot product with $\vec{b}$ and $\vec{c}$,

$$\vec{b} \cdot (\vec{a} \times \vec{b}) + \vec{b} \cdot (\vec{b} \times \vec{c}) = p(\vec{a} \cdot \vec{b}) + q|\vec{b}|^2 + r(\vec{b} \cdot \vec{c})$$

$$\therefore 0 = \frac{p}{2} + q + \frac{r}{2}. \text{ So, } 0 = p + 2q + r \quad (2)$$

and $\vec{c} \cdot (\vec{a} \times \vec{b}) + \vec{c} \cdot (\vec{b} \times \vec{c}) = p(\vec{a} \cdot \vec{c}) + q(\vec{b} \cdot \vec{c}) + r|\vec{c}|^2$

$$\therefore [\vec{c} \quad \vec{a} \quad \vec{b}] = \frac{p}{2} + \frac{q}{2} + r$$

$$\therefore [\vec{a} \quad \vec{b} \quad \vec{c}] = \frac{p}{2} + \frac{q}{2} + r \quad (3)$$

From (1) and (3),

$$p + \frac{q}{2} + \frac{r}{2} = \frac{p}{2} + \frac{q}{2} + r$$

$$\therefore p = r$$

From (2), $p = -q$. So, $r = -q$

Now $\frac{p^2 + 2q^2 + r^2}{q^2} = \frac{q^2 + 2q^2 + q^2}{q^2} = 4$

Ans. : (D)

- (128) If $|\vec{a}| = 2, |\vec{b}| = 3$ and $|2\vec{a} - \vec{b}| = 5$, then $|2\vec{a} + \vec{b}| = \dots$ [JEE : 2014]
 (A) 17 (B) 7 (C) 5 (D) 1

Sol. : $|2\vec{a} - \vec{b}|^2 = 25$

$$\therefore 4|\vec{a}|^2 - 4\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 25$$

$$\therefore 16 + 9 - 25 = 4\vec{a} \cdot \vec{b}$$

$$\therefore 0 = \vec{a} \cdot \vec{b}$$

$$\therefore |2\vec{a} + \vec{b}|^2 = 4|\vec{a}|^2 + 4\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 16 + 9$$

$$\therefore |2\vec{a} + \vec{b}| = 5$$

Ans. : (C)

- (129) If $|\vec{c}|^2 = 60$ and $\vec{c} \times (\hat{i} + 2\hat{j} + 5\hat{k}) = \vec{0}$, then $\vec{c} \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) = \dots$ [JEE : 2014]
 (A) $4\sqrt{2}$ (B) 12 (C) 24 (D) $12\sqrt{2}$

Sol. : $\vec{c} \times (\hat{i} + 2\hat{j} + 5\hat{k}) = \vec{0}$

$$\therefore \vec{c} \text{ and } \hat{i} + 2\hat{j} + 5\hat{k} \text{ are collinear.}$$

$$\therefore \vec{c} = \lambda(\hat{i} + 2\hat{j} + 5\hat{k}), \lambda \in \mathbb{R} - \{0\}$$

$$\therefore |\vec{c}|^2 = |\lambda|^2 (30). \text{ So, } 60 = 30 |\lambda|^2$$

$$\therefore |\lambda|^2 = 2 \quad (|\vec{c}| = 60)$$

$$\therefore \lambda = \pm\sqrt{2}$$

$$\therefore \vec{c} = \pm\sqrt{2}(\hat{i} + 2\hat{j} + 5\hat{k})$$

$$\text{Now, } \vec{c} \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) = \pm\sqrt{2}(\hat{i} + 2\hat{j} + 5\hat{k}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) = \pm 12\sqrt{2} \quad \text{Ans. : (D)}$$

- (130) If $\vec{x} = 3\hat{i} - 6\hat{j} - \hat{k}$, $\vec{y} = \hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{z} = 3\hat{i} - 4\hat{j} - 12\hat{k}$, then the magnitude of projection of $\vec{x} \times \vec{y}$ on $\vec{z}$ is [JEE : 2014]
 (A) 12 (B) 15 (C) 14 (D) 13

Sol. : The magnitude of projection of $\vec{x} \times \vec{y}$ on $\vec{z} = \frac{|(\vec{x} \times \vec{y}) \cdot \vec{z}|}{|\vec{z}|} = \frac{|[\vec{x} \ \vec{y} \ \vec{z}]|}{|\vec{z}|}$

$$\text{Now, } [\vec{x} \ \vec{y} \ \vec{z}] = \begin{vmatrix} 3 & -6 & -1 \\ 1 & 4 & -3 \\ 3 & -4 & -12 \end{vmatrix} = 3(-60) + 6(-3) - 1(-16) = -182$$

$$\text{The magnitude of projection of } \vec{x} \times \vec{y} \text{ on } \vec{z} = \frac{182}{13} = 14 \quad \text{Ans. : (C)}$$

(131) If $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a} + \vec{b}| = \sqrt{3}$ and $\vec{c} = \vec{a} + 2\vec{b} + 3(\vec{a} \times \vec{b})$, then $2|\vec{c}| = \dots$

[JEE Online : 2015]

(A) $\sqrt{55}$

(B) $\sqrt{37}$

(C) $\sqrt{51}$

(D) $\sqrt{43}$

Sol. : $|\vec{a} + \vec{b}|^2 = 3$

$$\therefore |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 3$$

$$\therefore 1 + 1 + 2\vec{a} \cdot \vec{b} = 3$$

$$\therefore \vec{a} \cdot \vec{b} = \frac{1}{2}$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 - |\vec{a} \cdot \vec{b}|^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\begin{aligned} \text{Now, } |\vec{c}|^2 &= |\vec{a} + 2\vec{b} + 3(\vec{a} \times \vec{b})|^2 \\ &= |\vec{a}|^2 + 4|\vec{b}|^2 + 9|\vec{a} \times \vec{b}|^2 + 4\vec{a} \cdot \vec{b} + 6\vec{a} \cdot (\vec{a} \times \vec{b}) + 12\vec{b} \cdot (\vec{a} \times \vec{b}) \\ &= 1 + 4 + 9\left(\frac{3}{4}\right) + 4\left(\frac{1}{2}\right) + 6(0) + 12(0) = \frac{55}{4} \end{aligned}$$

$$\therefore |\vec{c}| = \frac{\sqrt{55}}{2}. \text{ So, } 2|\vec{c}| = \sqrt{55}$$

Ans. : (A)

(132) In parallelogram ABCD, $|\vec{AB}| = a$, $|\vec{AD}| = b$, $|\vec{AC}| = c$, then $\vec{DA} \cdot \vec{AB} = \dots$ [JEE online : 2015]

(A) $\frac{1}{2}(a^2 + b^2 + c^2)$ (B) $\frac{1}{2}(a^2 - b^2 + c^2)$ (C) $\frac{1}{2}(a^2 + b^2 - c^2)$ (D) $\frac{1}{2}(b^2 + c^2 - a^2)$

Sol. : $\vec{AB} + \vec{AD} = \vec{AC}$

$$\therefore |\vec{AB} + \vec{AD}|^2 = |\vec{AC}|^2$$

$$\therefore |\vec{AB}|^2 + |\vec{AD}|^2 + 2\vec{AB} \cdot \vec{AD} = |\vec{AC}|^2$$

$$\therefore a^2 + b^2 + 2\vec{AB} \cdot \vec{AD} = c^2$$

$$\therefore a^2 + b^2 + 2\vec{AB} \cdot (-\vec{DA}) = c^2$$

$$\therefore a^2 + b^2 - c^2 = 2\vec{DA} \cdot \vec{AB}$$

$$\therefore \frac{1}{2}(a^2 + b^2 - c^2) = \vec{DA} \cdot \vec{AB}$$

Ans. : (C)

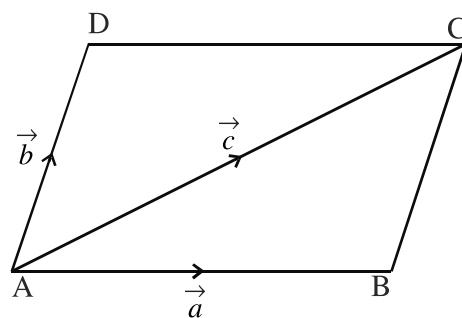


Figure 20.16

- (133) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero vectors such that no two of them are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$. If θ is the measure of the angle between $\vec{b}$ and $\vec{c}$, then $\sin\theta = \dots\dots$ [JEE Main : 2015]

(A) $\frac{2\sqrt{2}}{3}$ (B) $-\frac{\sqrt{2}}{3}$ (C) $\frac{2}{3}$ (D) $-\frac{2\sqrt{3}}{3}$

Sol. : $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$

$$\therefore (\vec{c} \cdot \vec{a}) \vec{b} - (\vec{c} \cdot \vec{b}) \vec{a} = \frac{1}{3} |\vec{b}| |\vec{c}| \vec{a}$$

$$\therefore -(\vec{c} \cdot \vec{b}) = \frac{1}{3} |\vec{b}| |\vec{c}| \quad (\vec{a}, \vec{b} \text{ are not collinear})$$

$$\therefore -|\vec{c}| |\vec{b}| \cos\theta = \frac{1}{3} |\vec{b}| |\vec{c}|$$

$$\therefore \cos\theta = -\frac{1}{3}. \text{ So, } \sin\theta = \frac{2\sqrt{2}}{3} \quad \theta \in [0, \pi] \quad \text{Ans. : (A)}$$

- (134) In ΔPQR , $\vec{a} = \vec{QR}$, $\vec{b} = \vec{RP}$, $\vec{c} = \vec{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$, $\vec{b} \cdot \vec{c} = 24$, then which of the following is true ? [JEE Adva. : 2015]

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$

(B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(C) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$

(D) $\vec{a} \cdot \vec{b} = -72$

Sol. : $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\therefore \vec{b} + \vec{c} = -\vec{a}$$

$$\therefore |\vec{b} + \vec{c}|^2 = |-\vec{a}|^2$$

$$\therefore |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c} = |\vec{a}|^2$$

$$\therefore 48 + |\vec{c}|^2 + 2(24) = 144$$

$$\therefore |\vec{c}|^2 = 48$$

$$\therefore |\vec{c}| = 4\sqrt{3}$$

$$\frac{|\vec{c}|^2}{2} - |\vec{a}| = \frac{48}{2} - 12 = 12$$

$\therefore$ Option (A) is true.

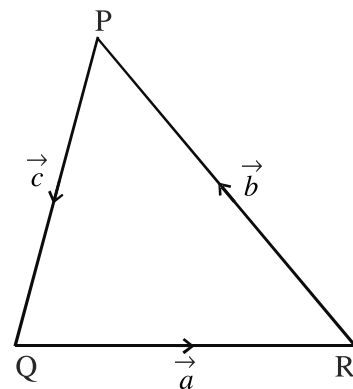


Figure 20.17

$$(B) \frac{|\vec{c}|^2}{2} + |\vec{a}| = \frac{48}{2} + 12 = 36 \neq 30$$

∴ Option (B) is false.

$$(D) \vec{a} + \vec{b} = -\vec{c}$$

$$\therefore |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$$

$$\therefore |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = |\vec{c}|^2$$

$$\therefore 2\vec{a} \cdot \vec{b} = 48 - 144 - 48$$

$$\therefore \vec{a} \cdot \vec{b} = -72$$

∴ Option (D) is true.

$$(C) \vec{b} + \vec{c} = -\vec{a}$$

Taking outer product with $\vec{a}$, $\vec{a} \times (\vec{b} + \vec{c}) = -\vec{a} \times \vec{a}$

$$\therefore \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\therefore \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

$$\text{Now, } |\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 2|\vec{a} \times \vec{b}|$$

$$= 2\sqrt{|\vec{a}|^2|\vec{b}|^2 - |\vec{a} \cdot \vec{b}|^2} = 2\sqrt{144 \times 48 - (-72)^2} = 48\sqrt{3}$$

∴ Option (C) is true.

Ans. : (A), (C), (D)

- (135) $\vec{p}$, $\vec{q}$ and $\vec{r}$ are non-coplanar vectors in R^3 . The components of $\vec{s}$ on $\vec{p}$, $\vec{q}$, $\vec{r}$ are 4, 3, 5 respectively. If the components of $\vec{s}$ in the directions of $-\vec{p} + \vec{q} + \vec{r}$; $\vec{p} - \vec{q} + \vec{r}$ and $-\vec{p} - \vec{q} + \vec{r}$ are x, y, z respectively, then $2x + y + z = \dots$ [JEE Adva. : 2015]

(A) 9

(B) 4

(C) 3

(D) 5

Sol. : Components of $\vec{s}$ on $\vec{p}$, $\vec{q}$, $\vec{r}$ are 4, 3, 5.

$$\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$$

$$\text{Let } \vec{s} = x(-\vec{p} + \vec{q} + \vec{r}) + y(\vec{p} - \vec{q} + \vec{r}) + z(-\vec{p} - \vec{q} + \vec{r})$$

$$\therefore 4\vec{p} + 3\vec{q} + 5\vec{r} = (-x + y - z)\vec{p} + (x - y - z)\vec{q} + (x + y + z)\vec{r}$$

$$\therefore -x + y - z = 4, x - y - z = 3 \text{ and } x + y + z = 5$$

$$\text{Solving above equations, } x = 4, y = \frac{9}{2}, z = -\frac{7}{2}$$

$$\therefore 2x + y + z = 9$$

Ans. : (A)

- (136) If the magnitude of projection of $\alpha \hat{i} + \beta \hat{j}$ on $\sqrt{3} \hat{i} + \hat{j}$ is $\sqrt{3}$ and $\alpha = (2 + \sqrt{3} \beta)$, then a possible value of $|\alpha|$ is [JEE Adva. 2015]

(A) 3 (B) 2 (C) 1 (D) -1

Sol. : The magnitude of projection of vector $\alpha \hat{i} + \beta \hat{j}$ on $\sqrt{3} \hat{i} + \hat{j}$ is $\sqrt{3}$

$$\therefore \frac{|(\alpha \hat{i} + \beta \hat{j}) \cdot (\sqrt{3} \hat{i} + \hat{j})|}{\sqrt{3+1}} = \sqrt{3}$$

$$\therefore |\sqrt{3} \alpha + \beta| = 2\sqrt{3}$$

$$\therefore \sqrt{3} \alpha + \beta = 2\sqrt{3} \text{ or } \sqrt{3} \alpha + \beta = -2\sqrt{3}$$

$$\text{Now, } \alpha = 2 + \sqrt{3} \beta$$

$$\therefore \sqrt{3}(2 + \sqrt{3}\beta) + \beta = 2\sqrt{3} \quad \text{or} \quad \sqrt{3}(2 + \sqrt{3}\beta) + \beta = -2\sqrt{3}$$

$$\therefore \beta = 0, \alpha = 2$$

$$\therefore |\alpha| = 2$$

$$\therefore 4\beta = -4\sqrt{3}$$

$$\beta = -\sqrt{3}, \alpha = -1$$

$$|\alpha| = 1$$

$$\therefore |\alpha| = 1 \text{ or } 2$$

Ans. : (B), (C)

- (137) With respect to the origin O, the position vectors of points P, Q, R are $\sqrt{3} \hat{i} + \hat{j}$, $\hat{i} + \sqrt{3} \hat{j}$ and $\beta \hat{i} + (1 - \beta) \hat{j}$ respectively. If the distance of point R from the angle bisector of $\vec{OP}$ and $\vec{OQ}$ is $\frac{3}{\sqrt{2}}$, then a possible value (values) of $|\beta|$ is [JEE Adva. : 2015]

(A) 5 (B) 4 (C) 1 (D) 2

Sol. : The equation of the angle bisector of $\vec{OP}$ and $\vec{OQ}$ is $x - y = 0$.

The position vector of R is $\beta \hat{i} + (1 - \beta) \hat{j}$.

The coordinates of R are $(\beta, 1 - \beta)$.

Distance of R from $x - y = 0$ is $\frac{3}{\sqrt{2}}$

$$\therefore \frac{|\beta - (1 - \beta)|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\therefore 2\beta - 1 = 3 \quad \text{or} \quad 2\beta - 1 = -3$$

$$\therefore \beta = 2 \quad \text{or} \quad \beta = -1$$

$$\therefore |\beta| = 1 \text{ or } 2$$

Ans. : (C), (D)

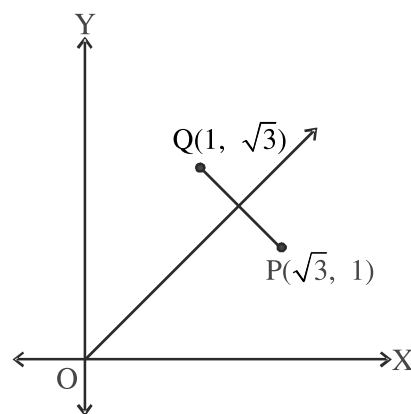


Figure 20.18

: Questions with More Than One Correct Answers :

(138) Unit vectors $\vec{a}$ and $\vec{b}$ are non-collinear. If $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then $|\vec{v}|$ =

- (A) $|\vec{u}|$ (B) $|\vec{u}| + |\vec{u} \cdot \vec{a}|$ (C) $|\vec{v}| + |\vec{u} \cdot \vec{b}|$ (D) $|\vec{u}| + \vec{u} \cdot (\vec{a} + \vec{b})$

Sol. $|\vec{a}| = |\vec{b}| = 1$

Since $\vec{a}$ and $\vec{b}$ are non-collinear, $(\vec{a} \wedge \vec{b}) \neq 0, \pi$

Let $(\vec{a} \wedge \vec{b}) = \theta$

Now, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$

$\therefore \vec{a} \cdot \vec{b} = \cos\theta$

Now, $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$

$$|\vec{u}|^2 = |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{b})^2 |\vec{b}|^2$$

$$= 1 - 2(\vec{a} \cdot \vec{b})^2 + (\vec{a} \cdot \vec{b})^2$$

$$= 1 - (\vec{a} \cdot \vec{b})^2 = 1 - \cos^2 \theta = \sin^2 \theta = |\vec{a} \times \vec{b}|^2 \quad (|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta = \sin\theta)$$

$$|\vec{v}| = |\vec{u}| \quad (\vec{v} = \vec{a} \times \vec{b})$$

$\therefore$ (A) is true.

Now, $\vec{u} \cdot \vec{a} = \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{b}) = |\vec{a}|^2 - (\vec{a} \cdot \vec{b})^2 = 1 - \cos^2 \theta = \sin^2 \theta$

$$\therefore |\vec{u}| + |\vec{u} \cdot \vec{a}| = |\vec{v}| + |\vec{v}|^2 \neq |\vec{v}|$$

$\therefore$ (B) is not true.

$$\vec{u} \cdot \vec{b} = \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{b}) = \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b}) |\vec{b}|^2 = 0$$

$$|\vec{v}| + |\vec{u} \cdot \vec{b}| = |\vec{u}| = |\vec{v}|$$

$\therefore$ (C) is true.

$$\vec{u} \cdot (\vec{a} + \vec{b}) = \vec{u} \cdot \vec{a} + \vec{u} \cdot \vec{b} = |\vec{v}|^2 + 0$$

$$\therefore |\vec{u}| + \vec{u} \cdot (\vec{a} + \vec{b}) = |\vec{v}| + |\vec{v}|^2$$

$\therefore$ (D) is not true.

Ans. : (A) (C)

- (139) The vector which is coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is

(A) $\hat{j} - \hat{k}$ (B) $-\hat{i} + \hat{j}$ (C) $\hat{i} - \hat{j}$ (D) $-\hat{j} + \hat{k}$

Sol. Vector which is coplanar with $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to

$$\begin{aligned}\vec{c} = \hat{i} + \hat{j} + \hat{k} \text{ is } \vec{r} &= \lambda (\vec{c} \times (\vec{a} \times \vec{b})) \\ &= \lambda ((\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}) \\ &= \lambda (4\hat{i} + 4\hat{j} + 8\hat{k} - 4\hat{i} - 8\hat{j} - 4\hat{k}) \\ &= \lambda (-4\hat{j} + 4\hat{k})\end{aligned}$$

Considering the options given, taking $\lambda = \pm \frac{1}{4}$,

$$\vec{r} = -\hat{j} + \hat{k} \text{ or } \vec{r} = \hat{j} - \hat{k}$$

Ans. : (A), (D)

- (140) If vectors $\vec{p} = a\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{q} = \hat{i} + b\hat{j} - c\hat{k}$ are collinear, then $(a, b, c) = \dots$

(A) $(-1, 2, 5)$ (B) $(\frac{1}{2}, -4, -10)$ (C) $(1, -2, -5)$ (D) $(-\frac{1}{2}, 4, 10)$

Sol. $\vec{p}$ and $\vec{q}$ are collinear.

$$\therefore (a\hat{i} - 2\hat{j} + 5\hat{k}) = m(\hat{i} + b\hat{j} - c\hat{k})$$

Comparing coefficients on both sides,

$$a = m, -2 = mb, 5 = -mc$$

Taking $m = 1, a = 1, b = -2, c = -5$

So, (C) is true.

Taking $m = \frac{1}{2}, a = \frac{1}{2}, b = -4, c = -10$

So, (B) is true.

Taking $m = -1, a = -1, b = 2, c = 5$

So, (A) is true.

Taking $m = -\frac{1}{2}, a = -\frac{1}{2}, b = 4, c = 10$

So, (D) is true.

Ans. : (A), (B), (C), (D)

Second method :

$$\vec{p} \times \vec{q} = \vec{0} \Rightarrow \frac{b}{c} = \frac{2}{5}, ab = -2, ac = -5, a = 1, \frac{1}{2}, -1, -\frac{1}{2}. \text{ So, all options are correct.}$$

Ans. : (A), (B), (C), (D)

- (141) Out of the following options, is meaningless.

(A) $\vec{a} \times (\vec{b} \times \vec{c})$ (B) $\vec{a} \times (\vec{b} \cdot \vec{c})$ (C) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ (D) $(\vec{a} \cdot \vec{b}) \vec{c}$

Sol. $\vec{b} \cdot \vec{c}$ is a real constant whose dot product or cross product with any vector is not possible.

So, options (B) and (C) are not possible.

$(\vec{a} \cdot \vec{b}) \vec{c}$ is product of $\vec{c}$ by a scalar, and so, (D) is possible. $\vec{a} \times (\vec{b} \times \vec{c})$ is the vector triple product and so (A) is possible.

Ans. : (B), (C)

(142) If $\vec{p}$, $\vec{q}$ and $\vec{r}$ are non-coplanar and $\vec{x} = \frac{\vec{q} \times \vec{r}}{[\vec{p} \ \vec{q} \ \vec{r}]}$, $\vec{y} = \frac{\vec{r} \times \vec{p}}{[\vec{p} \ \vec{q} \ \vec{r}]}$ and $\vec{z} = \frac{\vec{p} \times \vec{q}}{[\vec{p} \ \vec{q} \ \vec{r}]}$,

then

(A) $\vec{x} \cdot \vec{p} + \vec{y} \cdot \vec{q} + \vec{z} \cdot \vec{r} = 3$

(B) $\vec{x} \cdot \vec{p} + \vec{y} \cdot \vec{q} + \vec{z} \cdot \vec{r} = 0$

(C) $\vec{p} \times \vec{x} + \vec{q} \times \vec{y} + \vec{r} \times \vec{z} = \vec{0}$

(D) $\vec{p} \times \vec{x} + \vec{q} \times \vec{y} + \vec{r} \times \vec{z} = \vec{p} \times (\vec{q} \times \vec{r})$

Sol. $\vec{x} \cdot \vec{p} + \vec{y} \cdot \vec{q} + \vec{z} \cdot \vec{r} = \frac{1}{[\vec{p} \ \vec{q} \ \vec{r}]} [(\vec{q} \times \vec{r}) \cdot \vec{p} + (\vec{r} \times \vec{p}) \cdot \vec{q} + (\vec{p} \times \vec{q}) \cdot \vec{r}]$

$$= \frac{1}{[\vec{p} \ \vec{q} \ \vec{r}]} [[\vec{q} \ \vec{r} \ \vec{p}] + [\vec{r} \ \vec{p} \ \vec{q}] + [\vec{p} \ \vec{q} \ \vec{r}]] = 3$$

$\therefore$ (A) is true, (B) is not true.

Now, $\vec{p} \times \vec{x} + \vec{q} \times \vec{y} + \vec{r} \times \vec{z} = \frac{1}{[\vec{p} \ \vec{q} \ \vec{r}]} [\vec{p} \times (\vec{q} \times \vec{r}) + \vec{q} \times (\vec{r} \times \vec{p}) + \vec{r} \times (\vec{p} \times \vec{q})]$

$$= \frac{1}{[\vec{p} \ \vec{q} \ \vec{r}]} [(\vec{p} \cdot \vec{r})\vec{q} - (\vec{p} \cdot \vec{q})\vec{r} + (\vec{q} \cdot \vec{p})\vec{r} - (\vec{q} \cdot \vec{r})\vec{p} + (\vec{r} \cdot \vec{q})\vec{p} - (\vec{r} \cdot \vec{p})\vec{q}] = \vec{0}$$

$\therefore$ (C) is true, (D) is not true.

Ans. : (A), (C)

(143) If vectors $(x^2 - 6)\hat{i} + (x - 2y)\hat{j} + (y^2 - 5)\hat{k}$ and $\hat{i} + 2\hat{k}$ are parallel, then $(x, y) = \dots\dots$

(A) $(-2, 1)$

(B) $(2, 1)$

(C) $(-2, -1)$

(D) $(2, -1)$

Sol. Vectors $(x^2 - 6)\hat{i} + (x - 2y)\hat{j} + (y^2 - 5)\hat{k}$ and $\hat{i} + 2\hat{k}$ are parallel vectors.

$$\therefore (x^2 - 6)\hat{i} + (x - 2y)\hat{j} + (y^2 - 5)\hat{k} = \lambda(\hat{i} + 2\hat{k})$$

$$\therefore x^2 - 6 = \lambda, \quad x - 2y = 0, \quad y^2 - 5 = 2\lambda$$

Now, $y^2 - 5 = 2\lambda$

$$\therefore y^2 - 5 = 2(x^2 - 6)$$

$$\therefore y^2 = 2x^2 - 7$$

$$\therefore y^2 = 8y^2 - 7$$

(Taking $x = 2y$)

$$\therefore 7y^2 = 7$$

$$\therefore y = \pm 1. \text{ Hence } x = \pm 2$$

$$(x = 2y)$$

$$\therefore (x, y) = (2, 1) \text{ or } (-2, -1)$$

Ans. : (B), (C)

(144) If vectors $5\hat{i} + x\hat{j} - 2\hat{k}$ and $-y\hat{i} + \hat{j} + z\hat{k}$ are collinear, then $(x, y, z) = \dots$

- (A) $(1, -5, -2)$ (B) $\left(\frac{-1}{2}, -10, -4\right)$ (C) $\left(\frac{-1}{2}, 10, 4\right)$ (D) $\left(2, \frac{-5}{2}, -4\right)$

Sol. $5\hat{i} + x\hat{j} - 2\hat{k}$ and $-y\hat{i} + \hat{j} + z\hat{k}$ are collinear.

$$\therefore 5\hat{i} + x\hat{j} - 2\hat{k} = \lambda (-y\hat{i} + \hat{j} + z\hat{k})$$

$$\therefore 5 = -\lambda y, \quad x = \lambda, \quad -2 = \lambda z$$

$$y = \frac{-5}{\lambda}, \quad x = \lambda \text{ and } z = \frac{-2}{\lambda}$$

$$\text{Taking } \lambda = 1, \quad x = 1, \quad y = -5, \quad z = -2$$

$\therefore$ (A) is true.

$$\text{Taking } \lambda = \frac{-1}{2}, \quad x = \frac{-1}{2}, \quad y = 10, \quad z = 4$$

$\therefore$ (C) is true but (B) is not true.

$$\text{Taking } \lambda = 2, \quad x = 2, \quad y = \frac{-5}{2}, \quad z = -1 \neq -4$$

$\therefore$ (D) is not true.

Ans. : (A), (C)

Note : It can also be verified by taking cross product as $\vec{0}$.

(145) The adjacent sides of a parallelogram are $\vec{x} = -\vec{a} + 3\vec{b}$ and $\vec{y} = 3\vec{a} + \vec{b}$. If $|\vec{a}| = |\vec{b}| = 2$ and $(\vec{a}, \vec{b}) = \frac{\pi}{3}$, then the lengths of the diagonals are

- (A) $4\sqrt{7}$ (B) $4\sqrt{5}$ (C) $4\sqrt{3}$ (D) $4\sqrt{2}$

Sol. Let in parallelogram ABCD,

$$\vec{AB} = \vec{x} \text{ and } \vec{AD} = \vec{y}.$$

$$|\vec{AC}|^2 = |\vec{x} + \vec{y}|^2$$

$$= |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x} \cdot \vec{y}$$

$$= |-\vec{a} + 3\vec{b}|^2 + |3\vec{a} + \vec{b}|^2 + 2(-\vec{a} + 3\vec{b}) \cdot (3\vec{a} + \vec{b})$$

$$= |\vec{a}|^2 + 9|\vec{b}|^2 - 6\vec{a} \cdot \vec{b} + 9|\vec{a}|^2 + |\vec{b}|^2 + 6\vec{a} \cdot \vec{b} - 6|\vec{a}|^2 + 6|\vec{b}|^2 + 16\vec{a} \cdot \vec{b}$$

$$= 4 + 36 + 36 + 4 - 24 + 24 + 16|\vec{a}| |\vec{b}| \cos \frac{\pi}{3}$$

$$= 80 + 16(2)(2) \left(\frac{1}{2}\right) = 80 + 32 = 112$$

$$\therefore |\vec{AC}| = 4\sqrt{7}$$

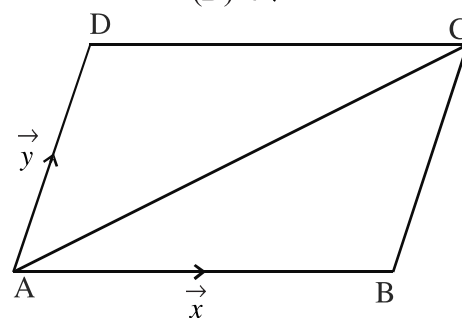


Figure 20.19

$$\text{Now, } |\vec{BD}|^2 = |\vec{x} - \vec{y}|^2$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x} \cdot \vec{y}$$

$$= |-\vec{a} + 3\vec{b}|^2 + |3\vec{a} + \vec{b}|^2 - 2(-\vec{a} + 3\vec{b}) \cdot (3\vec{a} + \vec{b})$$

$$= 80 - 32 = 48$$

$$|\vec{BD}| = 4\sqrt{3}$$

Ans. : (A), (C)

(146) If the vectors $(a+8)\hat{i} + (a+7)\hat{j} + (a+6)\hat{k}$; $(a+5)\hat{i} + (a+4)\hat{j} + (a+3)\hat{k}$;

$(a+2)\hat{i} + (a+1)\hat{j} + a\hat{k}$ are coplanar, then a

(A) > 0

(B) < 0

(C) $\notin \mathbb{R}$

(D) $\in \mathbb{R}$

Sol. Given vectors are coplanar.

$$\therefore \begin{vmatrix} a+8 & a+7 & a+6 \\ a+5 & a+4 & a+3 \\ a+2 & a+1 & a \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ a+2 & a+1 & a \end{vmatrix} = 0$$

(Taking $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$)

$\therefore$ It is true for every $a \in \mathbb{R}$.

Ans. : (A), (B), (D)

(147) $\vec{x}$, $\vec{y}$ and $\vec{z}$ are non-coplanar vectors such that $\vec{a} = \vec{x} + \vec{y} + \vec{z}$, $\vec{b} = \vec{x} + \vec{y} - \vec{z}$,

$\vec{c} = \vec{x} - \vec{y} + \vec{z}$ and $\vec{d} = 4\vec{x} - 3\vec{y} + 5\vec{z}$. If $\vec{d} = p\vec{a} + q\vec{b} + r\vec{c}$, then

(A) $p = 1$

(B) $p + q = \frac{3}{2}$

(C) $-p + q + r = 3$

(D) $q + r = 3$

Sol. $\vec{d} = 4\vec{x} - 3\vec{y} + 5\vec{z}$

$$\therefore p(\vec{x} + \vec{y} + \vec{z}) + q(\vec{x} + \vec{y} - \vec{z}) + r(\vec{x} - \vec{y} + \vec{z}) = 4\vec{x} - 3\vec{y} + 5\vec{z}$$

$\vec{x}$, $\vec{y}$, $\vec{z}$ being non-coplanar,

$$p + q + r = 4 \quad (1)$$

$$p + q - r = -3 \quad (2)$$

$$p - q + r = 5 \quad (3)$$

Solving equations (1), (2) and (3),

$p = 1$, $q = \frac{-1}{2}$, $r = \frac{7}{2}$. So, (A) is true.

$p + q = \frac{1}{2}$. So, (B) is false.

$-p + q + r = 2$. So, (C) is false.

$q + r = 3$. So, (D) is true.

Ans. : (A), (D)

(148) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors and $\vec{a}$ and $\vec{b}$ are non-collinear.

If $\sqrt{2}(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} + \vec{b}$, then

(A) the angle between $\vec{a}$ and $\vec{c}$ has measure $\frac{3\pi}{4}$

(B) the angle between $\vec{a}$ and $\vec{c}$ has measure $\frac{\pi}{4}$

(C) the angle between $\vec{b}$ and $\vec{c}$ has measure $\frac{3\pi}{4}$

(D) the angle between $\vec{b}$ and $\vec{c}$ has measure $\frac{\pi}{4}$

Sol. $\sqrt{2}(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} + \vec{b}$

$$\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = \frac{\vec{a}}{\sqrt{2}} + \frac{\vec{b}}{\sqrt{2}}$$

$$\therefore (\vec{a} \cdot \vec{c}) = \frac{1}{\sqrt{2}} \text{ and } (\vec{b} \cdot \vec{c}) = \frac{-1}{\sqrt{2}} \quad (\vec{a} \text{ and } \vec{b} \text{ are non-collinear.})$$

$$\therefore \angle(\vec{a}, \vec{c}) = \frac{\pi}{4} \quad \angle(\vec{b}, \vec{c}) = \frac{3\pi}{4}$$

(B) is true, (C) is true.

Ans. : (B), (C)

(149) For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, if $[\vec{a} \ \vec{b} \ \vec{c}] = 0$, then

(A) $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

(B) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar

(C) $\vec{a}$, $\vec{b}$, $\vec{c}$ are collinear

(D) $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent

Sol. $[\vec{a} \ \vec{b} \ \vec{c}] = 0$. So, $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar and so, $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.

Ans. : (A), (D)

(150) A vector perpendicular to $2\hat{i} - 3\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - 2\hat{k}$ and whose magnitude is 3 is

(A) $\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (B) $-\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ (C) $\sqrt{3}(\hat{i} + \hat{j} - \hat{k})$ (D) $-\sqrt{3}(\hat{i} + \hat{j} - \hat{k})$

Sol. A vector perpendicular to $\vec{x} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{y} = \hat{i} + \hat{j} - 2\hat{k}$ is $\pm \lambda(\vec{x} \times \vec{y})$.

$$\text{Now, } \vec{x} \times \vec{y} = 5\hat{i} + 5\hat{j} + 5\hat{k}$$

$$\therefore |\vec{x} \times \vec{y}| = 5\sqrt{3}$$

$$\begin{aligned} \therefore \text{The unit vector perpendicular to } \vec{x} \text{ and } \vec{y} &= \pm \frac{(\vec{x} \times \vec{y})}{|\vec{x} \times \vec{y}|} \\ &= \pm \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

$$\begin{aligned} \text{The vector with magnitude 3 and perpendicular to } \vec{x} \text{ and } \vec{y} &= \pm 3 \frac{(\vec{x} \times \vec{y})}{|\vec{x} \times \vec{y}|} \\ &= \pm \sqrt{3} (\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

Ans. : (A), (B)

(151) For non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}, (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = \dots$

(A) $(\vec{a} - \vec{b}) \times (\vec{c} - \vec{b})$

(B) $(\vec{a} - \vec{b}) \times (\vec{a} - \vec{c})$

(C) $(\vec{a} - \vec{c}) \times (\vec{c} - \vec{b})$

(D) $\frac{1}{2}[(\vec{a} \times (\vec{b} - \vec{c}) + \vec{b} \times (\vec{c} - \vec{a}) + \vec{c} \times (\vec{a} - \vec{b}))]$

Sol. $(\vec{a} - \vec{b}) \times (\vec{c} - \vec{b}) = \vec{a} \times \vec{c} - \vec{a} \times \vec{b} - \vec{b} \times \vec{c} \neq \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$
 $\therefore$ (A) is not true.

(Note : $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is not the zero vector. Verify)

Now, $(\vec{a} - \vec{b}) \times (\vec{a} - \vec{c}) = -\vec{a} \times \vec{c} - \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = \vec{c} \times \vec{a} + \vec{a} \times \vec{b} + \vec{b} \times \vec{c}$

$\therefore$ (B) is true.

$$(\vec{a} - \vec{c}) \times (\vec{c} - \vec{b}) = \vec{a} \times \vec{c} - \vec{a} \times \vec{b} + \vec{c} \times \vec{b} = -(\vec{c} \times \vec{a} + \vec{a} \times \vec{b} + \vec{b} \times \vec{c})$$

$\therefore$ (C) is not true.

Also, $\frac{1}{2}[(\vec{a} \times (\vec{b} - \vec{c}) + \vec{b} \times (\vec{c} - \vec{a}) + \vec{c} \times (\vec{a} - \vec{b}))]$

$$= \frac{1}{2}[\vec{a} \times \vec{b} - \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{b} \times \vec{a} + \vec{c} \times \vec{a} - \vec{c} \times \vec{b}]$$

$$= \frac{1}{2}[\vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{b} \times \vec{c}]$$

$$= \frac{1}{2}(2)(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$

$$= \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

$\therefore$ (D) is true.

Ans. : (B), (D)

(152) If for distinct non-collinear vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}, \vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and

$\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then

(A) $\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{d}$ (B) $\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d} \neq \vec{b} \cdot \vec{d} + \vec{a} \cdot \vec{c}$

(C) $(\vec{a} - \vec{d}) \parallel (\vec{b} - \vec{c})$ (D) $(\vec{a} + \vec{d}) \parallel (\vec{b} - \vec{c})$

Sol. $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$$\therefore (\vec{a} \times \vec{b}) - (\vec{c} \times \vec{d}) = \vec{0} \text{ and } (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{d}) = \vec{0}$$

$$\therefore (\vec{a} \times \vec{b}) - (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{d})$$

$$\therefore (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{d}) - (\vec{c} \times \vec{d}) = \vec{0}$$

$$\therefore \vec{a} \times (\vec{b} - \vec{c}) + (\vec{b} - \vec{c}) \times \vec{d} = \vec{0}$$

$$\therefore \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = \vec{0}$$

$$\therefore (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$$

$$\therefore (\vec{a} - \vec{d}) \parallel (\vec{b} - \vec{c})$$

$\therefore$ (C) is true, (D) is not true.

Now, since $(\vec{a} - \vec{d})$ and $(\vec{b} - \vec{c})$ are parallel, they are not perpendicular to each other.

$$\therefore (\vec{a} - \vec{d}) \cdot (\vec{b} - \vec{c}) \neq 0$$

$$\therefore \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} - \vec{d} \cdot \vec{b} + \vec{d} \cdot \vec{c} \neq 0$$

$$\therefore \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d} \neq \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{d}$$

$\therefore$ (A) is not true, (B) is true.

Ans. : (B), (C)

(153) If for non-zero vectors $\vec{a}, \vec{b}, \vec{c}$; $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$, then

(A) $(\vec{a} \times \vec{b}) \times \vec{c} = 0$

(B) $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

(C) $\vec{a}$ and $\vec{b} \times \vec{c}$ are collinear.

(D) $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Sol. $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector, but 0 is scalar. So, (A) is not true.

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$$

Taking dot product with $\vec{c}$,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{b} \times \vec{c}) \cdot \vec{c} + (\vec{c} \times \vec{a}) \cdot \vec{c} = \vec{0} \cdot \vec{c}$$

$$\therefore [\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{c} \vec{c}] + [\vec{c} \vec{a} \vec{c}] = 0$$

$$\therefore [\vec{a} \vec{b} \vec{c}] = 0$$

$\therefore (\vec{a} \times \vec{b}) \cdot \vec{c} = 0$ and $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

$\therefore$ (B) and (D) are true. (1)

$\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and from (D) $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$ too.

$\therefore$ (C) is not true. Ans. : (B), (D)

Note : If $\vec{b} \times \vec{c} = \lambda \vec{a}$, then $\vec{a} \cdot (\vec{b} \times \vec{c}) = \lambda \vec{a} \cdot \vec{a} = \lambda |\vec{a}|^2 = 0$ since $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

But $\vec{a} \neq \vec{0}$. So, $\lambda = 0$. Hence $\vec{b} \times \vec{c} = \vec{0}$, which is not possible.

(154) If for vectors $\vec{a} = 2\hat{i} + x\hat{j} + x^2\hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + \hat{k}$, $\vec{c} = -3\hat{i} + (3x + 11)\hat{j} + (x - 9)\hat{k}$,

$(\vec{a}, \vec{b})$ is acute and $(\vec{a}, \vec{c})$ is obtuse, then $x \in \dots\dots$

(A) $(-\infty, 1)$ (B) $(2, 3)$ (C) $(-\infty, 1) \cup (2, 3)$ (D) $(-\infty, 2)$

Sol. $(\vec{a}, \vec{b})$ is acute and $(\vec{a}, \vec{c})$ is obtuse.

$$\therefore \vec{a} \cdot \vec{b} > 0$$

$$\therefore x^2 - 3x + 2 > 0$$

$$\therefore (x - 2)(x - 1) > 0$$

$$\therefore x \in (-\infty, 1) \cup (2, \infty)$$

$$\vec{a} \cdot \vec{c} < 0$$

$$\therefore -6 + 3x^2 + 11x + x^3 - 9x^2 < 0$$

$$\therefore x^3 - 6x^2 + 11x - 6 < 0$$

$$\therefore (x - 1)(x^2 - 5x + 6) < 0$$

$$\therefore (x - 1)(x - 2)(x - 3) < 0$$

$$\therefore x \in (-\infty, 1) \cup (2, 3)$$

Ans. : (A), (B), (C)

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

(A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.

(B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is false.

(D) Statement 1 is false, statement 2 is true.

(155) **Statement 1:** If $(p + a)\hat{i} + b\hat{j} + c\hat{k}$, $a\hat{i} + (q + b)\hat{j} + c\hat{k}$ and $a\hat{i} + b\hat{j} + (r + c)\hat{k}$ are coplanar vectors and $pqr \neq 0$, then $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1$

Statement 2: If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Sol. $\vec{x} = (p + a)\hat{i} + b\hat{j} + c\hat{k}$, $\vec{y} = a\hat{i} + (q + b)\hat{j} + c\hat{k}$, $\vec{z} = a\hat{i} + b\hat{j} + (r + c)\hat{k}$ are coplanar.

So, $[\vec{x} \ \vec{y} \ \vec{z}] = 0$

$$\therefore \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} p & -q & 0 \\ 0 & q & -r \\ a & b & r+c \end{vmatrix} = 0 \quad (R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3)$$

$$\therefore p(qr + qc + br) + q(ar) = 0$$

$$\therefore pqr + pqc + prb + qra = 0$$

$$\frac{c}{r} + \frac{b}{q} + \frac{a}{p} = -1 \quad (\text{Divide by } pqr \neq 0)$$

$\therefore$ Statement 1 is true.

$\vec{a}, \vec{b}, \vec{c}$ are coplanar. So, $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ i.e. $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

$\therefore$ Statement 2 is true and is the correct explanation for statement 1. **Ans. : (A)**

(156) **Statement 1:** $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 2\hat{k}$ and $\vec{c}$ are vectors such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 10$. Then unit vector along $\vec{c}$ is $\frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$.

Statement 2: Any vector in R^3 can be written as a linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Sol. : Suppose $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{Now } \vec{a} \times \vec{c} = \vec{b}$$

$$\therefore (z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k} = 2\hat{i} - 2\hat{k}$$

$$\therefore z - y = 2, \quad x - z = 0, \quad y - x = -2$$

$$\text{Also, } \vec{a} \cdot \vec{c} = 10$$

$$\therefore x + y + z = 10$$

$$\text{Taking } x = z \text{ and } y = z - 2, \quad z + z - 2 + z = 10$$

$$\therefore z = 4 = x \text{ and } y = 2$$

$$\therefore \vec{c} = 4\hat{i} + 2\hat{j} + 4\hat{k}$$

$$|\vec{c}| = \sqrt{16+4+16} = 6$$

$$\therefore \text{The unit vector in the direction of } \vec{c} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$$

So, statement 1 is true. Statement 2 is also true, but it does not give correct explanation of statement 1. **Ans. : (B)**

(157) $\vec{PQ}, \vec{QR}, \vec{RS}, \vec{ST}, \vec{TU}$ and $\vec{UP}$ are the sides of a regular hexagon.

$$\text{Statement 1: } \vec{PQ} \times (\vec{RS} + \vec{ST}) \neq \vec{0}$$

$$\text{Statement 2: } \vec{PQ} \times \vec{RS} = \vec{0} \text{ and } \vec{PQ} \times \vec{ST} \neq \vec{0}$$

Sol. : $\vec{PQ} \times (\vec{RS} + \vec{ST}) \neq \vec{0}$

$\therefore \vec{PQ} \times \vec{RT} \neq \vec{0}$

$\therefore \vec{PQ}$ and $\vec{RT}$ are not parallel vectors.

$\therefore$ Statement 1 is true.

and $\vec{PQ} \times \vec{RS} = \vec{0}$

So, $\vec{PQ} \parallel \vec{RS}$ is not possible.

and $\vec{PQ} \times \vec{ST} \neq \vec{0}$, from which $\vec{PQ}$ and $\vec{ST}$ are not parallel vectors. But $\vec{PQ} \parallel \vec{ST}$.

$\therefore$ Statement 2 is false.

Ans. : (C)

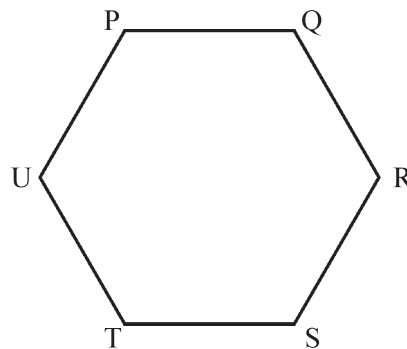


Figure 20.20

(158) **Statement 1:** If $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -3\hat{i} + 3\hat{j} - 6\hat{k}$, then $\vec{a} \times \vec{b} = \vec{0}$

Statement 2: For non-zero, non-collinear vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin(\angle \vec{a}, \vec{b})$

Sol. : $\vec{a} \times \vec{b} = (\hat{i} - \hat{j} + 2\hat{k}) \times (-3\hat{i} + 3\hat{j} - 6\hat{k})$
 $= -3(\hat{i} - \hat{j} + 2\hat{k}) \times (\hat{i} - \hat{j} + 2\hat{k}) = \vec{0}$

Statement 1 is true.

Statement 2 is also true. But statement 2 does not give correct explanation for statement 1.

Ans. : (B)

(159) **Statement 1 :** $\vec{a} = 3\hat{j} - \hat{k}$ and $\vec{b} = -3\hat{i} + 2\hat{j} + \hat{k}$. If $\vec{b} = \vec{b}_1 + \vec{b}_2$ is a vector such that $\vec{b}_1$ is parallel to $\vec{a}$ and $\vec{b}_2$ is perpendicular to $\vec{a}$, then $\vec{b}_2 = -3\hat{i} + \hat{j} + 3\hat{k}$

Statement 2: If $\vec{b}_1$ and $\vec{b}_2$ are non-zero, non-collinear vectors, then $\vec{b}$ can be written as

$\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1$ is parallel to $\vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Sol. : $\vec{b}_1$ is parallel to $\vec{a}$.

$\therefore \vec{b}_1 = m(3\hat{j} - \hat{k}) = 3m\hat{j} - m\hat{k}$

Now, $\vec{b} = \vec{b}_1 + \vec{b}_2$

$\therefore \vec{b}_2 = \vec{b} - \vec{b}_1$
 $= -3\hat{i} + 2\hat{j} + \hat{k} - 3m\hat{j} + m\hat{k}$
 $= -3\hat{i} + (2 - 3m)\hat{j} + (m + 1)\hat{k}$

$\vec{b}_2 \perp \vec{a}$. So, $\vec{b}_2 \cdot \vec{a} = 0$

$\therefore (2 - 3m)(3) + (m + 1)(-1) = 0$

$$\therefore 6 - 9m - m - 1 = 0$$

$$\therefore 5 = 10m. \text{ So, } m = \frac{1}{2}$$

$$\therefore \vec{b}_1 = \frac{3}{2} \hat{j} - \frac{1}{2} \hat{k}$$

$$\text{and } \vec{b}_2 = -3\hat{i} + \frac{\hat{j}}{2} + \frac{3}{2}\hat{k} \neq -3\hat{i} + \hat{j} + 3\hat{k}$$

$\therefore$ Statement 1 is false. Statement 2 is true.

Ans. : (D)

(160) **Statement 1:** If $|\vec{x}| = 4$, $|\vec{y}| = 3$ and $|\vec{x} + \vec{y}| = 5$, then $|\vec{x} - \vec{y}| = 5$

Statement 2: $|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}|$ where $\vec{a}$ and $\vec{b}$ are non-zero vectors.

$$\text{Sol. : } |\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x} \cdot \vec{y}$$

$$\therefore 25 = 16 + 9 + 2\vec{x} \cdot \vec{y}$$

$$\therefore \vec{x} \cdot \vec{y} = 0$$

$$\text{Now } |\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x} \cdot \vec{y} = 16 + 9 - 0 = 25$$

$$\therefore |\vec{x} - \vec{y}| = 5$$

$\therefore$ Statement 1 is true.

$$|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}| \text{ is true only if } \vec{a} \perp \vec{b}$$

$\therefore$ Statement 2 is false.

Ans. : (C)

(161) **Statement 1:** If $\vec{x} \cdot \vec{a} = 0$, $\vec{x} \cdot \vec{b} = 0$, $\vec{x} \cdot \vec{c} = 0$, ($\vec{x} \neq \vec{0}$), then $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ is possible.

Statement 2: If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, then $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

$$\text{Sol. : } \vec{x} \cdot \vec{a} = 0 \Rightarrow \vec{x} \perp \vec{a}$$

$$\vec{x} \cdot \vec{b} = 0 \Rightarrow \vec{x} \perp \vec{b}$$

$$\vec{x} \cdot \vec{c} = 0 \Rightarrow \vec{x} \perp \vec{c}$$

So, $\vec{x}$ is perpendicular to $\vec{a}$, $\vec{b}$ and $\vec{c}$ and $\vec{x} \neq \vec{0}$

$\therefore \vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Now, $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar. From this, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ where atleast one of α , β , γ is non-zero.

Taking $\alpha = \beta = \gamma = 1$,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0} \text{ is a possibility.}$$

$\therefore$ Statement 1 is true.

$\therefore$ Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

$\therefore$ Statement 2 is true. But statement 2 is not enough to explain statement 1. **Ans. : (B)**

(162) **Statement 1:** If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ are coplanar.

$$\text{Statement 2: } [\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}] = 2[\vec{a} \ \vec{b} \ \vec{c}]$$

$$\begin{aligned} \text{Sol. : } [\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}] &= (\vec{a} \times \vec{b}) \cdot ((\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})) \\ &= (\vec{a} \times \vec{b}) \cdot [((\vec{b} \times \vec{c}) \cdot \vec{a})\vec{c} - ((\vec{b} \times \vec{c}) \cdot \vec{c})\vec{a}] \\ &= (\vec{a} \times \vec{b}) \cdot ([\vec{b} \ \vec{c} \ \vec{a}]\vec{c} - [\vec{b} \ \vec{c} \ \vec{c}]\vec{a}) \\ &= [\vec{a} \ \vec{b} \ \vec{c}] (\vec{a} \times \vec{b}) \cdot \vec{c} - 0 \\ &= [\vec{a} \ \vec{b} \ \vec{c}]^2 \end{aligned}$$

$\therefore$ Statement 2 is false. (Here itself the answer is decided as option (C))

Now, If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then

$$[\vec{a} \ \vec{b} \ \vec{c}] = 0$$

$$\therefore [\vec{a} \ \vec{b} \ \vec{c}]^2 = 0$$

$$\therefore [\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}] = 0$$

$\therefore \vec{a} \times \vec{b}, \vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are coplanar.

$\therefore$ Statement 1 is true.

Ans. : (C)

(163) **Statement 1 :** If $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are three points, where $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$,
 $\vec{b} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{c} = 5\hat{i} + 8\hat{j} + 13\hat{k}$, then OABC is a tetrahedron,
 where O is the origin.

Statement 2: If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar and they respectively are position vectors of A, B, C, then OABC will be tetrahedron where O represents origin.

$$\begin{aligned} \text{Sol. : } [\vec{OA} \ \vec{OB} \ \vec{OC}] &= [\vec{a} \ \vec{b} \ \vec{c}] \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 5 & 8 & 13 \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 5 & 8 & 0 \end{vmatrix} = 0 \quad (C_3 \rightarrow C_3 - C_1 - C_2)$$

So, OABC is not a tetrahedron.

∴ Statement 1 is false.

Now, If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a} \ \vec{b} \ \vec{c}] \neq 0$

∴ OABC represents a tetrahedron.

∴ Statement 2 is true.

Ans. : (D)

(164) **Statement 1:** $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$

Statement 2: If $\vec{a} = \vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}|^2 = |\vec{b}|^2$

Sol. : If $\vec{a} = \vec{b}$, then $|\vec{a}| = |\vec{b}|$, but if $|\vec{a}| = |\vec{b}|$, then $\vec{a} = \vec{b}$ is not true.

because if $|\vec{a}| = |\vec{b}|$, then $\vec{a} = -\vec{b}$ is also possible for event $\vec{a} = (3, 4), \vec{b} = (-4, 3)$.

$\vec{a} \neq \vec{b}$. But $|\vec{a}| = |\vec{b}|$

So, statement 1 is false.

Now, $\vec{a} = \vec{b} \Rightarrow \vec{a} \cdot \vec{a} = \vec{a} \cdot \vec{b} \Rightarrow |\vec{a}|^2$

and $\vec{a} = \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{b} = |\vec{b}|^2$

∴ $\vec{a} = \vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}|^2 = |\vec{b}|^2$

∴ Statement 2 is true.

Ans. : (D)

(165) **Statement 1:** The edges of a parallelopiped are $\hat{j} + x\hat{k}, \hat{i} + x\hat{j} + \hat{k}$ and $x\hat{i} + \hat{k}$. If

$x = -\frac{1}{\sqrt{3}}$, then the volume of parallelopiped will be maximum. $x \in [0, 1]$

Statement 2: If $\vec{a}, \vec{b}, \vec{c}$ are edges of parallelopiped, then its volume is $\frac{1}{6} |[\vec{a} \ \vec{b} \ \vec{c}]|$.

Sol. : Volume of parallelopiped = $|V|$, where

$$V = [\vec{a} \ \vec{b} \ \vec{c}]$$

$$= \begin{vmatrix} 0 & 1 & x \\ 1 & x & 1 \\ x & 0 & 1 \end{vmatrix} = -1(1-x) + x(-x^2) = -1 + x - x^3 = -1 + x(1-x^2) < 0$$

$$V = 1 - x + x^3$$

$$\text{Now, } \frac{dV}{dx} = 3x^2 - 1 \text{ and } \frac{d^2V}{dx^2} = 6x$$

$$\text{For maximum } V, \frac{dV}{dx} = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

$$\text{and } \left(\frac{d^2V}{dx^2} \right)_{\left(x = -\frac{1}{\sqrt{3}} \right)} = \frac{-6}{\sqrt{3}} < 0$$

∴ Volume will be maximum for $x = -\frac{1}{\sqrt{3}}$.

∴ Statement 1 is true.

Now, volume of parallelopiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ is $|\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}|$.

∴ Statement 2 is false.

Ans. : (C)

- (166) **Statement 1:** If the angle between unit vectors $\vec{x}$ and $\vec{y}$ has measure θ and the angle bisector is a unit vector, then $\vec{a} = \frac{\vec{x} + \vec{y}}{2\cos\frac{\theta}{2}}$

Statement 2: If in the isosceles triangle ABC, $AB = AC = 1$, then the bisector of $\angle A$ will be

$$\vec{AE} = \frac{\vec{AB} + \vec{AC}}{2}.$$

Sol. : Since $\triangle ABC$ is isosceles, the median drawn from A and the bisector of $\angle A$ will be the same.

The median $\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2}$.

So, the bisector of $\angle A$ is $\vec{AE} = \frac{\vec{AB} + \vec{AC}}{2}$

∴ Statement 2 is true.

If $\vec{x}$ and $\vec{y}$ are congruent sides of an isosceles triangle, the median

$$\vec{AD} = \frac{\vec{x} + \vec{y}}{2}$$

$$\therefore |\vec{AD}|^2 = \frac{1}{4} [|\vec{x}|^2 + |\vec{y}|^2 + 2|\vec{x}||\vec{y}|\cos\theta]$$

$$= \frac{1}{4} [1 + 1 + 2\cos\theta] = \frac{1}{2} (1 + \cos\theta) = \frac{1}{2} \times 2\cos^2 \frac{\theta}{2}$$

$$\therefore |\vec{AD}| = \cos \frac{\theta}{2} \quad \left(\text{Since } 0 < \theta < \pi, \cos \frac{\theta}{2} > 0 \right)$$

Now, the median $\vec{AD}$ will become the angle bisector $\vec{a}$.

If the angle bisector $\vec{a}$ is a unit vector.

$$\therefore \vec{a} = \frac{\vec{AD}}{|\vec{AD}|} = \frac{\vec{x} + \vec{y}}{2\cos\frac{\theta}{2}}$$

∴ Statement 1 is true and statement 2 is enough to explain statement 1.

Ans. : (A)

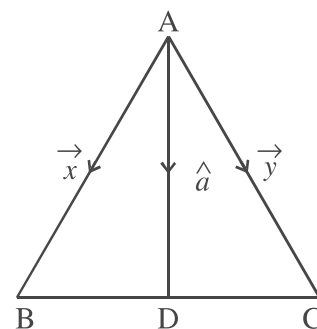


Figure 20.21

: ‘Match the Following’ Type Questions :

(167) Match the questions given in **Part I** with the answers in **Part II**.

Part I		Part II	
(1)	If $\vec{a}$, $\vec{b}$, $\vec{c}$ represent the sides $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ of ΔABC , then	(p)	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$
(2)	If the edges of a cube are $\vec{a}$, $\vec{b}$ and $\vec{c}$, then	(q)	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$
(3)	If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then	(r)	$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{3}{2}$
(4)	If $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$, then	(s)	$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

(A) (1) $\rightarrow$ (p), (2) $\rightarrow$ (r), (3) $\rightarrow$ (q), (4) $\rightarrow$ (s)

(B) (1) $\rightarrow$ (r), (2) $\rightarrow$ (p), (3) $\rightarrow$ (q), (4) $\rightarrow$ (s)

(C) (1) $\rightarrow$ (s), (2) $\rightarrow$ (q), (3) $\rightarrow$ (r), (4) $\rightarrow$ (p)

(D) (1) $\rightarrow$ (s), (2) $\rightarrow$ (r), (3) $\rightarrow$ (p), (4) $\rightarrow$ (q)

Sol. : (1) $\vec{a}$, $\vec{b}$, $\vec{c}$ are sides of ΔABC .

$$\therefore \text{The area of } \Delta ABC = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\vec{b} \times \vec{c}| = \frac{1}{2} |\vec{c} \times \vec{a}|$$

$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$ and $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are in the same direction.

(Direction perpendicular to the plane of the triangle.)

$$\therefore \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$\therefore$ (1) $\rightarrow$ (s)

(2) Since it is a cube, $|\vec{a}| = |\vec{b}| = |\vec{c}|$

$\therefore \Delta ABC$ is equilateral.

$$\therefore \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \frac{\pi}{3}$$

$$\vec{b} \cdot \vec{c} = |\vec{b}| |\vec{c}| \cos \frac{\pi}{3}$$

$$\vec{c} \cdot \vec{a} = |\vec{c}| |\vec{a}| \cos \frac{\pi}{3}$$

$$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$$

$\therefore$ (2) $\rightarrow$ (q)

(Actually, (1) $\rightarrow$ (s), (2) $\rightarrow$ (q) occurs only in option (C). So, Ans. (C))

$$(3) \quad |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\therefore 0 = 3 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\therefore \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-3}{2}$$

$$\therefore (3) \rightarrow (r)$$

$$(4) \quad \vec{a} \times \vec{b} = \vec{c} \Rightarrow \vec{c} \perp \vec{a} \text{ and } \vec{c} \perp \vec{b}$$

$$\vec{b} \times \vec{c} = \vec{a} \Rightarrow \vec{b} \perp \vec{a} \text{ and } \vec{c} \perp \vec{a}$$

$$\therefore \vec{a}, \vec{b}, \vec{c} \text{ are perpendicular to each other.}$$

$$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$$

$$\therefore (4) \rightarrow (p)$$

Ans. : (C)

(168) If $\vec{a} = \hat{i} - \hat{j} - \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{i} - \hat{j} + \hat{k}$, $\vec{d} = \hat{i} + \hat{j} - \hat{k}$, then match those given in **Part I** with that of **Part II** :

Part I	Part II	
(P) $\vec{b} \cdot \vec{c}$	(1)	3
(Q) $[\vec{b} \ \vec{c} \ \vec{d}]$	(2)	$\vec{a} \cdot \vec{b}$
(R) $\vec{c} \cdot \vec{d}$	(3)	$\vec{a} \cdot \vec{c}$
(S) $\vec{c} \times \vec{d}$	(4)	4
	(5)	$2\hat{j} - 2\hat{k}$
	(6)	$2\hat{j} + 2\hat{k}$

$$(A) \quad (P) \rightarrow (1), (Q) \rightarrow (4), (R) \rightarrow (2), (S) \rightarrow (5)$$

$$(B) \quad (P) \rightarrow (3), (Q) \rightarrow (4), (R) \rightarrow (2), (S) \rightarrow (6)$$

$$(C) \quad (P) \rightarrow (3), (Q) \rightarrow (4), (R) \rightarrow (1), (S) \rightarrow (5)$$

$$(D) \quad (P) \rightarrow (3), (Q) \rightarrow (1), (R) \rightarrow (2), (S) \rightarrow (6)$$

$$\text{Sol. : } \vec{b} \cdot \vec{c} = 1 - 1 + 1 = 1 \text{ and } \vec{a} \cdot \vec{c} = 1 + 1 - 1 = 1 = \vec{b} \cdot \vec{c}$$

$$\therefore (P) \rightarrow (3)$$

$$(Q) \quad [\vec{b} \ \vec{c} \ \vec{d}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1 - 1) - 1(-1 - 1) + 1(1 + 1) = 0 + 2 + 2 = 4$$

$$\therefore (Q) \rightarrow (4)$$

$$(R) \quad \vec{c} \cdot \vec{d} = 1 - 1 - 1 = -1 \text{ and } \vec{a} \cdot \vec{b} = 1 - 1 - 1 = -1$$

$$\therefore \vec{c} \cdot \vec{d} = \vec{a} \cdot \vec{b}$$

$$\therefore (R) \rightarrow (2)$$

(Here, it is clear that the answer is (B).)

$$\begin{aligned} \text{(S)} \quad \vec{c} \times \vec{d} &= (\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + \hat{j} - \hat{k}) \\ &= 2\hat{j} + 2\hat{k} \end{aligned}$$

$\therefore$ (S) $\rightarrow$ (6)

Ans. : (B)

Short method : (S) $\rightarrow$ (6)

(Q) $\rightarrow$ (4)

Ans. : (B)

(169) For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, match the concepts of **Part I** and **Part II** :

Part I		Part II	
(1)	$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$	(P)	$\vec{a} \perp \vec{b}$
(2)	If $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are unit vectors, then	(Q)	$ \vec{a} + \vec{b} = \vec{b} + \vec{a} $
(3)	$[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}]$	(R)	$\vec{a}, \vec{b}$ are unit vectors.
(4)	If $\vec{a}, \vec{b}$ are in the same direction, then	(S)	$[\vec{a} \quad \vec{b} \quad \vec{c}]$
		(T)	$[\vec{a} \quad \vec{b} \quad \vec{c}]^2$
		(U)	$\vec{a}$ and $\vec{c} - \vec{b}$ are collinear.
		(V)	$\vec{a}, \vec{b}, \vec{c}$ are collinear.

(A) (1) $\rightarrow$ (S), (2) $\rightarrow$ (Q), (3) $\rightarrow$ (T), (4) $\rightarrow$ (R)

(B) (1) $\rightarrow$ (U), (2) $\rightarrow$ (R), (3) $\rightarrow$ (S), (4) $\rightarrow$ (P)

(C) (1) $\rightarrow$ (U), (2) $\rightarrow$ (P), (3) $\rightarrow$ (V), (4) $\rightarrow$ (Q)

(D) (1) $\rightarrow$ (U), (2) $\rightarrow$ (P), (3) $\rightarrow$ (T), (4) $\rightarrow$ (Q)

Sol. : (1) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$$\therefore \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$$

$$\therefore \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$

$\therefore \vec{a}$ and $\vec{b} - \vec{c}$ are collinear.

$\therefore$ (1) $\rightarrow$ (U)

$$(2) |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$$

$$\therefore 1 + |\vec{a} \cdot \vec{b}|^2 = 1$$

$$\therefore \vec{a} \cdot \vec{b} = 0$$

$$\therefore \vec{a} \perp \vec{b}$$

$\therefore$ (2) $\rightarrow$ (P)

$$(3) [\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$

$$\therefore (3) \rightarrow (T)$$

(Here, it is decided that the answer is (D).)

$$(4) \vec{a} \text{ and } \vec{b} \text{ are vectors in the same direction.}$$

$$\therefore \vec{a} = k\vec{b}, k > 0$$

$$\text{Now, } |\vec{a} + \vec{b}| = |k\vec{b} + \vec{b}|$$

$$= |(k+1)| |\vec{b}|$$

$$= k|\vec{b}| + |\vec{b}|$$

$$(k > 0)$$

$$= |\vec{a}| + |\vec{b}|$$

$$(\text{Taking } k > 0)$$

$$\therefore (4) \rightarrow (Q)$$

Note : On finding first (2) and then (3), answer (D) is decided.

Ans. : (D)

Questions Based on Information :

- (170) The vertices of $\triangle ABC$ are $A(2, 0, 2)$, $B(1, 1, -1)$ and $C(4, -2, 1)$. The point D divides $\overline{AB}$ from A in the ratio 1:2 and E divides $\overline{AC}$ from C in the ratio 1:2. F is a point in space. The point of intersection of $\overleftrightarrow{CD}$ and $\overleftrightarrow{BE}$ is P. The distance of F from the plane of $\triangle ABC$ is $3\sqrt{2}$.

From the above given data, answer the following questions :

- (1) Position vector of P is

$$(A) 3\hat{i} + \hat{j} + \hat{k} \quad (B) -3\hat{i} + \hat{j} + \hat{k} \quad (C) 3\hat{i} - \hat{j} + \hat{k} \quad (D) -3\hat{i} - \hat{j} + 2\hat{k}$$

- (2) Volume of tetrahedron ABCF

$$(A) 7 \text{ cubic units} \quad (B) \frac{7}{5} \text{ cubic units} \quad (C) \frac{7}{3} \text{ cubic units} \quad (D) \frac{3}{5} \text{ cubic units}$$

Sol. : (1) D divides $\overline{AB}$ from A in the ratio 1:2.

$$\therefore D\left(\frac{5}{3}, \frac{1}{3}, 1\right)$$

$\therefore$ The equation of $\overleftrightarrow{CD}$, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, $\lambda \in \mathbb{R}$ (The equation of a line in space)

$$\begin{aligned} \vec{r} &= 4\hat{i} - 2\hat{j} + \hat{k} + \lambda\left(\left(\frac{5}{3}-4\right)\hat{i} + \left(\frac{1}{3}+2\right)\hat{j} + (1-1)\hat{k}\right) \\ &= 4\hat{i} - 2\hat{j} + \hat{k} + \frac{\lambda}{3}(-7\hat{i} + 7\hat{j}), \end{aligned}$$

E divides $\overline{AC}$ from C in the ratio 1:2.

$$\therefore E\left(\frac{10}{3}, \frac{-4}{3}, \frac{4}{3}\right)$$

$\therefore$ The equation of $\overleftrightarrow{BE}$ $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \mu\left(\left(\frac{10}{3}-1\right)\hat{i} + \left(\frac{-4}{3}-1\right)\hat{j} + \left(\frac{4}{3}+1\right)\hat{k}\right)$, $\mu \in \mathbb{R}$

$$= \hat{i} + \hat{j} - \hat{k} + \frac{\mu}{3}(7\hat{i} - 7\hat{j} + 7\hat{k}),$$

To obtain the point of intersection of $\overleftrightarrow{CD}$ and $\overleftrightarrow{BE}$, suppose,

the point $\left(4 - \frac{7\lambda}{3}, -2 + \frac{7\lambda}{3}, 1\right)$ of $\overleftrightarrow{CD}$ is on $\overleftrightarrow{BE}$.

$$\therefore \left(4 - \frac{7\lambda}{3}, -2 + \frac{7\lambda}{3}, 1\right) = \left(1 + \frac{7\mu}{3}, 1 - \frac{7\mu}{3}, -1 + \frac{7\mu}{3}\right)$$

$$\therefore 1 = -1 + \frac{7\mu}{3} \quad \text{and} \quad 4 - \frac{7\lambda}{3} = 1 + \frac{7\mu}{3}$$

$$\therefore \frac{6}{7} = \mu$$

$$\therefore 3 - \frac{7}{3}\left(\frac{6}{7}\right) = \frac{7\lambda}{3}$$

$$\therefore \frac{3}{7} = \lambda$$

$\therefore$ The point of intersection is $P(3, -1, 1)$

$\therefore$ The position vector of P is $3\hat{i} - \hat{j} + \hat{k}$

Ans. : (C)

(2) Volume of tetrahedron ABCF = $\frac{1}{3}$ (Area of base $\Delta ABC \times$ Altitude from F to ΔABC)

$$\begin{aligned} \text{Now, the area of } \Delta ABC &= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| \\ &= \frac{1}{2} |(-\hat{i} + \hat{j} - 3\hat{k}) \times (2\hat{i} - 2\hat{j} - \hat{k})| \\ &= \frac{1}{2} |-7\hat{i} - 7\hat{j}| \\ &= \frac{7\sqrt{2}}{2} \text{ square units} \end{aligned}$$

$$\begin{aligned} \therefore \text{Volume of tetrahedron} &= \frac{1}{3} \left(\frac{7\sqrt{2}}{2} \times 3\sqrt{2} \right) \\ &= 7 \text{ cubic units} \end{aligned}$$

Ans. : (A)

(171) If $\vec{b} \neq \vec{0}$, $\vec{a}$ can be written as $\vec{a} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1$ is parallel to $\vec{b}$ and $\vec{b}_2$ is perpendicular to $\vec{b}$.

$$\vec{b}_1 = \lambda \vec{b}, \lambda \neq 0$$

$$\therefore \vec{a} \cdot \vec{b} = \vec{b}_1 \cdot \vec{b} + \vec{b}_2 \cdot \vec{b} = \lambda |\vec{b}|^2 \quad \text{because } \vec{b}_2 \perp \vec{b}$$

$$\therefore \lambda = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}, \quad (\vec{b}_1 \text{ is projection of } \vec{a} \text{ on } \vec{b})$$

$$\text{Proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} \quad \text{or} \quad \text{Proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right) \frac{\vec{b}}{|\vec{b}|} = (\vec{a} \cdot \hat{b}) \hat{b}$$

Now, $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ is the component of $\vec{a}$ in the direction of $\vec{b}$ i.e. $\text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \vec{a} \cdot \hat{b}$

If $\vec{a} = k\vec{b}$, then $\vec{a} = b_1\vec{b}$ and $b_2 = 0$ and if $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} = b_2\vec{b}$ and $b_1 = 0$.

Answer the following questions, from the above data :

(1) If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = -4\hat{i} - 2\hat{j} + 4\hat{k}$, then $\text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \vec{a} \cdot \hat{b} = \dots\dots$

- (A) 1 (B) -1 (C) 0 (D) $-\frac{1}{6}$

(2) If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{k}$, then $\text{Proj}_{\vec{b}} \vec{a} = \dots\dots$

- (A) $\frac{1}{25}(4\hat{i} + 3\hat{k})$ (B) $\frac{1}{5}(4\hat{i} + 3\hat{k})$ (C) $-\frac{1}{5}(4\hat{i} + 3\hat{k})$ (D) $-\frac{1}{25}(4\hat{i} + 3\hat{k})$

(3) If $\vec{a} = 4\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{a} = \vec{b}_1 + \vec{b}_2$, then $\vec{b}_2 = \dots\dots$

- (A) $\frac{1}{3}(10\hat{i} - \hat{j} + 11\hat{k})$ (B) $\frac{2}{3}(\hat{i} + \hat{j} - \hat{k})$
(C) $\frac{1}{3}(10\hat{i} + \hat{j} + 11\hat{k})$ (D) $\frac{1}{3}(14\hat{i} + 5\hat{j} + 7\hat{k})$

Sol. : (1) $\text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-8-2+4}{\sqrt{16+4+16}} = \frac{-6}{6} = -1$

Ans. : (B)

(2) $\text{Proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} = \left(\frac{8-3}{16+9} \right) (4\hat{i} + 3\hat{k}) = \frac{1}{5}(4\hat{i} + 3\hat{k})$

Ans. : (B)

(3) $\vec{a} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1$ is the projection of $\vec{a}$ on $\vec{b}$.

$\therefore \vec{b}_1 = \text{Proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} = \frac{4+1-3}{(1+1+1)} (\hat{i} + \hat{j} - \hat{k}) = \frac{2}{3}(\hat{i} + \hat{j} - \hat{k})$

Now, $\vec{b}_2 = \vec{a} - \vec{b}_1$

$$\begin{aligned} &= 4\hat{i} + \hat{j} + 3\hat{k} - \frac{2}{3}(\hat{i} + \hat{j} - \hat{k}) \\ &= \frac{10}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{11}{3}\hat{k} \\ &= \frac{1}{3}(10\hat{i} + \hat{j} + 11\hat{k}) \end{aligned}$$

Ans. : (C)

(172) A(2, 1, -2), B(0, 1, 4) and C(1, -4, 1) are non-collinear points on a plane. N is the foot of perpendicular from B to $\overline{AC}$. Answer the following :

(1) Position vector of N

(A) $\frac{1}{7}(10\hat{i} + 13\hat{j} + 2\hat{k})$

(B) $\frac{1}{7}(10\hat{i} - 13\hat{j} - 2\hat{k})$

(C) $\frac{1}{7}(10\hat{i} - 13\hat{j} + 2\hat{k})$

(D) $\frac{1}{7}(10\hat{i} + 13\hat{j} + 2\hat{k})$

(2) Area of ΔBNA =

(A) $\frac{150\sqrt{6}}{49}$

(B) $\frac{20}{7\sqrt{10}}$

(C) $\frac{20\sqrt{10}}{7}$

(D) $\frac{20\sqrt{6}}{7}$

(3) $\cos(\angle NBC) = \dots\dots$

(A) $\frac{\sqrt{10}}{7}$

(B) $\frac{2}{7}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\sqrt{10}}{7}$

Sol. : (1) Suppose N divides $\overline{AC}$ from A in the ratio λ .

$\therefore$ Coordinates of N $\left(\frac{\lambda+2}{\lambda+1}, \frac{-4\lambda+1}{\lambda+1}, \frac{\lambda-2}{\lambda+1}\right)$

$\overrightarrow{BN} = \left(\frac{\lambda+2}{\lambda+1}, \frac{-5\lambda}{\lambda+1}, \frac{-3\lambda-6}{\lambda+1}\right)$

$\overrightarrow{BN} \perp \overrightarrow{AC}$

$\therefore \overrightarrow{BN} \cdot \overrightarrow{AC} = 0$

$\therefore \left(\frac{\lambda+2}{\lambda+1}, \frac{-5\lambda}{\lambda+1}, \frac{-3\lambda-6}{\lambda+1}\right) \cdot (-1, -5, 3) = 0$

$\therefore -\lambda - 2 + 25\lambda - 9\lambda - 18 = 0$

$\therefore 15\lambda = 20$

$\therefore \lambda = \frac{4}{3}$

$\therefore$ The position vector of N is $\frac{1}{7}(10\hat{i} - 13\hat{j} - 2\hat{k})$

Ans. : (B)

Sol. : (2) Area of $\Delta BNA = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BN}|$

$= \frac{1}{2}|(-2\hat{i} + 6\hat{k}) \times \frac{1}{7}(10\hat{i} - 20\hat{j} - 30\hat{k})|$

$= \frac{20}{14}|(-\hat{i} + 3\hat{k}) \times (\hat{i} - 2\hat{j} - 3\hat{k})|$

$= \frac{10}{7}|(6\hat{i} + 2\hat{k})|$

$= \frac{20\sqrt{10}}{7}$ square units

Ans. : (C)

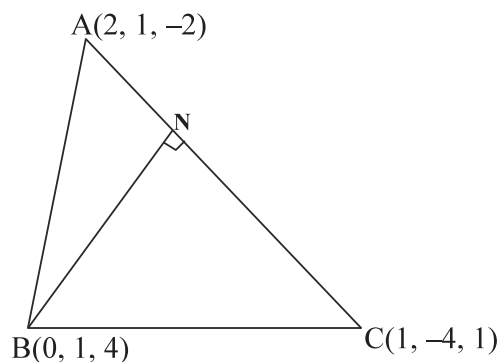


Figure 20.22

$$\begin{aligned}\text{Sol. : (3) } \cos \angle NBC &= \frac{\vec{BN} \cdot \vec{BC}}{|\vec{BN}| |\vec{BC}|} \\ &= \frac{\frac{1}{7}(10\hat{i} - 20\hat{j} - 30\hat{k}) \cdot (\hat{i} - 5\hat{j} - 3\hat{k})}{\frac{10}{7}\sqrt{1+4+9} \cdot \sqrt{1+25+9}} = \frac{20}{7\sqrt{2}\sqrt{5}} = \frac{2\sqrt{10}}{7}\end{aligned}$$

Ans. : (D)

(173) Suppose $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, $\vec{b} = -3\hat{i} + 6\hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 6\hat{j} - 2\hat{k}$. $\vec{x}$ is the projection of $\vec{a}$ on $\vec{b}$ and $\vec{y}$ is the projection of $\vec{x}$ on $\vec{c}$. Answer the following :

(1) $\vec{y} = \dots\dots$

(A) $\frac{943}{49}(3\hat{i} + 6\hat{j} - 2\hat{k})$

(B) $\frac{943}{(49)^2}(3\hat{i} + 6\hat{j} - 2\hat{k})$

(C) $\frac{-943}{49}(3\hat{i} + 6\hat{j} - 2\hat{k})$

(D) $\frac{-943}{(49)^2}(3\hat{i} + 6\hat{j} - 2\hat{k})$

(2) $\vec{x} \cdot \vec{b} = \dots\dots$

(A) $\frac{-41}{7}$

(B) -41

(C) $\frac{41}{7}$

(D) 41

(3) Which of the following options is correct ?

(A) $\vec{a}$, $\vec{x}$ and $\vec{b}$ are coplanar.

(B) $\vec{a}$ and $\vec{y}$ are collinear.

(C) $\vec{a}$, $\vec{x}$ and $\vec{y}$ are coplanar.

(D) $\vec{x}$ and $\vec{c}$ are collinear.

$$\text{Sol. : (1) } \vec{x} = \text{Proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} = \frac{-41}{49}(-3\hat{i} + 6\hat{j} + 2\hat{k})$$

$$\vec{y} = \text{Proj}_{\vec{c}} \vec{x} = \left(\frac{\vec{x} \cdot \vec{c}}{|\vec{c}|^2} \right) \vec{c} = \frac{-41}{49} \left(\frac{23}{49} \right) (3\hat{i} + 6\hat{j} - 2\hat{k}) = \frac{-943}{(49)^2} (3\hat{i} + 6\hat{j} - 2\hat{k})$$

Ans. : (D)

$$(2) \vec{x} \cdot \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} (\vec{b} \cdot \vec{b}) = \vec{a} \cdot \vec{b} = -41$$

Ans. : (B)

(3) Since $\vec{x}$ and $\vec{b}$ are collinear, $\vec{a}$, $\vec{x}$, $\vec{b}$ are coplanar.

No two of $\vec{a}$, $\vec{x}$ and $\vec{y}$ are collinear and $[\vec{a} \ \vec{x} \ \vec{y}] \neq 0$

$[\vec{a} \ \vec{x} \ \vec{y}] = \text{non-zero constant.}$

$$[\vec{a} \ \vec{x} \ \vec{y}] = \left(\frac{-41}{49} \right) \left(\frac{-943}{(49)^3} \right) 36 \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \neq 0$$

$\therefore$ Only option (A) is true.

Ans. : (A)