2023

Sample Questions

Junior Research Fellowship (JRF) in Computer Science (CS)

Test Code: CSB (Afternoon Examination)

Note that all questions in the sample set are not at the same level of difficulty, and may not carry equal marks in an examination.

Computer Science

C1. Let A be an array containing n distinct integers, and w be a positive integer. Your task is to select the *maximum number of elements* of A such that the sum of the selected elements is less than or equal to w. Devise $O(n)$ time algorithms (with justification) for this task for the following two cases:

(a) the array A is sorted;

(b) the array A is unsorted.

HINT: For case (b), you may consider a divide-and-conquer type algorithm, and use the fact that the median of an array of n elements can be found in $O(n)$ time.

C2. Suppose a lockdown has been announced at your place. You plan to spend the very first day of the lockdown by watching movies on a TV with no recording facility. You have received a schedule of movies to appear on all the channels on the said day. You do not like to switch over from one movie to another, and your aim is to watch as many complete movies as possible. Suppose there are n distinct movies for which the corresponding start and end times are available across all the channels. Write an efficient program (using pseudo-code) to find the maximum number of complete movies that you can watch on the day. Analyze the time complexity of your algorithm.

C3. Consider a hard disk whose storage space S is divided into a number of logical sectors (segments) of equal length l. A batch of requests for storing data into it is served in such a way that no piece of data is fragmented to accommodate the same over more than one sectors under the assumption that the length of each piece of data is less than the sector size and a single sector can accommodate multiple pieces of data. Write an algorithm to store a given set of m successive storage requests $\{s_1, s_2, ..., s_m\}$ using the minimum number of sectors, where s_i is the size of i-th data request, $s_i < l$, for each i and $\sum_i s_i < S$. For example, consider the set of data storage requests {2kb, 5kb, 4kb, 7kb, 1kb, 3kb, 8kb} into a hard disk of sector length 10kb. The minimum number of sectors required for the storage is 3 and a possible distribution of the data pieces into 3 sectors are {2kb, 8kb}, {4kb, 1kb, 5kb} and {7kb, 3kb}.

C4. Consider the task of multiplying two integer matrices A and B, each of size 500×500. Each matrix can have at most 500 non-zero entries. Any row of A or B that contains at least one non-zero element, must have no less than 50 non-zero elements.

(a) Propose a space-efficient data structure for this task.

(b) Based on your proposed data structure, design an efficient algorithm to multiply A and B. The resulting matrix should be stored separately.

(c) What is the maximum number of integer multiplications needed to complete this task as per your implementation?

C5. Consider a set A containing n positive integers $a_0, a_1, \ldots, a_{n-1}$ such that $a_i > \sum_{j=0}^{i-1}\big((j+1)a_j\big)$ for $i = 1, 2, \ldots, n-1$. You are also given a target positive integer T.

(a) Write an efficient algorithm to determine whether there is any subset of A, such that the sum of the elements in this subset is T.

(b) Analyse the worst case time complexity of your algorithm.

C6. Let $S = \{1, 2, \ldots, n\}$ and $f : S \to S$. If $R \subseteq S$, define $f(R) = \{f(x) \mid x \in R\}$.

(a) Suppose you are given the values of $f(1), f(2), \ldots, f(n)$, in order. Propose a data structure for storing these values so that, for a given x, the values of $f(x)$ and $f^{-1}(x) = \{y \mid f(y) = x\}$ can be determined efficiently.

(b) Suppose that we call a set $R \subseteq S$ "*special*" if $f(R) = R$. For $n = 3$, give an example of a function f and subsets $R_i \subseteq S$ such that R_i is special, and $|R_i| = i$ for $i = 1, 2, 3$.

(c) Using your data structure from (a), design an efficient algorithm to find a set $R \subseteq S$ that is *special* (as defined in (b)). You will get full credit only if your algorithm takes $O(n)$ time in the worst case.

C7. There are n students standing in a line. The students have to rearrange themselves in ascending order of their roll numbers. This rearrangement must be accomplished only by successive swapping of adjacent students.

(a) Design an algorithm for this purpose that minimises the number of swaps required.

(b) Derive an expression for the number of swaps needed by your algorithm in the worst case.

C8. Let $S = \{x_1, x_2, \ldots x_n\}$ be a set of n integers. A pair (x_i, x_j) (where $i \neq j$) is said to be the closest pair if $|x_i - x_j| \leq |x_{i'} - x_{j'}|$, for all possible pairs $(x_{i'}, x_{j'})$, $i', j' = 1, 2, \ldots, n, i' \neq j'$.

(a) Describe a method for finding the closest pair among the set of integers in S using $O(n \log_2 n)$ comparisons.

(b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S, the current closest pair can be reported in $O(\log_2 n)$ time.

(c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.

C9. Let A be an $n \times n$ matrix of integers whose element in the i-th row and j-th column is denoted by a_{ij}. Further, we shall denote by μ_i the maximum value that appears in the i-th row, i.e. $\mu(i) = \max\{a_{ij} \colon j \in \{1, 2, \ldots, n\}\}$. Suppose that for any i, j such that $i < j$, if $a_{ik} = \mu(i)$

and $a_{jl} = \mu(j)$, then $k \leq l$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

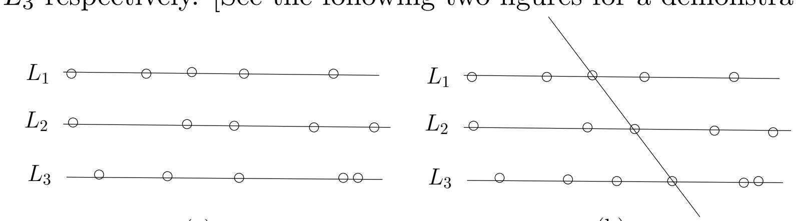
(a) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n \log n)$.

(b) Establish its correctness, and justify the time complexity of the proposed algorithm.

C10. Let M be an $(n \times n)$ matrix where each element is a distinct positive integer. Construct another matrix M' by permuting the rows and/or permuting the columns, such that the elements of at least one row appear in *increasing* order (while looking from left to right) and those of at least one column appear in *decreasing* order (while looking from top to bottom). Describe an $O(n^2)$ time algorithm for constructing M'. Justify your analysis.

C11. Consider three parallel lines L_1, L_2 and L_3. On each line L_i, a set of n points $\{p_{i1}, p_{i2}, \ldots, p_{in}\}$ is placed.

The objective is to identify a triplet of indices (k, ℓ, m) (if exists) such that a straight line can pass through p_{1k}, $p_{2\ell}$ and p_{3m} on L_1, L_2 and L_3 respectively. [See the following two figures for a demonstration.]



(a) (b)

In Figure (a), there does not exist any triplet (k, ℓ, m) such that a straight line can pass through $p_{1k}, p_{2\ell}$ and p_{3m}. In Figure (b), the triplet $(3, 3, 4)$ is a solution since a straight line passes through p_{13}, p_{23} and p_{34}.

Present an efficient algorithm for solving this problem. Justify its correctness and worst case time complexity. It can be assumed that the x, y-coordinates of the points are given as input.

[Full credit will be given if your algorithm is correct, and of worst case time complexity $O(n^2)$.]

C12. A least expensive connection route among n houses needs to be designed for a cable-TV network. Consider the following algorithm $\mathcal{A}$ for finding a spanning tree.

```
Algorithm A
Input: G = (V, E)
Output: Set of edges M ⊆ E
Sort E in decreasing order of cost of edges;
i ← 0;
while i < |E| do
begin
    Let temp = (u1, u2) be the i-th edge E[i] in E;
    Delete E[i], i.e., replace E[i] by φ;
    if u1 is disconnected from u2 then
        restore temp in list E as E[i];
    i ← i + 1;
end
return the edges in E which are not φ;
```

(a) Prove that the algorithm $\mathcal{A}$ can be used to correctly find a least cost connection route, given a set of n houses and information about the cost of connecting any pair of houses.

(b) What is the worst case time complexity of $\mathcal{A}$?

(c) If all the edges have distinct cost, how many solutions can there be?

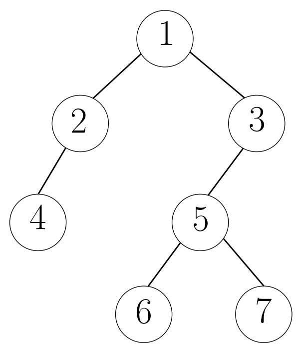
C13. You are given k sorted lists, each containing m integers in ascending order. Assume that (i) the lists are stored as singly-linked lists with one integer in each node, and (ii) the head pointers of these lists are stored in an array.

(a) Write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(k)$ additional storage.

(b) Next, write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(1)$ additional storage.

(c) Analyse the time complexity of your algorithm for each of the above two cases.

C14. Let $\mathcal{B}$ be a rooted binary tree of n nodes. Two nodes of $\mathcal{B}$ are said to be a sibling pair if they are the children of the same parent. For example, given the binary tree in the figure below, the sibling pairs are (2, 3) and (6, 7). Design an $O(n)$ time algorithm that prints all the sibling pairs of $\mathcal{B}$.



C15. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complete binary trees that are heaps as well. Assume $\mathcal{H}_1$ and $\mathcal{H}_2$ are max-heaps, each of size n. Design and analyze an efficient algorithm to merge $\mathcal{H}_1$ and $\mathcal{H}_2$ to a new max-heap $\mathcal{H}$ of size $2n$.

C16. Let A and B be two arrays, each containing n distinct integers. Each of them is sorted in increasing order. Let $C = A \cup B$. Design an algorithm for computing the median of C as efficiently as you can.

C17. Let $G = (V, E)$ be an undirected weighted graph with all edge weights being positive. Design an efficient algorithm to find the *maximum* spanning tree of G.

C18. The diameter of a tree $T = (V, E)$ is given by $\max_{u,v \in V}\{\delta(u, v)\}$, where $\delta(u, v)$ is the shortest path distance (i.e., the length of the shortest path) between vertices u and v. So, the diameter is the largest of all shortest path distances in the tree.

(a) Write pseudo-code for an efficient algorithm to compute the diameter of a given tree T.

(b) Analyze the time complexity of your algorithm.

(c) What is its space complexity?

(d) Clearly mention the data structure(s) used by your algorithm.

(e) A vertex c is called a center of a tree T if the distance from c to its most distant vertex is the minimum among all vertices in V. Write an algorithm to determine and report a center of the given tree T.

C19. A connected, simple, undirected planar graph $G(V, E)$ is given where V denotes the set of vertices and E denotes the set of edges. In V, there is a designated source vertex s and a designated destination vertex t. Let $P(v)$ denote the shortest path (may contain repetition of nodes/edges) from s to t that passes through v, and let $l(v)$ denote the path length (i.e., the number of edges) of $P(v)$.

(a) Describe an $O(|V|)$ time algorithm that determines the value of τ where $\tau = \max_{\forall v \in V} l(v)$. Justify your analysis.

(b) Propose a data structure that supports your algorithm.

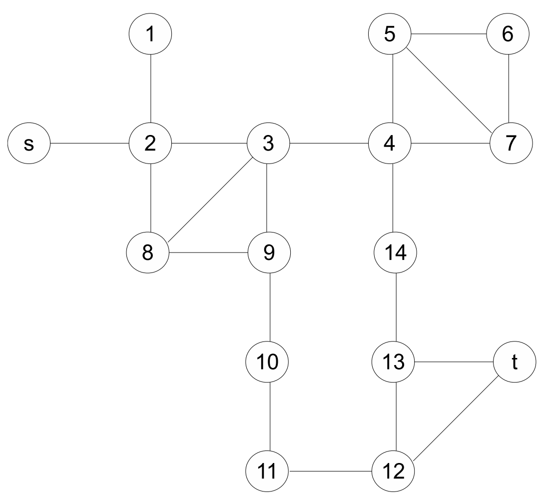


Figure 1: The example graph.

[For example, in the graph shown in Figure 1, $\tau = 10$, which corresponds to $P(6) : s \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 6 \rightarrow 7 \rightarrow 4 \rightarrow 14 \rightarrow 13 \rightarrow t$.]

C20. A vertex cover of a graph $G = (V, E)$ is a set of vertices $V' \subseteq V$ such that for any edge $(u, v) \in E$, either u or v (or both) is in V'. Write an efficient algorithm to find the minimum vertex cover of a given tree T. Establish its correctness. Analyse its time complexity.

C21. (a) Assume you have a chocolate bar containing a number of small identical squares arranged in a rectangular pattern. Our job is to split the bar up until each piece is one of the small squares by breaking along the lines between the squares. We obviously want to do it with the minimum number of breakings. How many breakings will it take?

(b) Consider that the chocolate bar has n breaking lines along the length and m breaking lines along the breadth. Write the pseudocode of an algorithm that will take n, m as inputs and print the line numbers of the lines along which your strategy does the breaking of the chocolate.

C22. Let $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 2$.

(a) Express 63 as a sum of distinct a_i's.

(b) Write an algorithm to express any positive integer k as a sum of at most $\lceil \log_2 k \rceil$ many distinct a_i's.

(c) Prove the correctness of your algorithm.

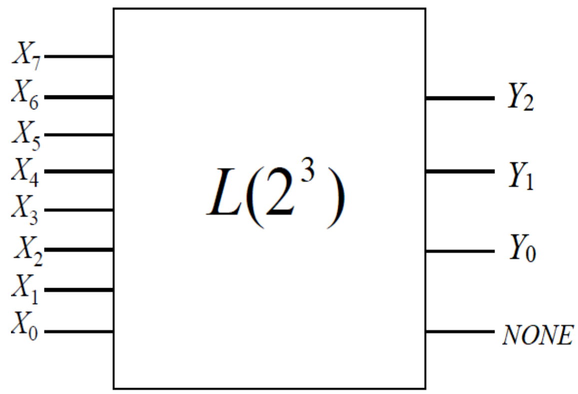
C23. A Boolean logic circuit $L(2^N)$ takes a 2^N-bit binary number $X = X_{2^N-1}X_{2^N-2}\dots X_0$ as its input, and produces an N-bit binary output $Y = Y_{N-1}Y_{N-2}\dots Y_0$ indicating the most significant bit of X that is TRUE. In case none of the inputs are TRUE, it produces an output $NONE = 1$.

For example, if $N = 2$,

(a) for $X = 0110$, output $Y = 10$ and $NONE = 0$, and

(b) for $X = 0000$, output $Y =$ Don't care and $NONE = 1$.

Consider the block diagram of $L(2^3)$ shown in the figure below. Give a simplified Boolean equation for each output and draw the circuit using the minimum number of gates. Use gates with 4 or less inputs.



C24. Consider a hypothetical gate $G(A, B, C)$ for which the Karnaugh map is given in the following figure.

C \ AB	00	01	11	10
0	0	0	1	0
1	0	1	0	1

Figure 2: Karnaugh map for the hypothetical gate G.

(a) Implement the NOT, 2-input AND, and 2-input OR operations using minimum number of G gates. Assume that the logical value 1 is available as an input.

(b) Consider the switching function $f(w, x, y, z)$ defined as follows

$$f(w, x, y, z) = \sum(0, 1, 2, 4, 7, 8, 9, 10, 12, 15).$$

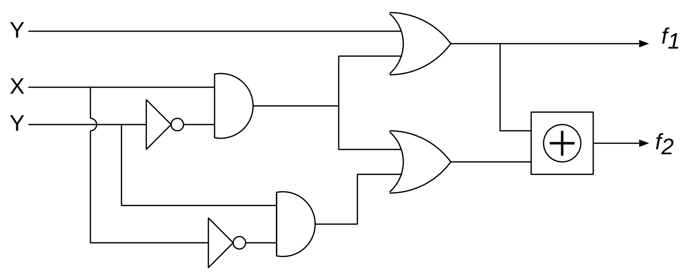
Realize f by means of minimum number of G gates.

C25. Suppose instead of a decoder with n input bits (n is even) to access a memory of size 2^n, one uses two decoders of input sizes k bits and $(n-k)$ bits. Explain how these two decoders can be used to access the memory of size 2^n.

Determine the value of k that achieves minimum address decoding time. Justify your answer. Assume that the time complexity of the decoder is measured by the number of output lines of that decoder.

Explain with a diagram how to solve the aforementioned decoding problem if $n = 17$.

C26. A Boolean function f is said to be positive unate if f can be expressed in a form where all the variables appear in uncomplimented form, and only the AND and OR Boolean operators are used. For example, the function $g_1 = X_1X_2 + X_2X_3$ is positive unate but $g_2 = X_1X_2 + \bar{X}_2X_3$ is not. Consider the following circuit and determine which of the two functions f_1 and f_2 are positive unate.



C27. Let C denote a logic block that is capable of comparing two 4-bit 2's complement numbers A (a_3, a_2, a_1, a_0) and B (b_3, b_2, b_1, b_0), where $a_i, b_i \in \{0, 1\}$ for $i = 0, 1, 2, 3$. The circuit C has eight input lines $a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0$, and three output lines E, L, G (Equal: E; Less than: L; Greater than: G). For example, if $A > B$, then the outputs should be $E = 0, L = 0$, and $G = 1$.

Write the Boolean equations for the three outputs E, L, and G.

C28. Let us consider a household, as shown in the following figure, with two lights. One of these lights is at porch gate and the other is inside the room. The lights are controlled by three switches A, B and C; out of which two are inside the house and one is outside the house. Both the lights are OFF when all the switches are OFF. Both the lights are ON when all the three switches are ON. If any two switches are ON then the porch light is ON. If exactly one of A, B or C is ON, the light inside the room is ON. Design a circuit using only NAND gates to realize the above scenario. Justify whether this circuit involves the minimum number of NAND gates.

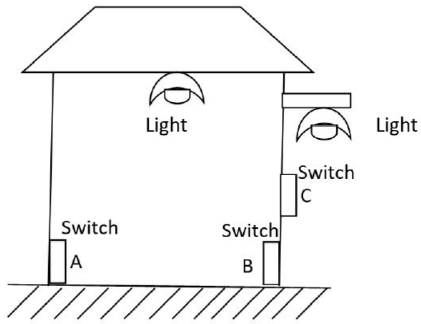


Figure 3: A household with a pair of lights.

C29. Let α be the percentage of a program code which can be executed simultaneously by n processors in a computer system. Assume that the remaining code must be executed sequentially by a single processor. Each processor has an execution rate of x MIPS, and all the processors are of equal capability.

(a) Derive an expression for the effective MIPS rate when the code is executed using n processors, in terms of n, α and x.

(b) Given $n = 16$ and $x = 4$ MIPS, determine the value of α to yield a system performance of 40 MIPS.

C30. (a) Write the smallest real number greater than 6.25 that can be represented in the IEEE-754 single precision format (32-bit word with 1 sign bit and 8-bit exponent).

(b) Convert the sign-magnitude number **10011011** into a 16-bit 2's complement binary number.

(c) The CPU of a machine is requesting the following sequence of memory accesses given as word addresses: 1, 4, 8, 5, 20, 17, 19, 56. Assuming a direct-mapped cache with 8 one-word blocks, that is initially empty, trace the content of the cache for the above sequence.

C31. A machine $\mathcal{M}$ has the following *five* pipeline stages; their respective time requirements in nanoseconds (ns) are given within parentheses:
F-stage — instruction fetch (9 ns),
D-stage — instruction decode and register fetch (3 ns),
X-stage — execute/address calculation (7 ns),
M-stage — memory access (9 ns),
W-stage — write back to a register (2 ns).
Assume that for each stage, the pipeline overhead is 1 ns. A program P having 100 machine instructions runs on $\mathcal{M}$, where every 3^{rd} instruction needs a 1-cycle stall before the X-stage. Calculate the CPU time in micro-seconds for completing P.

C32. The CPU of a computer has a ripple-carry implementation of a 2's complement adder that takes two 8-bit integers $A = a_7a_6 \ldots a_0$ and $B = b_7b_6 \ldots b_0$ as inputs, and produces a sum $S = s_7s_6 \ldots s_0$, where $a_i,\ b_i,\ c_i \in \{0, 1\}$ for $(0 \leq i \leq 7)$.

Let $A = 1001\ 1001$ and $B = 1000\ 0110$. What will be the output S of the adder? How will the value of S be interpreted by the machine?

C33. Add the following two floating point numbers A and B given in IEEE-754 single precision format and show the sum S in the same format.

A: 0000011000100 0000 000000000000001
B: 1000011000100 0000 000000000000001

C34. Draw a complete binary tree T with $(N - 1)$ nodes where $N = 2^n$. Suppose each node in T is a processor and each edge of T is a physical link between two processors through which they can communicate. Given M arrays $A_i = \{e_{1i}, e_{2i}, \ldots, e_{Ni}\}$ for $1 \leq i \leq M$, develop an algorithm for the given architecture to compute the sum of each array $SUM_i = \sum_{j=1}^{N} e_{ji}$ for all i in $O(\log N + M)$ time.

C35. Consider a machine with four registers (one of which is the accumulator `A`) and the following instruction set.

- `LOAD R` and `STORE R` are indirect memory operations that load and store using the address stored in the register `R`. Thus, `LOAD R` loads the contents of memory[R] into `A` and `STORE R` stores the contents of `A` in memory[R].

- `MOV R1 R2` copies the contents of register `R1` into register `R2`.
- `ADD R` and `SUB R` operate on the accumulator and one other register `R`, such that `A = A op R`.
- `LDC n` stores the 7-bit constant `n` in the accumulator.
- `BRA`, `BZ`, and `BNE` are branch instructions, each taking a 5-bit offset.

Design an instruction encoding scheme that allows each of the above instructions (along with operands) to be encoded in 8 bits.

C36. The access time of a cache memory is 100 ns and that of main memory is 1000 ns. It is estimated that 80% of the memory requests are for read and the remaining 20% are for write. The hit ratio for read access is 0:9. A write through procedure is used.

(a) What is the average access time of the system considering only memory read cycles?

(b) What is the average access time of the system considering both read and write requests?

C37. A program P consisting of 1000 instructions is run on a machine at 1 *GHz* clock frequency. The fraction of floating point (FP) instructions is 25%. The average number of clock-cycles per instruction (CPI) for FP operations is 4.0 and that for all other instructions is 1.0.

(a) Calculate the average CPI for the overall program P.

(b) Compute the execution time needed by P in seconds.

C38. The average memory access time for a microprocessor with first level cache is 3 clock cycles.

- If data is present in the cache, it is found in 1 clock cycle.
- If data is not found in the cache, 100 clock cycles are needed to get it from off-chip memory.

It is desired to obtain a 50% improvement in average memory access time by adding a second level cache.

- This second level cache can be accessed in 6 clock cycles.
- The addition of this second level cache does not affect the first level cache.

- Off-chip memory accesses still require 100 clock cycles.

To obtain the desired speedup, how often must data be found in the second level cache?

C39. Two modules M_1 and M_2 of an old machine are being replaced by their improved versions M_3 and M_4, respectively in a new machine. With respect to the old machine, the speed-up of these modules (M_3 and M_4) are 30 and 20, respectively. Only one module is usable at any instant of time. A program P, when run on the old machine, uses M_1 and M_2 for 30% and 20% of the total execution time, respectively. Calculate the overall speed-up of P when it is executed on the new machine.

C40. The page size in a system is increased while keeping everything else (including the total size of main memory) the same. For each of the following metrics below, explain whether its value is generally expected to increase, decrease, or not change as a result of this increase in page size:

(a) size of the page table of a process;
(b) TLB hit rate.

C41. The byte-addressable logical address space in a lightweight computing platform consists of 128 segments, each of capacity 32 pages, each of which can hold 4096 bytes. The physical memory (also byte-addressable) consists of 2^{14} page frames, each 4096 bytes long.

(a) Formulate the logical and physical address formats.
(b) If segment table entries and page table entries both occupy 4 bytes each, compute the size in bytes of (i) the segment table and (ii) page tables for any process.
(c) Assume that no swap space is available. If the kernel takes up 12×2^{20} bytes, and the rest of the physical memory may be used to hold processes, compute the maximum number of processes that the kernel can handle concurrently in the worst case.

C42. The following figure pictorially shows the layout of a file descriptor of a file system. It uses 4 direct, 1 indirect, and 1 doubly indirect block addresses. If the size of a disk block is 64 bytes and the size of each disk block address is 8 bytes, calculate the maximum possible file size this file system may have. Give the answer in the form of "$a \times 2^A$", where a (not divisible by 2) and A both denote integers.

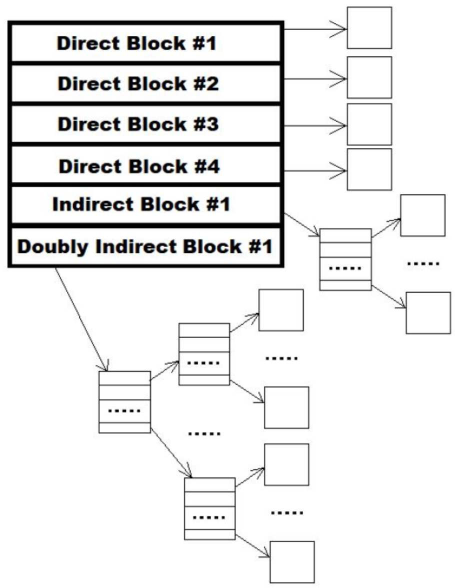


Figure 4: A file system with direct and indirect blocks.

C43. In a Buddy memory allocation system, a process is allocated an amount of memory whose size is the smallest power of 2 that is greater than or equal to the amount requested by the process.

A system using buddy memory allocation has 1MB memory. For a given sequence of nine processes, their respective memory requirements in KB are: 50, 150, 90, 130, 70, 80, 120, 180, 68.

(a) Illustrate with an allocation diagram to justify whether all the requests, in the given order, can be complied with. Assume that memory once allocated to a process is no longer available during the entire span of the above sequence.

(b) Calculate the total memory wasted due to fragmentation in your memory allocation by the above scheme.

C44. Two processes P_1 and P_2 have a common shared variable `count`. While P_1 increments it, P_2 decrements it. Given that `R0` is a register, the corresponding assembly language codes are:

P_1:	count++		P_2:	count-	
MOV	count	R0	MOV	count	R0
ADD	#1	R0	SUB	#1	R0
MOV	R0	count	MOV	R0	count

Give an example to justify whether a race condition may occur if P_1 and P_2 are executed concurrently.

C45. Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1, share the following variables:

```
char flag[2] = {0,0};
char turn = 0;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
do {
 flag[i] = 1;
 while (flag[j])
       if (turn == j) {
          flag[i] = 0;
          while (turn == j) {};
       }
 ...
 CRITICAL SECTION
 ...
 turn = j;
 flag[i] = 0;
 ...
 REMAINDER SECTION
 ...
} while (1);
```

Does this solution satisfy the mutual exclusion, progress and bounded waiting properties?

C46. Suppose that an operating system provides two functions, `block()` which puts the calling process on the blocked queue, and `wakeup(P)` which moves process P to the runnable queue if it is currently on the blocked queue (otherwise, its behaviour is unpredictable).

Consider two processes A and B running the code given below. The intended behaviour of the code is to have A and B run forever, alternately printing their names on the screen.

```
void A()                          void B()
{ while(1) {                      { while(1) {
    block();                          printf("B");
    printf("A");                      wakeup(A);
    wakeup(B);                        block();
  }                                 }
}                                 }
```

(a) Construct a scenario in which the intended behaviour would not be observed.

(b) Redesign the code using semaphore(s) so that it works correctly. You should show the initialisation of the semaphore(s), and the calls to `wait()` and `signal()` made by A and B.

C47. A system has 4 processes A, B, C, D and 5 allocatable resources R_1, R_2, R_3, R_4, R_5. The maximum resource requirement for each process and its current allocation are as follows.

Process	Maximum	Allocation
	R_1, R_2, R_3, R_4, R_5	R_1, R_2, R_3, R_4, R_5
A	1, 1, 2, 1, 3	1, 0, 2, 1, 1
B	2, 2, 2, 1, 0	2, 0, 1, 1, 0
C	2, 1, 3, 1, 0	1, 1, 0, 1, 0
D	1, 1, 2, 2, 1	1, 1, 1, 1, 0

Suppose the currently available count of resources is given by 0, 0, X, 1, 1. What is the minimum value of X for which this is a safe state? Justify your answer.

C48. Consider a uniprocessor system with four processes having the following arrival and burst times:

	Arrival Time	CPU Burst Time
P1	0	10
P2	1	3
P3	2.1	2
P4	3.1	1

(a) Calculate the average waiting time and also the average turnaround time if shortest (remaining) job first (SJF) scheduling policy is used with pre-emption. Assume that the context switching time is zero. Note that in SJF, if at any point there is a tie, then the job that arrived earlier will get priority.

(b) Now consider the continuous arrival of new jobs at times 4, 5, 6, 7, . . . following $P4$, with CPU burst times of 2 units each. In this case, what will be the turnaround time of $P1$? Justify your answer.

C49. Let us consider the relation schema <CTHRSG>, where the attributes are C ≡ Course, T ≡ Teacher, H ≡ Class Hour, R ≡ Classroom, S ≡ Student, G ≡ Grade. We assume the following:

- Each course is taught by one teacher.
- Only one course can be taught in a classroom at one time.
- A teacher can be in only one classroom at one time.
- Each student has one grade in each course.
- A student can be in only one classroom at one time.

(a) Write down the above statements in terms of functional dependencies.

(b) Determine a candidate key for the schema <CTHRSG>.

(c) Show that the schema <CTHRSG> is not in BCNF. Decompose it into schemas which are in BCNF.

(d) Justify whether the above decomposition is dependency preserving or not.

C50. (a) Consider the following relations:

Student(<u>*snum*: integer</u>, *sname*: string, *major*: string, *level*: string, *age*: integer)

Class(<u>*name*: string</u>, *meets at*: string, *room*: string, *fid*: integer)

Enrolled(<u>*snum*: integer, *cname*: string</u>)

Faculty(<u>*fid*: integer</u>, *fname*: string, *deptid*: integer).

Write an SQL query to find the names of faculty members who teach in every room in which some class is taught.

(b) Let R be a relation with functional dependencies $\mathcal{F}$. For any subset of attributes $X \subset R$, the closure of X is defined as the set

$$X^+ = \{A \in R | X \in A \text{ holds with respect to } \mathcal{F}\}.$$

For two non-empty attribute sets Y and Z in R, prove or disprove each of the following statements:

(i) $(Y^+Z)^+ = (YZ)^+$;

(ii) $(YZ)^+ = Y^+Z^+$.

C51. Let $R = (A, B, C, D, E, F)$ be a schema with the set $F = \{A \to BC, CD \to E, B \to D, E \to A\}$ of functional dependencies. Suppose R is decomposed into two schemata $R_1 = (A, B, C)$ and $R_2 = (A, D, E, F)$

(a) Is this decomposition loss-less? Justify.

(b) Is this decomposition dependency preserving? Justify.

(c) Identify all the candidate keys for R.

(d) Decompose R into normalized sets of relations using 3NF.

(e) If a new dependency $A \twoheadrightarrow F$ (multi-valued dependency) is introduced, what would be the new set of normalized relations?

C52. Consider the relations $r1(A, B, C)$, $r2(C, D, E)$ and $r3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r1)$ be 500, $V(C, r2)$ be 1000, $V(E, r2)$ be 50, and $V(E, r3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear in relation r for attribute X . If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.

C53. A school database maintains the following relations for its students, teachers and subjects:

- Student(st_name, st_address, class, section, roll_no, regn_no)
- Teacher(t_name, t_address, tel_no)
- Subject(s_name, t_name, text_book, class)

Consider the following constraints on the existing data.

- A student after admission to the school is assigned with a unique regn_no. However, a student also gets a roll_no that starts from 1 for each class and section. A class can have many sections and a student is placed in only one class and section as expected in a school.
- In the school a teacher's name (t_name) has been found to be unique. However, more than one teacher may stay at the same address and the tel_no is a land line connection where an address will have only one such telephone.
- A subject name (s_name) is unique but the same subject may be taught in many classes (for example, History may be taught in many classes with different contents but s_name remains the same). Every subject has a set of standard text_books for a class and there may be more than one teacher who can teach the subject. Any teacher may use any of the standard text books to teach a subject.

(a) Considering the above constraints, identify the functional/multi-valued dependencies present and normalize the relations.

(b) Using the normalized set of relations answer the following query using relational algebra or SQL: List all the teachers (t_name) who can teach History in Class V and reside in "Baranagar" (name of a locality). Consider that any address offers a locality name.

C54. Two queries equivalent to each other are specified for a relation $R(A, B, C, D, E, F)$. The queries are:

- $\pi_{A,B,C}(\sigma_{B>500}(R))$
- $\sigma_{B>500}(\pi_{A,B,C}(R))$

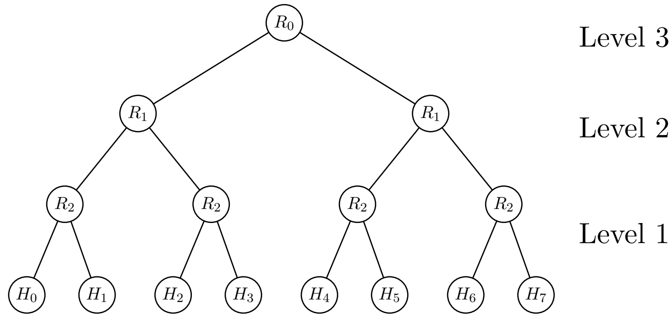
The system maintains a $B+$ tree index for (A, B, C) on R. However, the index is unclustered. The relation R occupies 100 pages and the index structure needs 5 pages only. Compute the number of disk accesses required for each of the queries and thereby decide which one of the two queries will be preferred by the query optimizer for minimum cost of execution. The cost of query execution is primarily dependent on the number of disk accesses.

C55. Recall the `stop-and-wait` and `sliding window` protocols for a packet-based data transmission. The stop and wait protocol sends one packet and waits for the acknowledgment to arrive. The waiting mechanism results in wastage of network bandwidth. In order to improve the bandwidth utilization, the sliding window protocol utilizes pipelined communication where multiple packets are sent without waiting for acknowledgements. In the latter case, each packet is stamped with a sequence number. The sequence number in each packet helps the sender to decide whether it should continue the transmission or wait for acknowledgement. It also helps the receiver to arrange the packets in order and to prepare acknowledgements.

Consider that a sliding window protocol is used for data transmission between two stations, say P and Q, located at a distance of d km. Assume that all the packets are k bits long. The propagation delay is t sec/km and the channel capacity is b bits/sec. Calculate the minimum number of bits required for the sequence number field in a packet for maximum utilization of the bandwidth. Assume that the processing delay is negligible.

C56. A block of bits with n rows and m columns uses horizontal and vertical parity bits for error detection. If exactly 4 bits are in error during transmission, derive an expression for the probability that the error will be detected.

C57. In a computer network, 2^k host computers $H_0, H_1, \ldots, H_{2^k-1}$ are connected via routers that are connected in the form of a binary tree. An example is shown in the figure below for $k = 3$. Each node labelled R_i corresponds to a router at level $3 - i \quad (i = 1, 2, 3)$; the nodes at the leaf level correspond to the hosts.



All the lines are full-duplex, and routers are faster than the links. Also, all links at the same level are of the same capacity. Traffic is always routed via shortest path. Suppose the capacity of each link at level 1 is 1 MBps.

(a) Find the minimum capacities to be assigned to the links at upper layers i ($1 < i \leq k$), such that the router R_0 is never congested. Prove the correctness of your result.

(b) With the link capacities found in (i), show that there always exist traffic patterns that may congest any router R_i other than R_0. Justify your answer.

C58. Consider a 100mbps token ring network with 10 stations having a ring latency of 50 μs (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is 200 μs.

(a) Express the maximum efficiency of this network when only a single station is active in the network.

(b) Find an upper bound on the token rotation time when all stations are active.

(c) Calculate the maximum throughput rate that one host can achieve in the network.

C59. Station A is sending data to station B over a full duplex error free channel. A sliding window protocol is being used for flow control. The send and receive window sizes are 6 frames each. Each frame is 1200 bytes long and the transmission time for such a frame is 70 μS. Acknowledgment frames sent by B to A are very small and require negligible transmission time. The propagation delay over the link is 300 μS. What is the maximum achievable throughput in this communication?

C60. Consider a large number of hosts connected through a shared communication channel. The hosts transmit whenever they have any data. However, if two data packets try to occupy the channel at the same time, there will be a collision and both will be garbled. The hosts retransmit these packets that suffered collisions. Assume that the generation of new packets by a host is a Poisson process and is independent of other hosts. The total number of transmissions (old and new packets combined) per packet time follows Poisson distribution with mean 2.0 packets per packet time. Compute the throughput of the channel. (Packet time is the amount of time needed to transmit a packet.)

C61. A heavily loaded 1 km long, 10 Mbps token ring network has a propagation speed of 200 meter per micro-second. Fifty stations are uniformly spaced around the ring. Each data packet is 256 bits long, including 32 bits of header. The token is of 8 bits. What is the effective data rate of the network assuming the stations always have packets to transmit?

Mathematics for Computer Science

MC1. Let p and q be distinct primes and $m = p^q + q^p$. What is the remainder if m is divided by pq?

MC2. Let $H = \{1 + 4k : k \in \mathbb{Z},\ k \geq 0\}$. An element $x \in H$ is called *H-prime* if $x \neq 1$ and x cannot be written as product of two strictly smaller elements of H.

(a) Show that $xy \in H$ for all $x, y \in H$.

(b) Prove that every $x \in H$ greater than 1, can be factored as a product of H-primes but unique factorisation does not hold.

MC3. Let $f(x) = \sum_{j=0}^{n} a_j x^j$ and $b(x) = \sum_{j=0}^{n} b_j x^j$, where $a_j, b_j \in \mathbb{Z},\ j = 0, \ldots, n$. For $m \in \mathbb{N}$, we say $f \equiv g \mod m$ if $a_j = b_j \mod m$ for $0 \leq j \leq n$. For an odd prime $p > 0$, let

$$f(x) = x^{p-1} - 1 \quad \text{and} \quad g(x) = (x-1)(x-2)\cdots(x-p+1).$$

Prove that

(a) the polynomial $f(x) - g(x)$ has degree $p - 2$, and

(b) $f \equiv g \mod p$.

MC4. For $n \geq 1$, let $f_n = 2^{2^n} + 1$. Show that if m and n are relatively prime then f_m and f_n are relatively prime. Hence show that there are infinitely many primes.

MC5. Show that for $n \geq 2$, $1 + 1/2 + \cdots + 1/2n$ is not an integer. You may use the fact that there is a prime p between n and $2n$ for $n \geq 2$.

(HINT: Show that such a prime divides the denominator of $1 + 1/2 + \cdots + 1/2n$.)

MC6. Let p be a prime and let $A = \{1, 2, \ldots, p-1\}$. Prove that there is a bijection f from A onto A such that $kf(k) \equiv 1 \pmod{p}$ for each $k \in A$. Hence deduce that if p is an odd prime, then p divides the numerator of $1 + (1/2) + \cdots + (1/(p-1))$.

MC7. Let p be a prime number. What is the largest power of p that divides $p^2!$?

MC8. Show that the 100 digit number $11\cdots 1$ is divisible by the prime number 101.

MC9. Let p and q be prime numbers. If q divides $2^p - 1$ then show that p divides $q-1$.

MC10. If a and b are integers such that 9 divides $a^2 + ab + b^2$ then show that 3 divides both a and b.

MC11. Let c be a 3^n digit number whose digits are all equal. Show that 3^n divides c.

MC12. Show that, for distinct positive integers m and n, the greatest common divisor of $2^{2^m}+1$ and $2^{2^n}+1$ is 1. Deduce that the number of prime integers is infinite.

MC13. Let p be a prime and r an integer, $0 < r < p$. Show that $\frac{(p-1)!}{r!(p-r)!}$ is an integer.

MC14. Show that if a prime number is divided by 30, then the remainder is either one or a prime number.

MC15. Prove that $\sqrt{n-1} + \sqrt{n+1}$ is an irrational number for all positive integers n.

MC16. Show that n^5 and n have the same last digit for all positive integers n.

MC17. For all integers $n \geq 1$, show that $2^{2n} - 3n - 1$ is divisible by 9.

MC18. Find all the integers n for which $\sqrt{\frac{3n-5}{n+1}}$ is also an integer.

MC19. Let p, q be distinct odd primes. Prove that the equation $x^2 = 1$ has exactly four (distinct) solutions in the ring of integers modulo pq.

MC20. For any $x \in \mathbb{R}$, let $[x]$ be the greatest integer not exceeding x. Let m be a positive integer. Show that

$$[(1+\sqrt{3})^{2m+1}] = (1+\sqrt{3})^{m+1} + (1-\sqrt{3})^{m+1}.$$

MC21. Let m, n be natural numbers. Show that m and m^{4n+1} have same last digit in their decimal expansions.

MC22. Does there exist an integer x satisfying the following congruences?

$$\begin{aligned} 10x &= 1 \pmod{21} \\ 5x &= 2 \pmod{6} \\ 4x &= 1 \pmod{7} \end{aligned}$$

Justify your answer.

MC23. Let $A_1, A_2, \ldots, A_n$ be distinct subsets of a set X. Show that there exists a subset B of X having at most n elements such that $B \cap A_1, B \cap A_2, \ldots, B \cap A_n$ are all distinct.

(HINT: Use induction on n).

MC24. Find the number of squares in $[0,8] \times [0,8] \subseteq \mathbb{R}^2$ whose vertices have integer coefficients and the sides are parallel to the coordinate axes.

MC25. (a) Show that $\binom{n}{k} = \sum_{m=k}^{n} \binom{m-1}{k-1}$.

(b) Prove that

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n-1}\frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

MC26. Prove that any set of $n^2 + 1$ points in a square with side length one contains two points with distance between them less than or equal to $\frac{\sqrt{2}}{n}$.

MC27. Let n be a positive integer and p a prime. Show that the number of ordered basis of a vector space of dimension n over a field of order p is

$$(p^n - 1)(p^n - p)(p^n - p^2)\dots(p^n - p^{n-1}).$$

Deduce that this number is a multiple of $n!$.

MC28. If p is an odd prime, show that

$$\binom{2p-1}{p-1} \equiv 1 \bmod p^2.$$

MC29. Let X be a set consisting of 5 elements. Let

$$A = \{f : X \to X; f(f(x)) = x \text{ for each } x \in X\}.$$

Compute the number of elements of A.

MC30. Let L and L' be two sets of parallel lines in $\mathbf{R}^2$ consisting of m and n lines, respectively. Assume that every line in L intersects every line in L'. Find the number of parallelograms formed by the lines in $\mathsf{L} \cup \mathsf{L}'$.

MC31. A square in the plane is subdivided into n^2 squares $S_1, \cdots, S_{n^2}$ of equal size by horizontal and vertical lines. Determine the number of squares in the plane that are unions of a subcollection $S_{i_1}, \cdots, S_{i_k}$.

MC32. Find all sets of consecutive natural numbers whose sum is 54.

MC33. If 70% of students in a class pass the algebra test, 75% pass the analysis test, 80% pass the geometry test, and 85% pass the number theory test, what is the least percentage of students who pass all four subjects?

MC34. Find the number of elements in the set

$$S = \{(a, b, c) \in \mathbf{N}^3 : a + b + c = 25\},$$

where $\mathbf{N}$ is the set of natural numbers.

MC35. Let r be a positive real number. Prove that among the numbers $r, 2r, \ldots, (n-1)r$ there is at least one number which differs from an integer by at most $\frac{1}{n}$.

MC36. How many of the $n!$ terms in the expansion of a determinant of order n remain, if all the elements of the main diagonal are set to zero ?

MC37. Find the divisors between 60 and 65 of $2^{48} - 1$.

MC38. Show that every member of the sequence

$$1107, 10017, 100117, \ldots$$

is divisible by 53.

MC39. There are 102 distinct points in the unit square $[0,1] \times [0,1]$. Show that there are at least two distinct points such that the square of the distance between them is at most 0.02.

MC40. Suppose that there are n boxes labelled $1, 2, \ldots, n$ and there are n balls also labelled similarly. The balls are thrown into boxes completely randomly so that each box receives one ball.

(a) How many possible arrangements of balls in boxes is possible?

(b) Find the probability that the ball labelled 1 goes into the box labelled 1.

(c) Find the probability that at least one ball is in the box with the same label.

MC41. Show that the number of bijections f of $\{1, 2, \ldots, n\}$ such that $f(i) \neq i$ for any i is equal to

$$\sum_{j=0}^{n} (-1)^j \frac{n!}{j!}.$$

MC42. Find the smallest integer $n > 0$ such that 2012 divides $9^n - 1$. (Hint: 251 and 503 are prime numbers).

MC43. Let p_k be the k-th prime number. Show that there are infinitely many k such that

$$p_{k+1} - p_k > 2.$$

MC44. For a real number x, $[x]$ denotes the largest integer less than or equal to x.

(a) For reals x, y show that $[x+y] - [x] - [y]$ is zero or one.

(b) For $m \geq 1$ and a prime p, let $\alpha_p(m)$ be the largest power of p which divides m (i.e., $p^{\alpha_p(m)}$ divides m but $p^{\alpha_p(m)+1}$ does not.) It is known that $\alpha_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$. Show that $\alpha_p\left(\binom{2n}{n}\right) \leq \frac{\log 2n}{\log p}$.

(HINT: How many terms in the formula for $\alpha_p(n!)$ are non-zero?)

MC45. In a LAN, n^2 routers are connected in an $n \times n$ mesh such that $R(i, j)$ represents a router in the i-th row and j-th column of the mesh.

(a) Find how many distinct shortest paths exist between two routers $R(i_1, j_1)$ and $R(i_2, j_2)$ $(1 \leq i_1, j_1, i_2, j_2 \leq n)$. Two paths are distinct if they differ in at least one link.

(b) At most how many of these distinct shortest paths will be node disjoint, i.e., with no common node except the source and the destination? Justify your answer.

MC46. Let G be a graph on n vertices and having m edges. Prove the following statements.

(a) If the maximum degree of a vertex in G is Δ, then there is no independent set in G containing more than $n - \frac{m}{\Delta}$ vertices.

(b) There is a bipartite subgraph of G that contains at least $m/2$ edges.

MC47. Construct a tree such that no matter how you colour its vertices properly using two colours (i.e. two adjacent vertices do not get the same colour), every maximum independent set of the tree contains vertices of both colours.

MC48. Let G be a connected graph that cannot be disconnected by the removal of fewer than k edges. If every vertex of G has even degree and S is a set of k edges that disconnects G, then show that k is even.

MC49. Let G be a graph in which every vertex has odd degree. Let u be a vertex and C a connected component of $G - u$. Prove that u has an odd number of neighbours in C if and only if $|C|$ is odd.

MC50. A planar graph is said to be an *outerplanar* graph if it has a planar drawing in which every vertex appears on the boundary of the outer face. Let G be a planar graph. Show that if it is possible to add edges to G in order to make it a planar graph that contains a Hamiltonian cycle, then there exist two subgraphs G_1, G_2 of G such that G_1 and G_2 are both outerplanar and $E(G) = E(G_1) \cup E(G_2)$.

MC51. Construct a planar graph with as many vertices as possible that has the property that it can be obtained by removing three edges from a complete graph.

MC52. Let G be a directed acyclic graph in which there is no directed path containing more than k vertices. Then prove that the vertices of G can be coloured with k colours such that no two adjacent vertices get the same colour.

MC53. Let $T = (V, E)$ be a rooted tree, and let $v \in V$ be any vertex of T.

- The *eccentricity* of v is the maximum distance from v to any other vertex in T.
- The *centre* C of T is the set of vertices which have the minimum eccentricity among all vertices in T.
- The *weight* of v is the number of vertices in the largest subtree of v.
- The *centroid* G of T is the set of vertices with the minimum weight among all vertices in T.

Construct a tree T that has disjoint centre and centroid, each having two vertices, i.e., $C \cap G = \varnothing$ and $|C| = |G| = 2$.

MC54. Consider an urn containing 10 red balls, 10 white balls, and 10 black balls. Balls are drawn at random with replacement one by one. Let T be the minimum number of draws required to get balls of three colours. Find the distribution of T.

MC55. Suppose X is a non-negative integer-valued random variable such that

$$P(X = k+1) = \frac{6+k}{3k+1} P(X = k) \text{ for } k = 0, 1, 2, \ldots,$$

and $P(X = 0) = 1/4$. Find $\mathrm{E}(X)$.

MC56. Suppose an urn contains w white balls, b black balls, and r red balls; $w, b, r > 0$. Balls are drawn one by one without replacement and at random until all balls are exhausted. Find the probability that, among the three colours, the white balls are exhausted first.

MC57. (a) Consider an alphabet $A = \{0, 1, \ldots, 9\}$. Design a Deterministic Finite Automaton (DFA) that accepts strings over A whose decimal equivalent leaves the remainder 3 if divided by 6.

(b) Using Pumping Lemma, prove that the language $\{a^n : n \text{ is a perfect square}\}$ is not regular.

MC58. Let $M = (Q, \Sigma, \delta, q_o, F)$ be a Deterministic Finite Automaton (DFA), and let $L = L(M)$ be the language accepted by M. Consider another DFA $M' = (Q, \Sigma, \delta, q_o, Q)$, and let $L' = L(M')$.

(a) Give an example of a language $L \subseteq \{0,1\}^*$ for which $L(M') = \textit{prefix}(L)$.

(b) Give an example of a language $L \subseteq \{0,1\}^*$ for which $L(M') \neq \textit{prefix}(L)$.

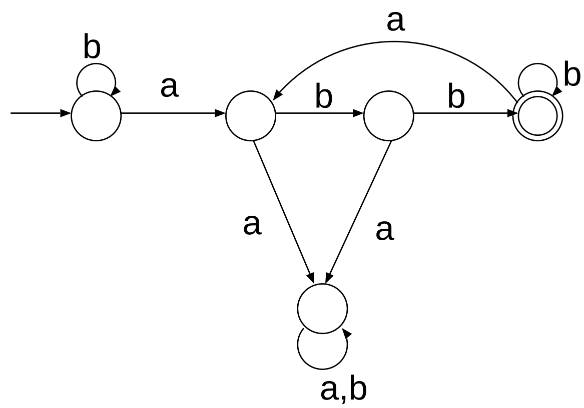
Justify your answer in each case.

Recall that for a given alphabet Σ, a string $\beta \in \Sigma^*$ is said to be a prefix of a string $\alpha \in \Sigma^*$ if $\alpha = \beta\gamma$ for some $\gamma \in \Sigma^*$. For example, the prefixes of 010 are $\varepsilon, 0, 01$ and 010. Given a language $L \subseteq \Sigma^*$, *prefix*(L) is defined as

$$\textit{prefix}(L) = \{x \in \Sigma^* \mid x \text{ is a prefix of some } y \in L\}.$$

MC59. Construct a finite state machine that accepts all the binary strings in which the number of 1's and number of 0's are divisible by 3 and 2, respectively.

MC60. Describe the language recognized by the following machine.



MC61. Consider the grammar $E \to E + n | E \times n | n$. For a sentence $n + n \times n$, find the handles in the right sentential form of the reductions.

MC62. Design a Turing machine that recognizes the unary language consisting of all strings of 0's whose length is a power of 2, i.e., $L = \{0^{2^n} | n \geq 0\}$.

MC63. Write a context-free grammar for the language consisting of all strings over $\{a, b\}$ in which the number of a's is not the same as that of b's.

MC64. Let the set $D = \{p \mid p$ is a polynomial over the single variable x, and p has a root which is an integer (positive or negative)$\}$.

(a) Design a Turing machine (TM) to accept D.

(b) Can D be decided by a TM? Justify your answer.

MC65. Give a context-free grammar G that generates $L = \{0^i 1^j 0^k \mid i+k = j\}$. Prove that $L = L(G)$.

MC66. Write a regular expression for all strings of 0's and 1's in which the total number of 0's to the right of each 1 is even. Justify your answer.

MC67. Construct a deterministic finite automaton accepting the following language:

$\{w \in \{0,1\}^* : w$ has an equal number of 01's and 10's $\}$.

For example, 101 is in the language because it contains one instance of 10 and one instance of 01 as well.

MC68. Consider the following statement:
For all languages $L \subseteq \{0,1\}^*$, if L^* is regular then L is regular.
Is the above statement true? Justify your answer.

Mathematics

Unless otherwise specified,
$\mathbb{N}$ denotes the set of positive integers;
$\mathbb{Z}$ denotes the set of integers;
$\mathbb{Q}$ denotes the set of rational numbers;
$\mathbb{R}$ denotes the set of real numbers;
$\mathbb{C}$ denotes the set of complex numbers.

M1. Let C be a closed, symmetric (i.e., if $x \in C$ then $-x \in C$) convex subset of $\mathbb{R}^n$ such that

$$\bigcup_{m \in \mathbb{N}} mC = \mathbb{R}^n, \text{ where } mC = \{mx \mid x \in C\}.$$

Prove that there exists $r > 0$ such that

$$B(0, r) = \{x \in \mathbb{R}^n \mid \|x\| < r\} \subseteq C.$$

M2. Define $f(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$ for $x > 0$. Show that $0 < f(x) < 1/x$ and $f(x)$ is monotonically decreasing for $x > 0$.

M3. (a) Let the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be given by

$$0 < b_1 < a_1, \quad a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2} \quad \text{for } n \in \mathbb{N}.$$

Show that both sequences are monotone and they have the same limit.

(b) Let the polynomial $f_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, $n \in \mathbb{N}$ with all integer coefficients be such that $f_n(b_1) = f_n(b_2) = f_n(b_3) = f_n(b_4) = f_n(b_5) = 19$ for five distinct integers b_1, b_2, b_3, b_4 and b_5. How many different integer solutions exist for $f_n(x) = 23$? Justify your answer.

(c) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.$$

How many square roots does $\mathbf{A}$ have? Justify your answer.

M4. (a) Suppose f is a real-valued continuous function defined on $[-2, 2]$ and is three times differentiable in $(-2, 2)$. If $f(2) = -f(-2) = 4$ and $f'(0) = 0$, then show that there exists $x \in (-2, 2)$ such that $f'''(x) \geq 3$.

(b) Suppose f is a real-valued continuous function defined on $[0, 1]$ such that $(2x-1)(2f(x)-x) > 0$ for all $x \neq 1/2$. If $f'(1/2)$ exists, then show that it cannot be smaller than $1/2$.

(c) Find all values of $\alpha \in \mathbb{R}$ for which the sum

$$\sum_{n \geq 1} n^{n^\alpha} e^{-n}$$

is convergent.

M5. (a) Compute the following limit with full justification:

$$\lim_{n\to\infty} \frac{\sin 1 + 2\,\sin\frac{1}{2} + 3\,\sin\frac{1}{3} + \cdots + n\,\sin\frac{1}{n}}{n}.$$

(b) Show that there exists no one-to-one function $f : \mathbb{R} \to \mathbb{R}$ with the property that for all $x \in \mathbb{R}$, $f(x^2) - (f(x))^2 \geq 1/4$.

(c) If f is real-valued continuous function defined on $[0, 1]$ such that

$$\int_0^1 f(x)\, x^n = 0$$

for all $n = 0, 1, 2, \ldots$, then show that $f(x) = 0$ for all $x \in [0, 1]$.

M6. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \ldots$ with $\mid f_n'(x) \mid \leq 1$ for all n and x. Assume

$$\lim_{n\to\infty} f_n(x) = g(x),$$

for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

(b) Let $f : [0, 1] \to \mathbb{R}$ be continuous over $[0, 1]$, differentiable over $(0, 1)$, $f(0) = 0$, and $0 \leq f'(x) \leq 2f(x)$.
Show that f is identically 0.

(c) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x) dx = f(1).$$

M7. (a) Let $\{a_n\}_{n\geq 0}$ be a non-negative sequence satisfying $a_{m+n} < a_m + a_n + 1$, for all positive integers m, n. Show that

(i) $a_{17} < 2a_8 + a_1 + 2$.

(ii) $\frac{a_n}{n}$ converges as $n \to \infty$.

(b) Let ε_n be the fractional part of $n!e$, where n is a positive integer.

(i) Show that $\frac{1}{n+1} < \varepsilon_n < \frac{1}{n}$ for all positive integers n.

(ii) Prove that $n\sin(2n!e\pi)$ converges to 2π as $n \to \infty$.

M8. (a) Find the number of positive integers with k digits, that do not contain the digit 7.

(b) Let S denote the set of all positive integers that do not contain the digit 7. Prove that the infinite series

$$\sum_{n\in S} \frac{1}{n}$$

converges.

M9. (a) Let x be an irrational number. Show that for any positive integer $k \geq 1$, there exists $\delta_k > 0$, such that the open interval $(x-\delta_k, x+\delta_k)$ does not contain any rational number of the form $\frac{p}{k}$, where p is an integer.

(b) Consider the function f defined on $[1,2]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in [1,2] \text{ is irrational} \\ \frac{1}{k} & \text{if } x \in [1,2] \text{ is a rational } \frac{p}{k} \text{ where } p \text{ and} \\ & k \text{ are integers prime to each other} \end{cases}$$

Show that f is discontinuous at every rational $x \in [1,2]$ and continuous at every irrational $x \in [1,2]$.

M10. (a) Show that for every $\theta \in (0, \pi/2)$, there exists a unique real x_θ such that

$$(\sin\theta)^{x_\theta} + (\cos\theta)^{x_\theta} = \frac{3}{2}.$$

(b) Find all possible real values of α for which the following improper integral exists:

$$\int_{-1}^{1} |x|^\alpha dx.$$

M11. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is differentiable at $c \in \mathbb{R}$. Let $\{y_n\}$ be a decreasing sequence of real numbers and $\{x_n\}$ be an increasing sequence of real numbers, both converging to c. Show that

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = f'(c).$$

M12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. If it is an injective function, show that f is monotone. Hence or otherwise, show that any continuous function f with $f(f(x)) = -x$ cannot be injective.

M13. Let n be a positive integer and $\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_n$ be the homomorphism defined by $\phi(1) = (n, \bar{1})$, where $\bar{1}$ denotes the set of integers congruent to 1 mod n. Prove that $(\mathbb{Z} \oplus \mathbb{Z}_n)/\text{Im}(\phi)$ is cyclic and find its order.

M14. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $X^4 + 1 \in \mathbb{Q}[X]$. Consider the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\alpha)$.

(a) With complete justification determine the degree of this extension.

(b) Find three distinct fields K_1, K_2, K_3 such that $\mathbb{Q} \subsetneq K_i \subsetneq \mathbb{Q}(\alpha)$ for $i = 1, 2, 3$.

M15. Let X be a set containing at least two elements and G be a group with identity element e. Suppose that there is a map $\alpha : G \times X \to X$. Let us write $\alpha(g, x)$ as $g \cdot x$. Assume that the following properties are satisfied by α:

- $e \cdot x = x$ for all $x \in X$;
- $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$;
- given $(x_1, x_2) \in X \times X$ and $(y_1, y_2) \in X \times X$ with $x_1 \neq x_2$, $y_1 \neq y_2$, there is a $g \in G$ such that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$. (The case x_i is same as some y_j for $i, j = 1, 2$ is allowed).

For $x \in X$, define

$$G_x := \{g \in G \ : g \cdot x = x\}.$$

Prove the following:

(a) Given any pair $a, b \in X$, there is a $g \in G$ such that $g \cdot a = b$.

(b) $G_x \neq G$ for any $x \in X$.

(c) Let $x \in X$ and $g \in G \setminus G_x$. Then $G = G_x \cup G_x \, g \, G_x$.

(d) For any $x \in X$, there is no proper subgroup of G properly containing G_x.

M16. Let R be an integral domain. For $a, b \in R$, by "*a divides b*" we mean that there is an $x \in R$ such that $ax = b$. We recall the following definitions.

An element $d \in R$ is a gcd of $a, b \in R$ if:

- d divides both a and b, and
- if $c \in R$ divides both a and b, then c divides d.

An element $l \in R$ is an lcm of $a, b \in R$ if:

- both a and b divide l, and
- if both a and b divide $m \in R$, then l divides m.

Assuming that any two elements in R have an lcm, prove the following.

(a) Any two elements in R have a gcd.

(b) Any irreducible element in R is prime.

M17. Let G be a group of order n, H a subgroup of G of order m, $k = \dfrac{n}{m}$ and S_k the symmetric group on k symbols. Assuming $\dfrac{k!}{2} < n$, show that G has a nontrivial proper normal subgroup.

M18. Justify whether a group of order 48 is simple or not.

M19. (a) Let $R = \{f : [0,1] \longrightarrow \mathbb{R} \,|\, f \text{ is continuous }\}$ be a ring. In R, consider the ideal $I = \{f \in R \,|\, f(\frac{1}{2}) = 0 = f(\frac{1}{3})\}$. Verify whether I is a prime ideal.

(b) Let R be a commutative ring with unity and P be a prime ideal of R such that P does not contain any non-zero zero-divisor. Show that R is an integral domain.

(c) Determine whether $X^6 + 4X^3 + 1$ is irreducible in $\mathbb{Q}[X]$.

M20. (a) Let k be a field and α be transcendental over k. Let $E(\neq k)$ be a field such that

$$k \subsetneq E \subseteq k(\alpha).$$

Prove that $k(\alpha)$ is a finite extension of E.

(b) Let k be a field of characteristic not equal to 2. Let $a, b \in k$ be such that neither a nor b is a square in k. Prove that $k(\sqrt{a}, \sqrt{b}) = k(\sqrt{a} + \sqrt{b})$. Also show that the degree of $k(\sqrt{a}, \sqrt{b})$ over k is 2, if ab is a square in k; and it is 4, if ab is not a square in k.

M21. Find an ideal I in $A = \dfrac{\mathbb{Z}[X]}{(X^4 + X^2 + 1)}$ such that ${}^{A}\big/_{I}$ is a finite field with 25 elements.

M22. Examine whether there is a polynomial $f(X) \in \mathbb{R}[X]$ such that $\dfrac{\mathbb{R}[X]}{(f(X))}$ is isomorphic **as rings** to the product ring $\mathbb{C} \times \mathbb{C}$.

M23. Prove that the following two groups are isomorphic:

- $\mathbb{Z}[X]$, the group of polynomials with integer coefficients under addition, and
- $\mathbb{Q}_{>0}$, the group of positive rational numbers under multiplication.

HINT: Fundamental theorem of Arithmetic

M24. Let $A = \dfrac{\mathbb{C}[X, Y]}{(X^2 + Y^2 - 1)}$, and x, y denote the images of X, Y in A respectively. If $u = x + iy$, then prove that

(a) u is a unit in A, and

(b) $u - i$ generates a maximal ideal of A.

M25. Find all ring automorphisms of $\mathbb{Q}[X]$ where $\mathbb{Q}$ denotes the field of rational numbers.

M26. In the additive group of rational numbers, show that the group generated by any two elements is cyclic.

M27. Give two non-isomorphic field extensions of degree 2 of the field $\mathbb{Q}$ of rational numbers. Justify your answer.

M28. Let $\mathbb{Z}[X]$ denote the ring of polynomials in X with integer coefficients. Find an ideal I in $\mathbb{Z}[X]$ such that $\mathbb{Z}[X]/I$ is a field of order 4.

M29. Let F be a field whose multiplicative group is cyclic. Show that there exists a prime number p such that $px = 0$ for each x in F.

M30. Let G be a finite group admitting no nontrivial automorphisms. Show that G has atmost two elements.

M31. If m and n are distinct natural numbers, then show that the abelian groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not isomorphic.

M32. Let I be the ideal of the polynomial ring $\mathbb{Z}[X]$ generated by X^2+X+1 and 5. Is the quotient ring $\mathbb{Z}[X]/I$ a field? Justify your answer.

M33. Let I be the ideal generated by $X - 3$ and 7 in the polynomial ring $\mathbb{Z}[X]$. Show that, for each $f(X) \in \mathbb{Z}[X]$, there exists a unique integer $a, 0 \leq a \leq 6$, such that $f(X) - a \in I$.

M34. If G is a group of order 111 with exactly two elements of order 3 then show that G is cyclic.

M35. A ring R is said to be of characteristic m (m a positive integer) if $x + \cdots + x$(m times) $= 0$ for all $x \in R$. Show that any ring of characteristic pq (p, q prime) is the direct sum of a ring of characteristic p and one of characteristic q.

M36. Let F be an algebraically closed field of finite characteristic p. Let $q = p^e, e \geq 1$. Show that $x \rightarrow x^q$ is an automorphism of F into itself. Conclude that there is a unique subfield F_q of F having q elements and that F_q consists of the zeroes of the polynomial $X^q - X$.

M37. Show that the additive group of complex numbers is isomorphic to the additive group of real numbers.

(Hint : Consider these as vector spaces over an appropriate field.)

M38. Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$ and let k be an integer between 1 and n. Find the number of elements x in S_n such that the cycle containing 1 in the cycle decomposition of x has length k.

M39. Let σ be a permutation of $\{1, 2, 3, \ldots, n\}, n$ odd. Show that

$$(\sigma(1) - 1)(\sigma(2) - 2) \ldots (\sigma(n) - n)$$

is even.

M40. Let $\phi : G \to \mathbb{C}^*$ be a non-constant group homomorphism from a finite group G to the multiplicative group of non-zero complex numbers. Show that

$$\sum_{g \in G} \phi(g) = 0.$$

M41. Let ψ be an automorphism of the additive group $\mathbb{Z}/77\mathbb{Z}$ of the integers $\bmod 77$. Show that ψ^{60} is the identity automorphism (here ψ^{60} means the 60-fold composition of ψ with itself).

M42. Let a, b be two elements of a field F with $a \neq 0$. Prove that a polynomial $f(x) \in F[x]$ is irreducible if and only if the polynomial $f(ax + b)$ is irreducible.

M43. Let f, g be two polynomials in $\mathbb{Z}[x]$. Prove that the ideal generated by them in $\mathbb{Z}[x]$ contains a non-zero integer if and only if they are relatively prime in $\mathbb{Q}[x]$.

M44. Show that the polynomial $x^3 - 2x + 3$ can not be split as a product of two non-constant polynomials in $\mathbb{Q}[x]$.

M45. Let f be an irreducible cubic polynomial with rational coefficients and let K be an extension of $\mathbb{Q}$ of degree ten. Prove or disprove: f remains irreducible in $K[x]$.

M46. Determine the group of automorphisms of the permutation group S_3 on 3 elements.

M47. Find the number of elements of order 6 in the permutation group S_6 on six symbols.

M48. Show that the ideal generated by 2 and x in the ring $\mathbb{Z}[x]$ of polynomials over the ring of integers cannot be generated by one element.

M49. Can the ideal $I = \left\{f : [0,1] \to \mathbb{R},\ f\left(\frac{1}{2}\right) = 0\right\}$ in the ring R of all functions from $[0,1]$ to $\mathbb{R}$ be generated by one element?

M50. Let K be a subfield of $\mathbb{C}$ that is not contained in $\mathbb{R}$. Show that K is dense in $\mathbb{C}$.

M51. Let g be an element of order 143 in a group G. Show that there are unique elements g_1, g_2 in G satisfying:

I. $g = g_1 g_2 = g_2 g_1$,
II. Order of $g_1 = 11$, Order of $g_2 = 13$.

M52. Let A be any ring without zero divisors; that is, $xy = 0$ implies either $x = 0$ or $y = 0$. If $a, b \in A$ satisfy $a^m = b^m, a^n = b^n$ for two fixed relatively prime natural numbers m, n prove that $a = b$.

M53. If F is a finite extension field of a field E and if $E \subseteq R \subseteq F$ for some subring R, then prove that R is also a field.

M54. Determine the additive group of the field of 4 elements.

M55. Let $\mathbb{Q}$ denote the field of rational numbers. Let I be the principal ideal in the polynomial ring $\mathbb{Q}[x]$ generated by the polynomial $f(x) = x^4 - 1$, i.e.,

$$I = \{g(x)f(x)|g(x) \in \mathbb{Q}[x]\}.$$

Show that I is contained in exactly three prime ideals of $\mathbb{Q}[x]$.

M56. Consider the following subgroup of S_4.

$$B = \{Id, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

Show that B is a normal subgroup and S_4/B is isomorphic to S_3.

M57. Let R be a Ring with identity. Let I and J be two ideal in R. Suppose $\Phi : R \to R/I \times R/J$ defined by $\Phi(a) = (a + I, a + J)$ for $a \in R$, is a ring isomorphism. Show that $I \cap J = \{0\}$ and $I + J = R$.

M58. Show that if the multiplicative group of a field in cyclic, then the field in finite.

M59. Show that every finitely generated subgroup of the additive group of rational numbers is cyclic.

M60. If I is a maximal ideal of the polynomial ring $\mathbb{Z}[X]$, show that $\mathbb{Z}[X]/I$ is a finite field.

M61. Let S_n be the group of all permutations of the set $\{1, 2, \ldots, n\}$. Find the number of conjugates of the n-cycle taking i to $i + 1$ for $1 \leq i \leq (n-1)$ and n to 1.

M62. Let $\mathbb{C}$ be the field of complex numbers and $\varphi : \mathbb{C}[X, Y, Z] \to \mathbb{C}[t]$ be the ring homomorphism such that

$$\begin{aligned} \varphi(a) &= a \text{ for all } a \text{ in } \mathbb{C}, \\ \varphi(X) &= t, \\ \varphi(Y) &= t^2, \text{ and} \\ \varphi(Z) &= t^3. \end{aligned}$$

Determine the kernel of φ.

M63. Show that there is no field isomorphism between $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$. Are they isomorphic as vector spaces over $\mathbb{Q}$?

M64. Let $\mathbb{Q}$, $\mathbb{R}$ denote the fields of rational numbers and real numbers respectively. Which of the following rings are *not* isomorphic.

(a) $\mathbb{Q}[x]/ < x^2 + 1 >$ and $\mathbb{Q}[x]/ < x^2 + x + 1 >$.

(b) $\mathbb{R}[x]/ < x^2 + 1 >$ and $\mathbb{R}[x]/ < x^2 + x + 1 >$.

Justify your answer.

M65. Let S_4 denote the group of permutations of $\{1, 2, 3, 4\}$ and let H be a subgroup of S_4 of order 6. Show that there exists an element $i \in \{1, 2, 3, 4\}$ which is fixed by each element of H.

M66. Let $V_1, V_2 \subseteq V$ be two subspaces of a vector space V and $b_1, b_2 \in V$. Show that the affine subspaces $W_1 = V_1 + b_1$ and $W_2 = V_2 + b_2$ have a nonempty intersection if and only if $\pi(b_1) = \pi(b_2)$, where $\pi : V \to V/(V_1 + V_2)$ is the quotient map.

M67. Let L_1 and L_2 be fields, $L_1 \subseteq L_2$, and $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k} \in L_1^n \subseteq L_2^n$. Show that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ are linearly independent in L_1^n over L_1 implies $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ are linearly independent in L_2^n over L_2. (HINT: Consider the case $k = n$.)

M68. (a) Suppose that V is a 4-dimensional vector space over $\mathbb{R}$. Let $T : V \to V$ be an invertible linear map and W be a proper subspace of V, such that

$$V = W \oplus TW.$$

If $T^2W \subseteq W$, then show that $T^2W = W$.

(b) Suppose that A and B are two $n \times n$ real symmetric matrices and A is also non-negative definite. Show that all eigenvalues of the matrix AB are real.

M69. Let $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = x_2,\ x_3 = x_4 = x_5\}$ be the subspace of $\mathbb{R}^5$. Find the dimension of W.
Prove that there does not exist a linear map T from $\mathbb{R}^5$ to $\mathbb{R}^2$, such that its null space $N(T) = W$.

M70. Let A be a diagonalisable $m \times m$ matrix with entries from $\mathbb{C}$. Define $\theta exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ assuming A^0 is the identity matrix. Prove that $\det(\theta exp(A)) = e^{\theta Tr(A)}$.

M71. Let A be a real symmetric $m \times m$ matrix with m distinct eigenvalues and $v_1, \ldots, v_m$ be the corresponding eigenvectors. Let C be an $m \times m$ matrix satisfying $\langle Cv_j \, , v_j \rangle = 0$ for $1 \leq j \leq m$. Prove that there exists an $m \times m$ matrix X such that $AX - XA = C$.

M72. Let $u = (u_1, u_2, \ldots, u_n)$ be a fixed non-zero vector in $\mathbb{R}^n$ such that $u_1 + u_2 + \cdots + u_n = 0$, and let A be the $n! \times n$ matrix whose rows are the $n!$ permutations of u. Show that $\text{rank}(A) = n - 1$.

M73. Let $u = (u_1, u_2, \ldots, u_n)^T$ be a non-zero column vector in $\mathbb{C}^n$, and u^* be the row vector $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$. Let $A = uu^*$. Find all the eigenvalues of A and identify the corresponding eigen subspaces.

M74. Let $A = ((a_{ij}))$ be an $n \times n$ complex matrix. For $1 \leq k \leq n$, let $D_k \subseteq \mathbb{C}$ be the closed disc with center a_{kk} and radius $\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{jk}|$. If λ is any eigenvalue of A, then show that $\lambda \in D_k$ for at least one k.

(HINT: If $\underline{x}$ is an eigenvector corresponding to the eigenvalue λ, look at the coordinate position k for which $|x_k|$ is largest.)

M75. Let $\mathbb{R}^n$ denote the n dimensional real vector space. Let A, B be two linear transformations on $\mathbb{R}^n$ such that $A \circ A = A$ and $B \circ B = B$, where $\circ$ stands for composition of linear transformations. Show that the ranges of $A - A \circ B$ and $B - A \circ B$ have only the zero vector in common.

M76. Let V be the real vector space consisting of continuous real valued functions on $\mathbb{R}$. Show that the subset $\{\sin x, \ \cos x, \ e^x\}$ of V is linearly independent.

M77. Let L and T be two linear transformations from a real vector space V to $\mathbb{R}$ such that $L(v) = 0$ implies $T(v) = 0$. Show that $T = cL$ for some real number c.

M78. Let x_1, x_2, x_3, x_4 be vectors in $\mathbb{R}^n$ such that the inner products $\langle x_i, x_j \rangle$, $i \neq j$, are strictly negative. Show that any three of the vectors x_1, x_2, x_3, x_4 are linearly independent.

M79. Let A be an $n \times n$ square matrix such that A^2 is the identity. Show that any $n \times 1$ vector can be expressed as a sum of atmost two eigenvectors of A.

M80. Let A be an $m \times n$ matrix of rank 1 and G be an $n \times m$ matrix such that $AGA = A$. Show that the trace of AG is 1.

M81. Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Show that, for each nonzero scalar $\lambda, (\lambda I - B)^{-1} = P_\lambda(B)$ for some polynomial $P_\lambda(X)$ of degree 3.

M82. Let C be an invertible, real matrix of order n. If each row sum of C is 1, then show that each row sum of C^{-1} also is 1.

M83. Let x and y be non-zero vectors in an inner product space over the field of real numbers. Then $\|x + y\| = \|x\| + \|y\|$ if and only if $x = \alpha y$ for some positive real number α.

M84. Suppose that A is a 2×2 matrix over reals such that $A^n = I$ for some integer n. Show that trace of A is atmost 2.

M85. Find an $n \times n$ matrix over real numbers whose minimal polynomial is x^{n-1}.

M86. If A is a matrix with real entries such that $A^2 = A$ show that A is diagonalisable over $\mathbb{R}$.

M87. If A is an $n \times n$ upper triangular matrix with all diagonal entries 1, find the inverse of A as a polynomial in A.

M88. Let V_1, V_2, V_3 be three-dimensional subspaces of $\mathbf{R}^4$. Show that $V_1 \cap V_2 \cap V_3$ contains a non-zero vector.

M89. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a $n \times n$ matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1, j = 2, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that A is not a diagonalizable matrix over $\mathbb{C}$.

M90. Let M_2 be the space of all 2×2 complex matrices. Take

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{N} = \{X \in M_2 : PX = XP\}.$$

Show that if A, B are in $\mathcal{N}$ then $AB = BA$.

M91. Let D be a $n \times n$ real, diagonal matrix. Show that there exists a $x \in \mathbb{R}^n$ such that

$$\text{span}\ \{D^k x : 0 \leq k \leq n-1\} = \mathbb{R}^n$$

if and only if the eigen-values of D are distinct.

M92. Let A and B be $n \times n$, Hermitian matrices such that $AB = BA$. Suppose one of them has distinct eigenvalues. Show that A and B have a common eigenvector.

M93. Let A be an $m \times n$ real matrix with rank r and nullity k. Let V, W be the vector spaces of all real $n \times p, m \times p$ matrices, respectively. Define $L_A : V \to W$ by $L_A(X) = AX$(left multiplication by A). Compute the rank and nullity of L_A.

M94. Let B, C be two invertible $n \times n$ complex matrices such that $BC = qCB$ for some complex number q. Show that $q^m = 1$ for some $m \geq 1$. (Hint: Consider the eigenvalues of BC and CB.)

M95. Let A be an $n \times n$ complex matrix such that $AX = XA$ for every invertible complex matrix X. Show that $A = \lambda I$ for some complex number λ.

M96. Let A_0, A_1 be $n \times n$ complex matrices which may not commute. Let $m \geq 2$ and $\omega = e^{\frac{2\pi i}{m}}$. Show that

$$\begin{aligned} m(A_0^m + A_1^m) &= (A_0 + A_1)^m + (A_0 + \omega A_1)^m + (A_0 + \omega^2 A_1)^m + \\ &\quad \cdots + (A_0 + \omega^{m-1} A_1)^m. \end{aligned}$$

M97. Let A and B be $n \times n$ matrices with real entries. Show that the matrix $\begin{pmatrix} A & I \\ I & B \end{pmatrix}$ has rank n if and only if A is nonsingular and $B = A^{-1}$.

M98. Let n be a positive odd integer and let A be a symmetric $n \times n$ matrix of integer entries such that $a_{ii} = 0$, $i = 1, 2, \ldots, n$. Show that the determinant of A is even.

M99. Let A be a $n \times n$ real matrix with $A^2 = A^T$. Show that every real eigen-value of A is either 0 or 1.

M100. Suppose A, B, C are three $n \times n$ matrices such that A has n distinct eigenvalues. If $AB = BA$ and $AC = CA$, prove that $BC = CB$.

M101. Find a maximal linearly independent set in the following subset C of $\mathbb{R}^{k+1}$:

$$\mathsf{C} = \left\{ \left(1, t, t^2, \ldots, t^k\right) \in \mathbb{R}^{k+1} \mid t \in \mathbb{R} \right\}.$$

M102. Let A be a square matrix such that $A^k = 0$ for some $k > 0$. Prove that there exists an invertible matrix B such that BAB^{-1} is upper triangular with diagonal entries zero.

M103. Let S_n denote the group of all permutations of $\{1, 2, \ldots, n\}$. For any $\sigma \in S_n$, denote by P_σ the linear transformation on $\mathbb{R}^n$ given by

$$P_\sigma \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{bmatrix}$$

where σ^{-1} denotes the inverse of σ. Show that the correspondence $\sigma \to P_\sigma$ is a homomorphism from the group S_n into the group of all $n \times n$ orthogonal matrices. If $\epsilon(\sigma)$ denotes the signature (sign) of σ, show that $\sum_{\sigma \in S_n} \epsilon(\sigma) P_\sigma = 0$ for $n \geq 3$.