

- A complex number is of the form  $z = a + ib$ , where  $a, b \in \mathbb{R}$ .

$$a = \operatorname{Re}(z), b = \operatorname{Im}(z) \text{ and } i^2 = -1$$

- A complex number is denoted by  $(a, b)$  and it can be geometrically represented in  $\mathbb{R} \times \mathbb{R}$ . This representation is called Argand diagram.

X-axis is called real axis and Y-axis is called imaginary axis.

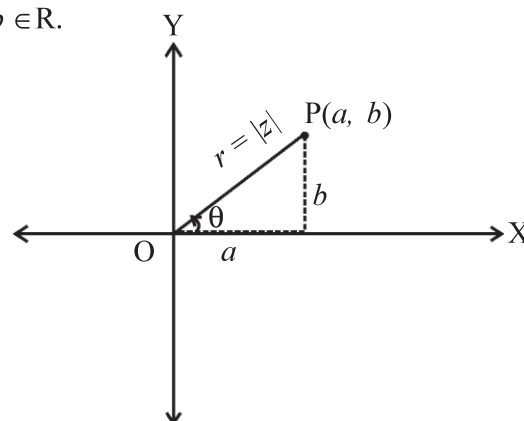


Figure 2.1

- The set of complex numbers is denoted by  $\mathbb{C}$ .

- The modulus of a complex number  $z = a + ib$  is  $|z| = \sqrt{a^2 + b^2}$ .

- Polar (Trigonometric) form of a complex number  $z = a + ib$  is

$$z = r (\cos \theta + i \sin \theta) = re^{i\theta}, \text{ where } r = \sqrt{a^2 + b^2} = |z|; e^{i\theta} = \cos \theta + i \sin \theta \text{ is the Euler form.}$$

- If  $\theta \in (-\pi, \pi]$ , then  $\theta$  is called the argument of  $z$  and it is denoted by  $\arg(z)$ . Conjugate of  $z = a + ib$  is  $\bar{z} = a - ib$ .

- If  $z = r (\cos \theta + i \sin \theta) = re^{i\theta}$  is the polar form, then  $\bar{z} = r (\cos \theta - i \sin \theta) = re^{-i\theta}$ .

- $|z| = |\bar{z}|$

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

- $z + \bar{z} = 2\operatorname{Re}(z), z - \bar{z} = 2i \operatorname{Im}(z)$

- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

- $\arg(\bar{z}) = -\arg(z)$

- $\arg\left(\frac{z}{z}\right) = 2\arg z$

- $z \bar{z} = |z|^2$

- $\arg(z^n) = n \arg(z)$

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

- $|z_1 + z_2| \leq |z_1| + |z_2|$

- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

- $|z_1 - z_2| \geq ||z_1| - |z_2||$

- $\left(\frac{\bar{z}_1}{z_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

- $|z_1 z_2| = |z_1| |z_2|$

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$

- If  $z = re^{i\theta}$  then  $z^n = r^n e^{in\theta}, n \in \mathbb{Q}$

(De'Movre's theorem)

- Cube roots of unity :**

$$1, \omega = \frac{-1 + \sqrt{3}i}{2}, \omega^2 = \frac{-1 - \sqrt{3}i}{2} \text{ are the cube roots of unity.}$$

- $1 + \omega + \omega^2 = 0, \omega^3 = 1.$
- In Argand diagram, the cube roots of unity are the vertices of an equilateral triangle.
- **Fourth roots of unity :**  
 $1, i, -1, -i$  are the fourth roots of unity.
- The  $n$ th roots of  $z = a + ib = r (\cos \theta + i \sin \theta)$  are  $r^{\frac{1}{n}} \left[ \cos \left( \frac{2k\pi + \theta}{n} \right) + i \sin \left( \frac{2k\pi + \theta}{n} \right) \right]$   
 where  $k = 0, 1, 2, \dots, n-1$
- $|z - z_0| = r$  represents a circle.  $z_0$  is the centre and  $r$  is the radius of the circle.
- If  $b^2 - 4ac < 0$ , for the quadratic equation  $ax^2 + bx + c = 0$ , where,  $a, b, c \in \mathbb{R}, a \neq 0$ , then the roots are  $\frac{-b \pm i\sqrt{4ac - b^2}}{2a}$ .
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\operatorname{Re}(z_1 \cdot \overline{z_2})$

If  $z = x + iy$ , then

$$\arg z = \begin{cases} \tan^{-1} \left( \frac{y}{x} \right), & x > 0, y \geq 0 \\ \tan^{-1} \left( \frac{y}{x} \right) + \pi, & x < 0, y \geq 0 \\ \tan^{-1} \left( \frac{y}{x} \right) - \pi, & x < 0, y < 0 \\ \frac{\pi}{2}, & x = 0, y > 0 \\ -\frac{\pi}{2}, & x = 0, y < 0 \end{cases}$$

If  $x = 0$  and  $y = 0$ , then  $\arg z$  is not defined.

If  $z_1, z_2$  are two fixed points in the Argand plane, then

- $|z - z_1| = |z - z_2| \Rightarrow$  Distances of  $z$  from  $z_1$  and  $z_2$  are equal.
- $|z - z_1| + |z - z_2| = |z_1 - z_2| \Rightarrow$  Locus of  $z$  is the line segment joining  $z_1$  and  $z_2$ .
- $|z - z_1| + |z - z_2| = \text{constant } (z_1 \neq z_2) (> |z_1 - z_2|) \Rightarrow$  Locus of  $z$  is an ellipse.
- $||z - z_1| - |z - z_2|| = |z_1 - z_2| \Rightarrow$  Locus of  $z$  is a straight line joining  $z_1$  and  $z_2$  but  $z$  does not lie between  $z_1$  and  $z_2$ .
- $||z - z_1| - |z - z_2|| = \text{constant } (z_1 \neq z_2) (< |z_1 - z_2|) \Rightarrow$  Locus of  $z$  is a hyperbola.
- $|z - z_1|^2 - |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow$  Locus of  $z$  is a circle with  $z_1$  and  $z_2$  as the extremities of a diameter.
- $|z - z_1| = k |z - z_2|, (k \neq 1) \Rightarrow$  Locus of  $z$  is a circle.
- $\arg \left( \frac{z - z_1}{z - z_2} \right) = \text{constant} \Rightarrow$  Locus of  $z$  is a segment of a circle.
- $\arg \left( \frac{z - z_1}{z - z_2} \right) = \pm \frac{\pi}{2} \Rightarrow$  Locus of  $z$  is a circle with  $z_1$  and  $z_2$  as the vertices of a diameter.

- $\arg \left( \frac{z-z_1}{z-z_2} \right) = 0 \text{ or } \pi \Rightarrow$  Locus of  $z$  is a straight line passing through  $z_1$  and  $z_2$ .
- If  $\left| z + \frac{1}{z} \right| = a$ , then the maximum and minimum values of  $|z|$  are  $\frac{a + \sqrt{a^2 + 4}}{2}$  and  $\frac{-a + \sqrt{a^2 + 4}}{2}$ .

If  $z_1, z_2, z_3$  are the vertices of  $\Delta ABC$  in the anticlockwise direction, then

$$AB = |z_1 - z_2|, AC = |z_3 - z_1| \text{ and}$$

$$BC = |z_3 - z_2| = |(z_3 - z_1) - (z_2 - z_1)|$$

$$\text{Also if } \arg \left( \frac{z_3 - z_1}{z_2 - z_1} \right) = \alpha \text{ and } \left| \frac{z_3 - z_1}{z_2 - z_1} \right| = \frac{AC}{AB},$$

$$\text{then } \frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB} (\cos \alpha + i \sin \alpha).$$

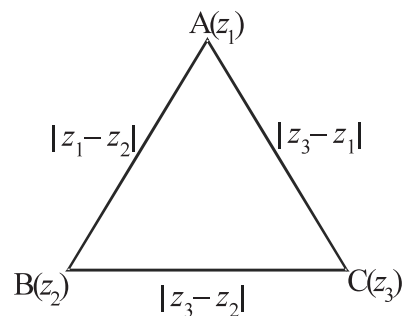


Figure 2.2

If  $z_1, z_2, z_3$  form an equilateral triangle, then  $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ .

### : Multiple Choice Questions :

- (1) If the biquadratic equation  $x^4 + ax^3 + bx^2 + cx + d = 0$  ( $a, b, c, d \in \mathbb{R}$ ) has no real roots and the sum of two of its roots is  $3 + 4i$  and the product of the other two roots is  $13 + i$ , then  $b = \dots$

- (A) 30 (B) 6 (C) 51 (D) 15

**Sol. :** Given equation is  $x^4 + ax^3 + bx^2 + cx + d = 0$ . (1)

Roots are  $\alpha \pm i\beta, \gamma \pm i\delta$

$\therefore$  Given equation is

$$(x - (\alpha + i\beta))(x - (\alpha - i\beta))(x - (\gamma + i\delta))(x - (\gamma - i\delta)) = 0$$

$$\therefore (x^2 - 2\alpha x + \alpha^2 + \beta^2)(x^2 - 2\gamma x + \gamma^2 + \delta^2) = 0 \quad (2)$$

We have to find the value of  $b$ . By comparing equation (1) and (2), the coefficient of  $x^2$  is  $b$ .

$$\therefore b = \alpha^2 + \beta^2 + 4\alpha\gamma + \gamma^2 + \delta^2 \quad (3)$$

Also given,  $\alpha + i\beta + \gamma + i\delta = 3 + 4i$

$$\therefore \alpha + \gamma + i(\beta + \delta) = 3 + 4i$$

$$\therefore \alpha + \gamma = 3 \text{ and } \beta + \delta = 4$$

$$\therefore \alpha^2 + \gamma^2 + 2\alpha\gamma = 9 \quad (4)$$

$$\text{and } \beta^2 + \delta^2 + 2\beta\delta = 16$$

$$\therefore \beta^2 + \delta^2 = 16 - 2\beta\delta \quad (5)$$

Substitute the values from the equations (4) and (5) in equation (3),

$$b = 9 + 2\alpha\gamma + 16 - 2\beta\delta \quad (6)$$

$$\text{Also, } (\alpha - i\beta)(\gamma - i\delta) = 13 + i. \text{ So, } (\alpha\gamma - \beta\delta) - i(\alpha\delta + \beta\gamma) = 13 + i$$

$$\therefore \alpha\gamma - \beta\delta = 13$$

$$\text{From equation (6), } b = 25 + 2(13) = 51$$

**Ans. : (C)**

- (2)  $z_1$  and  $z_2$  are distinct complex numbers and  $|z_1| = |z_2|$ . If the real part of  $z_1$  is a positive number and imaginary part of  $z_2$  is a negative number, then  $\frac{z_1 + z_2}{z_1 - z_2}$  is .....  $\left( \frac{z_1 + z_2}{z_1 - z_2} \neq 0 \right)$

- (A) purely imaginary (B) real and positive (C) real and negative (D) zero

**Sol. :** Let  $z_1 = a + ib$  and  $z_2 = c - id$ , where  $a > 0, d > 0$

$$|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2 \quad (1)$$

$$\text{Now, } \frac{z_1 + z_2}{z_1 - z_2} = \frac{(a+ib)+(c-id)}{(a+ib)-(c-id)} = \frac{[(a+c)+i(b-d)]}{[(a-c)+i(b+d)]}$$

$$= \frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$$

$$= \frac{(a^2 + b^2) - (c^2 + d^2) - 2i(ad + bc)}{a^2 + c^2 - 2ac + b^2 + d^2 + 2bd}$$

$$= \frac{-(ad + bc)i}{a^2 + b^2 - ac + bd} \quad (\text{Using (1)})$$

$\therefore \frac{z_1 + z_2}{z_1 - z_2}$  is a purely imaginary number.

**Ans. : (A)**

(3) If  $|z| = 1$  and  $z^{2n} + 1 \neq 0$ , then  $\frac{z^n}{z^{2n} + 1} - \frac{z^{-n}}{(\bar{z})^{2n} + 1} = \dots$

(A) 3 (B) 0 (C)  $i$  (D) 1

**Sol. :** Let  $z = \cos\theta + i \sin\theta$

$$\begin{aligned} \therefore \frac{z^n}{z^{2n} + 1} - \frac{z^{-n}}{(\bar{z})^{2n} + 1} &= \frac{\cos n\theta + i \sin n\theta}{1 + \cos 2n\theta + i \sin 2n\theta} - \frac{\cos n\theta - i \sin n\theta}{1 + \cos 2n\theta - i \sin 2n\theta} \\ &= \frac{\cos n\theta + i \sin n\theta}{2 \cos n\theta (\cos n\theta + i \sin n\theta)} - \frac{\cos n\theta - i \sin n\theta}{2 \cos n\theta (\cos n\theta - i \sin n\theta)} \\ &= \frac{1}{2 \cos n\theta} - \frac{1}{2 \cos n\theta} = 0 \end{aligned}$$

**Ans. : (B)**

**Another method :**  $|z| = 1 \Rightarrow z\bar{z} = 1$

$$\frac{z^n}{z^{2n} + 1} - \frac{z^{-n}}{(\bar{z})^{2n} + 1} = \frac{z^n}{z^{2n} + 1} - \frac{z^{-n}}{(\frac{1}{z})^{2n} + 1} = \frac{z^n}{z^{2n} + 1} - \frac{z^{2n} \cdot z^{-n}}{1 + z^{2n}} = \frac{z^n}{z^{2n} + 1} - \frac{z^n}{z^{2n} + 1} = 0$$

(4) The locus of the complex numbers satisfying the inequality  $\log_{\frac{1}{2}} \left( \frac{|z-1|+4}{3|z-1|-2} \right) > 1$

(where  $|z-1| \neq \frac{2}{3}$ ) is .....

(A) a circle (B) interior of a circle (C) exterior of a circle (D) a line

**Sol. :**  $\log_{\frac{1}{2}} \left( \frac{|z-1|+4}{3|z-1|-2} \right) > 1 = \log_{\frac{1}{2}} \left( \frac{1}{2} \right)$

$$\therefore \left( \frac{|z-1|+4}{3|z-1|-2} \right) < \frac{1}{2} \quad (\text{For } a < 1, \log_a x \text{ is a decreasing function})$$

$$\therefore 2|z-1| + 8 < 3|z-1| - 2$$

$$\therefore |z-1| > 10 \text{ which represents the exterior of a circle.}$$

**Ans. : (C)**

(5) If  $\alpha$  and  $\beta$  are the roots of  $x^2 - 2x + 2 = 0$ , then  $\alpha^n + \beta^n = \dots$

(A)  $2^{\frac{n}{2}} \cos \frac{n\pi}{4}$  (B)  $2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}$  (C)  $2^{\frac{n}{2}-1} \cos \frac{n\pi}{4}$  (D) 0

**Sol. :**  $x^2 - 2x + 2 = 0$

$$\therefore x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\therefore$  Let  $\alpha = 1 + i$ ,  $\beta = 1 - i$ ,

$$\alpha^n + \beta^n = (1 + i)^n + (1 - i)^n$$

$$= \left[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n + \left[ \sqrt{2} \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \right]^n$$

$$= 2^{\frac{n}{2}} \left[ \cos \left( \frac{n\pi}{4} \right) + i \sin \left( \frac{n\pi}{4} \right) + \cos \left( \frac{n\pi}{4} \right) - i \sin \left( \frac{n\pi}{4} \right) \right] = 2^{\frac{n}{2}} \cdot 2 \cos \left( \frac{n\pi}{4} \right) = 2^{\frac{n}{2}+1} \cos \left( \frac{n\pi}{4} \right)$$

**Ans. :** (B)

(6) If  $x^2 - x\sqrt{3} + 1 = 0$ , then  $\sum_{n=1}^{36} \left( x^n - \frac{1}{x^n} \right)^2 = \dots\dots$

(A) 0

(B) -72

(C) 72

(D) 36

**Sol. :**  $x^2 - x\sqrt{3} + 1 = 0$

$$x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2}$$

$\therefore$  Let  $x = \frac{\sqrt{3}}{2} + \frac{i}{2}$ . So,  $x = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$

$$\therefore x^{2n} = \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} \text{ and } x^{-2n} = \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3}$$

$$\therefore \left( x^n - \frac{1}{x^n} \right)^2 = x^{2n} + \frac{1}{x^{2n}} - 2 = x^{2n} + x^{-2n} - 2 = 2 \cos \frac{n\pi}{3} - 2$$

Now,  $\sum_{n=1}^{36} \left( x^n - \frac{1}{x^n} \right)^2 = \sum_{n=1}^{36} \left( -2 + 2 \cos \frac{n\pi}{3} \right)$

$$= -2 \times 36 + 2 \left[ \cos \frac{\pi}{3} + \cos \frac{2\pi}{3} + \dots + \cos \frac{36\pi}{3} \right]$$

$$= -72 + 2 \frac{\left[ \cos \left( \frac{\pi}{3} + \frac{35\pi}{6} \right) \sin \left( \frac{36\pi}{3 \times 2} \right) \right]}{\sin \frac{\pi}{6}}$$

$$\left( \cos \alpha + \cos (\alpha + \beta) + \dots \text{ upto } n \text{ terms} = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left( \alpha + \frac{(n-1)\beta}{2} \right) \right)$$

$$= -72 + 0$$

$$= -72$$

**Ans. :** (B)

(7) If  $a$  and  $b$  are the roots of  $x^2 + x + 1 = 0$ , then the equation whose roots are  $a^{19}$  and  $b^7$  is  $\dots\dots$

(A)  $x^2 - x + 1 = 0$

(B)  $x^2 - x - 1 = 0$

(C)  $x^2 + x - 1 = 0$

(D)  $x^2 + x + 1 = 0$

**Sol. :**  $a$  and  $b$  are the roots of  $x^2 + x + 1 = 0$ . So,  $a = \omega$  and  $b = \omega^2$ , where  $\omega = \frac{-1 + \sqrt{3}i}{2}$

Now,  $a^{19} = w^{19} = w^{18} w = w$ ,  $b^7 = (w^2)^7 = w^{14} = w^2$

and required equation is  $x^2 - (w + w^2)x + w \cdot w^2 = 0$

i.e.  $x^2 + x + 1 = 0$   $(w + w^2 = -1, w \cdot w^2 = w^3 = 1)$  **Ans. : (D)**

- (8) When  $f(z)$  is divided by  $z - i$ , the remainder is  $i$  and if  $f(z)$  is divided by  $z + i$ , the remainder is  $1 + i$ . If  $f(z)$  is divided by  $z^2 + 1$ , then the remainder is .....

(A)  $\frac{1}{2}(iz + 1)$  (B)  $\frac{1}{2}(iz + 1 + 2i)$  (C)  $iz + 1 + i$  (D) 0

**Sol. :** The remainder obtained while dividing  $f(z)$  by  $z - i$  is  $i$ .

$$\therefore f(i) = i. \quad (1)$$

$$\text{Similarly, } f(-i) = 1 + i. \quad (2)$$

$z^2 + 1$  is a quadratic polynomial. When  $f(z)$  is divided by  $z^2 + 1$ , we get remainder of the form  $az + b$   $(a, b \in \mathbb{C})$

$$\text{Let } f(z) = g(z)(z^2 + 1) + az + b \quad (3)$$

$$\therefore f(i) = g(i)(i^2 + 1) + ai + b$$

$$\therefore i = ai + b \quad (4)$$

$$\text{and } f(-i) = g(-i)((-i)^2 + 1) + a(-i) + b$$

$$1 + i = -ai + b \quad (5)$$

Solving (4) and (5), we get  $b = \frac{1}{2} + i$ ,  $a = \frac{i}{2}$

$\therefore$  The required remainder is  $az + b = \frac{1}{2} iz + \frac{1}{2} + i = \frac{1}{2}(iz + 1 + 2i)$  **Ans. : (B)**

- (9) For  $a = e^{\frac{i2\pi}{13}}$ ,  $\alpha = a + a^3 + a^4 + a^{-4} + a^{-3} + a^{-1}$  and  $\beta = a^2 + a^5 + a^6 + a^{-6} + a^{-5} + a^{-2}$  are the roots of the quadratic equation, .....

(A)  $x^3 - x - 3 = 0$  (B)  $x^2 - x + 2 = 0$  (C)  $x^2 + x + 3 = 0$  (D)  $x^2 + x - 3 = 0$

**Sol. :** The sum of the roots,

$$\begin{aligned} \alpha + \beta &= (a + a^3 + a^4 + a^{-4} + a^{-3} + a^{-1}) + (a^2 + a^5 + a^6 + a^{-6} + a^{-5} + a^{-2}) \\ &= (a + a^3 + a^4 + a^9 + a^{10} + a^{12}) + (a^2 + a^5 + a^6 + a^7 + a^8 + a^{11}) \quad (a^{13} = 1) \\ &= a + a^2 + a^3 + \dots + a^{12} \end{aligned}$$

$$= \frac{a(1 - a^{12})}{1 - a} = \frac{a(1 - a^{13} a^{-1})}{1 - a} = \frac{a(1 - a^{-1})}{1 - a} = -1 \quad (a^{13} = 1)$$

The product of the roots,

$$\begin{aligned} \alpha \beta &= (a + a^3 + a^4 + a^{-4} + a^{-3} + a^{-1})(a^2 + a^5 + a^6 + a^{-6} + a^{-5} + a^{-2}) \\ &= (a + a^3 + a^4 + a^9 + a^{10} + a^{12})(a^2 + a^5 + a^6 + a^7 + a^8 + a^{11}) \\ &= a^3(1 + a^2 + a^3 + a^8 + a^9 + a^{11})(1 + a^3 + a^4 + a^5 + a^6 + a^9) \\ &= a^3[1 + a^2 + 2(a^3 + a^5 + a^7 + a^9 + a^{13} + a^{15} + a^{17}) + a^4 + 3(a^6 + a^8 + a^{11} + a^{12} + a^{14}) + (a^2 + a^{25} + a^{18} + a^{20})] \\ &= 3(a + a^2 + a^3 + \dots + a^{12}) = 3(-1) = -3 \end{aligned}$$

$\therefore$  The required quadratic equation is  $x^2 - (-1)x - 3 = 0$

$$x^2 + x - 3 = 0$$

**Ans. : (D)**

(10) If  $\left| \frac{1-iz}{z-i} \right| = 1$ , then .....

(A)  $z$  is pure imaginary

(B)  $z$  is pure real.

(C)  $P(z)$  lies in the second quadrant

(D)  $P(z)$  lies in the third quadrant

**Sol. :** Let  $z = x + iy$ ,

$$|1 - iz|^2 = |z - i|^2$$

$$\therefore |1 - i(x + iy)|^2 = |x + iy - i|^2$$

$$\therefore |1 - ix + y|^2 = |x + i(y-1)|^2$$

$$\therefore (1 + y)^2 + x^2 = x^2 + (y-1)^2$$

$$\therefore y = 0$$

$\therefore z$  is purely real.

**Ans. :** (B)

(11) If  $z \neq 0$ , upper bound of  $|z^i|$  is .....

(A)  $|z|$

(B) 1

(C)  $e^{-\pi}$

(D)  $e^{\pi}$

**Sol. :** Let  $z = re^{i\theta}$ ,  $-\pi < \theta \leq \pi$ .

$$\therefore z^i = r^i e^{-\theta} = e^{\log r^i} \cdot e^{-\theta}$$

$$= (\cos(\log r) + i \sin(\log r)) e^{-\theta}$$

$$\therefore |z^i| = |e^{-\theta}| < e^{\pi} \quad -\pi < \theta \leq \pi, \quad -\pi \leq -\theta < \pi$$

$\therefore |z^i|$  has the upper bound  $e^{\pi}$ .

**Ans. :** (D)

(12) If  $z$  is a complex number and  $|z| \geq 2$ , then the minimum value of  $\left| z + \frac{1}{2} \right|$

(A) lies in (1, 2)

(B) is greater than  $\frac{5}{2}$

(C) lies between  $\frac{3}{2}$  and  $\frac{5}{2}$

(D) is  $\frac{5}{2}$

**Sol. : Method 1 :**

Geometrically,  $|z| = 2$  is a circle with centre (0, 0) and radius 2.

According to the figure 2.3,  $|z| \geq 2$  consists of points on the circle and exterior region of the circle.

$\left| z + \frac{1}{2} \right| = \left| z - \left( -\frac{1}{2} \right) \right|$ . So,  $\left( -\frac{1}{2}, 0 \right)$  lies inside the circle. Minimum

distance between this point and  $(-2, 0)$  is  $\frac{3}{2}$ .

**Method 2 :**

$$\left| z + \frac{1}{2} \right| = \left| z - \left( -\frac{1}{2} \right) \right| \geq \left| |z| - \left| -\frac{1}{2} \right| \right| = \left| |z| - \frac{1}{2} \right|$$

Minimum value  $|z|$  for  $|z| \geq 2$  is  $|z| = 2$ .

$$\therefore \left| z + \frac{1}{2} \right|_{\min} = \left| 2 - \frac{1}{2} \right| = \frac{3}{2}$$

**Ans. :** (A)

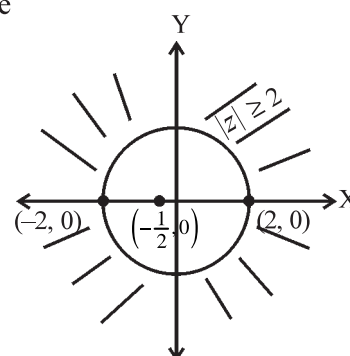


Figure 2.3

(13) How many complex numbers satisfy the equation  $\bar{z} = z^2$  ?

(A) 1

(B) 2

(C) 3

(D) 4

**Sol. :** Let  $z = x + iy$

Now  $\bar{z} = z^2$

$$\therefore x - iy = (x^2 - y^2) + i(2xy)$$

$$\therefore x = x^2 - y^2 \text{ and } -y = 2xy$$

From the second equation, we get  $y = 0$  or  $x = -\frac{1}{2}$

If  $y = 0$ ,  $x = x^2 - y^2$  gives  $x = x^2$

$$\therefore x = 0 \text{ or } x = 1$$

$$\therefore z = 0 \text{ or } z = 1$$

Now If  $x = -\frac{1}{2}$ , then  $x = x^2 - y^2$  gives  $-\frac{1}{2} = \frac{1}{4} - y^2$ . So,  $y = \pm \frac{\sqrt{3}}{2}$

$$\text{Hence } z = -\frac{1}{2} + \frac{\sqrt{3}i}{2} \text{ or } z = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

There are four complex numbers satisfying the given condition.

**Ans. :** (D)

(14) If  $|z|^2 \omega - |\omega|^2 z = z - \omega$  for distinct non-zero complex numbers  $z$  and  $\omega$ , then .....

(A)  $z = -\omega$

(B)  $z = \bar{\omega}$

(C)  $z\bar{\omega} = 1$

(D)  $z\omega = 1$

**Sol. :** By rearranging the terms of  $|z|^2 \omega - |\omega|^2 z = z - \omega$ ,

$$\frac{z}{\omega} = \frac{1 + |z|^2}{1 + |\omega|^2} \quad (1)$$

$\therefore \frac{z}{\omega}$  is a purely real number. Let  $\frac{z}{\omega} = k$  where  $k \in \mathbb{R}$

$$\therefore \frac{k\omega}{\omega} = \frac{1 + k^2 |\omega|^2}{1 + |\omega|^2} \quad (\text{from (1)})$$

$$\therefore k + k|\omega|^2 = 1 + k^2 |\omega|^2$$

$$\therefore k - 1 = k |\omega|^2 (k - 1)$$

$$\therefore (k|\omega|^2 - 1)(k - 1) = 0$$

$$\therefore k = 1 \text{ or } k = \frac{1}{|\omega|^2}$$

$$\therefore \frac{z}{\omega} = 1 \text{ or } \frac{z}{\omega} = \frac{1}{|\omega|^2} = \frac{1}{\omega\bar{\omega}}$$

$$z\bar{\omega} = 1$$

$$(z \neq \omega, \omega \neq 0)$$

**Ans. :** (C)

(15) If  $\left|z - \frac{4}{z}\right| = 2$ , then the difference between the maximum and the minimum values of  $|z|$  is .....

( $z \neq 0$ )

(A) 1

(B) 2

(C) 3

(D) 4

**Sol. :**  $\left|z - \frac{4}{z}\right| \geq \left||z| - \frac{4}{|z|}\right|$

$$\therefore \left||z| - \frac{4}{|z|}\right| \leq \left|z - \frac{4}{z}\right|$$

$$\therefore \left| |z| - \frac{4}{|z|} \right| \leq 2$$

$$\therefore -2 \leq |z| - \frac{4}{|z|} \leq 2$$

$$\therefore |z|^2 + 2|z| - 4 \geq 0 \text{ and } |z|^2 - 2|z| - 4 \leq 0$$

$$(|z|+1)^2 \geq 5 \text{ and } (|z|-1)^2 \leq 5$$

$$|z|+1 \geq \sqrt{5} \text{ and } |z|-1 \leq \sqrt{5}. \text{ But } |z|+1 = |z| + 1$$

$$\therefore |z|+1 \geq \sqrt{5} \text{ and } -\sqrt{5} \leq |z|-1 \leq \sqrt{5}$$

$$\therefore |z| \geq \sqrt{5}-1 \text{ and } 1-\sqrt{5} \leq |z| \leq \sqrt{5}+1$$

$$\sqrt{5}-1 \leq |z| \leq \sqrt{5}+1$$

$$\therefore \text{The maximum value of } |z| - \text{the minimum value of } |z| = (\sqrt{5}+1) - (\sqrt{5}-1) = 2$$

$$\therefore \text{The difference between the maximum and the minimum values of } |z| \text{ is } 2. \quad \text{Ans. : (B)}$$

(16) If  $z$  is a non-real complex number lying on the circle  $|z| = 1$ , then  $z = \dots$

$$(A) \frac{1-itan\left(\frac{\arg z}{2}\right)}{1+itan\left(\frac{\arg z}{2}\right)} \quad (B) \frac{1+itan\left(\frac{\arg z}{2}\right)}{1-itan\left(\frac{\arg z}{2}\right)} \quad (C) \frac{1-itan(\arg z)}{1+itan(\arg z)} \quad (D) \frac{1+itan(\arg z)}{1-itan(\arg z)}$$

**Sol. :**  $|z| = 1$ . So,  $z = \cos \alpha + i \sin \alpha$  So,  $\arg(z) = \alpha$

$$\therefore z = \frac{1-\tan^2 \frac{\alpha}{2}}{1+\tan^2 \frac{\alpha}{2}} + i \frac{2\tan \frac{\alpha}{2}}{1+\tan^2 \frac{\alpha}{2}}$$

$$= \frac{\left(1-\tan^2 \frac{\alpha}{2}\right) + i 2 \tan \frac{\alpha}{2}}{1+\tan^2 \frac{\alpha}{2}}$$

$$= \frac{\left(1+itan \frac{\alpha}{2}\right)^2}{\left(1+itan \frac{\alpha}{2}\right)\left(1-itan \frac{\alpha}{2}\right)} = \frac{1+itan \frac{\alpha}{2}}{1-itan \frac{\alpha}{2}}$$

$$= \frac{1+itan\left(\frac{\arg z}{2}\right)}{1-itan\left(\frac{\arg z}{2}\right)}$$

**Ans. : (B)**

(17)  $\cos y \sin y + \cos^2 y \sin 2y + \cos^3 y \sin 3y + \dots n \text{ terms} = \dots$

$$(A) \tan y (1 - \sin^n y \sin ny)$$

$$(B) \tan y (1 - \cos^n y \cos ny)$$

$$(C) \cot y (1 - \sin^n y \sin ny)$$

$$(D) \cot y (1 - \cos^n y \cos ny)$$

**Sol. :** Let  $\cos y = a$

$$\text{Let } S = a \sin y + a^2 \sin 2y + a^3 \sin 3y + \dots n \text{ terms}$$

$$\text{Let } C = a \cos y + a^2 \cos 2y + a^3 \cos 3y + \dots n \text{ terms}$$

$$\therefore C+iS = ae^{iy} + a^2 e^{i2y} + a^3 e^{i3y} + \dots n \text{ terms}$$

$$= \frac{ae^{iy} [1 - a^n e^{iny}]}{1 - ae^{iy}}$$

$$= \frac{ae^{iy} [1 - a^n e^{iny}] [1 - ae^{-iy}]}{1 - a(e^{iy} + e^{-iy}) + a^2 e^{iy} e^{-iy}}$$

$$= \frac{(ae^{iy} - a^2) [1 - a^n e^{iny}]}{1 - a(2 \cos y) + a^2}$$

$$\therefore C + iS = [\cos y (\cos y + i \sin y) - \cos^2 y] \cdot \frac{[1 - \cos^n y (\cos ny + i \sin ny)]}{1 - 2 \cos^2 y + \cos^2 y}$$

( $a = \cos y$ , and  $e^{i\theta} = \cos \theta + i \sin \theta$ )

$$= \frac{i \sin y \cos y}{\sin^2 y} [1 - \cos^n y \cos ny - i \cos^n y \sin ny]$$

S = imaginary part of the expression.

$$= \cot y (1 - \cos^n y \cos ny)$$

**Ans. : (D)**

- (18) The sum of the imaginary parts of the complex numbers satisfying both the conditions  $\left| \frac{z-12}{z-8i} \right| = \frac{5}{3}$  and  $\left| \frac{z-4}{z-8} \right| = 1$  is .....

(A) 25

(B) 28

(C) 32

(D) 35

**Sol. :** It is obvious that, the first equation represents a circle and the second equation represents the perpendicular bisector of the line segment joining the points (4, 0) and (8, 0).

$\therefore z$  is in the form  $6 + yi$ .

$$\left| \frac{z-12}{z-8i} \right| = \frac{5}{3}$$

$$\therefore \left| \frac{-6+yi}{6+(y-8)i} \right| = \frac{5}{3}$$

$$\therefore 9(6^2 + y^2) = 25(6^2 + (y-8)^2)$$

$$\therefore 16y^2 - 400y + 2176 = 0$$

$$\therefore y^2 - 25y + 136 = 0$$

$$\therefore y = 8 \text{ or } y = 17$$

$$\therefore z = 6+8i \text{ or } 6+17i$$

$$\therefore \text{The sum of the imaginary parts} = 8+17 = 25$$

**Ans. : (A)**

- (19) If  $f(x) = x^4 + ax^3 + bx^2 + cx + d$  with  $f(2i) = f(2+i) = 0$  is a polynomial with real coefficients, then  $a + b + c + d = \dots$

(A) 1

(B) 4

(C) 9

(D) 10

**Sol. :** We know that, if  $z = a + ib$  is a zero of the polynomial with real coefficients, then  $a - ib$  is also zero of the same polynomial.

Zeros of the given polynomial  $f(x)$  are  $2i, -2i, 2+i, 2-i$ .

$$\therefore f(x) = (x + 2i)(x - 2i)(x - 2-i)(x - 2 + i)$$

$$f(1) = (1 + 2i)(1 - 2i)(1 - 2 - i)(1 - 2 + i) = 5 \times 2 = 10$$

$$\therefore 1 + a + b + c + d = 10$$

$$\therefore a + b + c + d = 9$$

**Ans. : (C)**

- (20) If  $\omega$  is a cube root of 1, then  $2(1 + \omega)(1 + \omega^2) + 3(2\omega + 1)(2\omega^2 + 1) + \dots + (n+1)(n\omega + 1)(n\omega^2 + 1)$   
 $= \dots$  ( $\omega \neq 1$ )

(A)  $\frac{n^2(n+1)^2}{4}$  (B)  $\frac{n^2(n+1)^2}{4} + n$  (C)  $\frac{n^2(n+1)^2}{4} - n$  (D)  $\frac{n^2(n+1)}{2} + n$

**Sol. :**  $z = \sum_{r=1}^n (r+1)(1+r\omega)(1+r\omega^2)$

$$= \sum_{r=1}^n (r+1) [1 + r(\omega + \omega^2) + r^2 \omega^3]$$

$$= \sum_{r=1}^n (r+1) (1 - r + r^2) \quad (\omega^3 = 1, 1 + \omega + \omega^2 = 0)$$

$$= \sum_{r=1}^n (r^3 + 1) = \frac{n^2(n+1)^2}{4} + n$$

**Ans. : (B)**

- (21) If the complex number  $z$  ( $z \neq 2$ ) satisfies the equation  $z^2 = 4z + |z|^2 + \frac{16}{|z|^3}$ , then the value of  $|z|^4$  is .....

(A) 1 (B) 2 (C) 3 (D) 4

**Sol. :**  $z = re^{i\theta}$  implies  $\frac{1}{z} = \frac{1}{r} e^{-i\theta}$ . So,  $r = |z|$

Substitute the value of  $z$  and  $\frac{1}{z}$  in the given equation,  $z^2 = 4z + |z|^2 + \frac{16}{|z|^3}$

$$r^2 e^{2i\theta} = 4re^{i\theta} + r^2 + \frac{16}{r^3}$$

Comparing the real and imaginary parts, we get,

$$r^2 \cos 2\theta = 4r \cos \theta + r^2 + \frac{16}{r^3} \quad (1)$$

$$r^2 \sin 2\theta = 4r \sin \theta \quad (2)$$

$$\therefore 2r^2 \sin \theta \cos \theta - 4r \sin \theta = 0$$

$$\therefore r \sin \theta = 0 \text{ or } r \cos \theta = 2$$

**Case 1 :**  $r \sin \theta = 0$ . So,  $\sin \theta = 0$  ( $r \neq 0$ )

$$\therefore \cos \theta = \pm 1$$

Substitute  $\cos \theta = 1$  in equation (1),

$$r^2 (2\cos^2 \theta - 1) = 4r \cos \theta + r^2 + \frac{16}{r^3}$$

$$r^2 = 4r + r^2 + \frac{16}{r^3}$$

$$\therefore 4r + \frac{16}{r^3} = 0$$

$$\therefore 4r = \frac{-16}{r^3}$$

$$\therefore r^4 = -4 \text{ is not possible, as } r > 0.$$

**Case 2 :** Substitute  $\cos \theta = -1$  in equation (1),

$$r^4 = 4$$

$$\text{So, } |z|^4 = 4$$

**Case 3 :**  $r \cos \theta = 2$

From equation (1), since  $r > 0$ ,  $r^5 = -8$  is not possible. Since  $r = |z|$

$$\therefore r^4 = 4$$

$$\therefore |z|^4 = 4$$

**Ans. :** (D)

(22) If  $|u| < 1$ ,  $|v| < 1$  and  $z = \frac{u-v}{1-uv}$ , a lower bound of  $|z|$  is .....

(A)  $\frac{|u|-|v|}{1+|u||v|}$

(B)  $\frac{|u|+|v|}{1-|u||v|}$

(C)  $\frac{\|u|-|v\|}{1+|u||v|}$

(D)  $\frac{|u|+|v|}{1-|u||v|}$

**Sol. :**  $z = \frac{u-v}{1-uv}$

$$|u-v| > ||u| - |v|| \quad (1)$$

Let  $u = r e^{i\theta}$ ,  $v = \rho e^{i\phi}$ ;  $r, \rho < 1$

$$\therefore |u| = r, |v| = \rho$$

$$\begin{aligned} |1-uv|^2 &= |1-r\rho e^{i(\theta+\phi)}|^2 = |1-r\rho \cos(\theta+\phi) - i r\rho \sin(\theta+\phi)|^2 \\ &= 1 + r^2 \rho^2 \cos^2(\theta+\phi) + r^2 \rho^2 \sin^2(\theta+\phi) - 2r\rho \cos(\theta+\phi) \\ &= 1 + r^2 \rho^2 - 2r\rho \cos(\theta+\phi) \\ &\leq (1+r\rho)^2 = (1+|u||v|)^2 \quad (\cos(\theta+\phi) \geq -1) \end{aligned}$$

$$\therefore |1-uv| \leq (1+|u||v|) \quad (2)$$

From (1) and (2),

$$|z| = \left| \frac{u-v}{1-uv} \right| \geq \frac{\|u|-|v\|}{1+|u||v|}$$

**Second method :**

$$|u-v| \geq ||u| - |v||$$

$$\therefore |1-uv| = |1+(-uv)| \leq |1| + |-uv| = 1 + |u||v|$$

$$\therefore \frac{1}{|1-uv|} \geq \frac{1}{1+|u||v|}$$

$$\therefore |z| = \left| \frac{u-v}{1-uv} \right| \geq \frac{\|u|-|v\|}{1+|u||v|}$$

**Ans. :** (C)

(23) If the imaginary part of  $\frac{z-1}{e^{i\theta}} + \frac{e^{i\theta}}{z-1}$  is zero and  $\frac{z-1}{e^{i\theta}}$  is non-real, then  $z$  lies on .....

(A) a line

(B) a parabola

(C) a circle

(D) an ellipse

**Sol. :** Imaginary part of  $u + \frac{1}{u}$  is zero, where  $u = \frac{z-1}{e^{i\theta}}$

Now, if the imaginary part of complex number  $z$  is 0, then  $z - \bar{z} = 2iy = 0$

$$\therefore \left(u + \frac{1}{u}\right) - \overline{\left(u + \frac{1}{u}\right)} = 0$$

$$\therefore \left(u + \frac{1}{u}\right) - \left(\bar{u} + \frac{1}{\bar{u}}\right) = 0$$

$$\therefore (u - \bar{u}) + \left(\frac{1}{u} - \frac{1}{\bar{u}}\right) = 0$$

$$\therefore (u - \bar{u}) - \left( \frac{u - \bar{u}}{u\bar{u}} \right) = 0$$

$$\therefore (u - \bar{u}) \left( 1 - \frac{1}{|u|^2} \right) = 0 \quad (|u|^2 = u\bar{u})$$

$$\therefore |u|^2 = 1 \quad (u \neq \bar{u})$$

$$\therefore \left| \frac{z-1}{e^{i\theta}} \right|^2 = 1$$

$$\therefore |z-1|^2 = 1 \quad (|e^{i\theta}| = 1)$$

$$\therefore |z-1| = 1$$

$\therefore z$  lies on a circle.

**Ans. : (C)**

(24) The roots of the equation  $z^n = (z+1)^n$  .....

(A) lie on a circle with radius  $\frac{1}{2}$

(B) lie on the line  $2x+1=0$

(C) lie on a regular polygon with  $n$  sides

(D) lie on a regular polygon with  $2n$  sides

**Sol. :**  $z^n = (z+1)^n$

$$\therefore \left( 1 + \frac{1}{z} \right)^n = 1$$

$$\therefore 1 + \frac{1}{z} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \dots, n-1$$

$\therefore$  For  $k=0$ ,  $\frac{1}{z} = 0$  is not possible.

$$\therefore z = \frac{1}{-1 + \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}}, \quad k = 1, 2, \dots, n-1$$

$$= \frac{-1 + \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n}}{\left( -1 + \cos \frac{2k\pi}{n} \right)^2 + \sin^2 \frac{2k\pi}{n}}$$

$$= \frac{-\left( 1 - \cos \frac{2k\pi}{n} \right) - i \sin \frac{2k\pi}{n}}{2 \left( 1 - \cos \frac{2k\pi}{n} \right)} = \frac{-1}{2} - \frac{i \sin \frac{2k\pi}{n}}{2 \left( 1 - \cos \frac{2k\pi}{n} \right)} = \frac{-1}{2} - \left( \frac{-i}{2} \cot \frac{k\pi}{n} \right)$$

$$\therefore \operatorname{Re}(z) = \frac{-1}{2} \text{ gives } x = \frac{-1}{2}$$

The roots of the equation  $z^n = (z+1)^n$  lie on the line  $2x+1=0$ .

**Ans. : (B)**

(25) If  $z_1$  and  $z_2$  are two complex numbers such that  $\left| \frac{z_1 - 2z_2}{2 - z_1 z_2} \right| = 1$ , where  $|z_2| \neq 1$ , then point  $z_1$  lies .....

(JEE 2015)

(A) on a circle with radius 2

(B) on a circle with radius  $\sqrt{2}$

(C) on X-axis

(D) on Y-axis

**Sol. :** We prove if  $\left| \frac{\alpha - \beta}{1 - \alpha \bar{\beta}} \right| = 1$ ,  $|\beta| \neq 1$ , then  $|\alpha| = 1$

$$|\alpha - \beta| = |1 - \alpha \bar{\beta}|$$

$$\therefore |\alpha - \beta|^2 = |1 - \alpha \bar{\beta}|^2$$

$$\therefore |\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\alpha \bar{\beta}) = 1 + |\alpha|^2 |\beta|^2 - 2\operatorname{Re}(\alpha \bar{\beta})$$

$$\therefore 1 - |\alpha|^2 - |\beta|^2 + |\alpha|^2 |\beta|^2 = 0$$

$$\therefore (1 - |\alpha|^2)(1 - |\beta|^2) = 0$$

But  $|\beta| \neq 1$ . So,  $|\alpha| = 1$

Taking,  $\alpha = \frac{z_1}{2}$  and  $\beta = z_2$ ,  $\left| \frac{z_1 - 2z_2}{2 - z_1 z_2} \right| = 1$ . Since  $|z_2| \neq 1$ ,  $|z_1| = 2$ .

$\therefore z_1$  lies on a circle with radius 2.

**Ans. : (A)**

(26) If  $x_1 \leq x_2$  and  $y_1 \leq y_2$  for any two complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , then we say  $z_1 \cap z_2$ .

Let  $z$  be a complex number with  $1 \cap z$ . Then

$$(A) \frac{1-z}{1+z} \cap -i \quad (B) 1 \cap \frac{1-z}{1+z} \quad (C) \frac{1-z}{1+z} \cap 0 \quad (D) \frac{1+z}{1-z} \cap 0$$

**Sol. :** For each  $z = x + iy$ .  $1 \cap z$  such that  $1 \leq x$  and  $0 \leq y$ .

$$\begin{aligned} \text{Now, } \frac{1-z}{1+z} &= \frac{(1-x)-iy}{(1+x)+iy} = \frac{[(1-x)-iy][(1+x)-iy]}{(1+x)^2 + y^2} \\ &= \frac{1-(x^2+y^2)-2iy}{(1+x)^2 + y^2} \end{aligned}$$

$$\text{Since } x \geq 1 \text{ and } y \geq 0, \quad \frac{1-(x^2+y^2)}{(1+x)^2 + y^2} \leq 0 \text{ and } \frac{-2y}{(1+x)^2 + y^2} \leq 0$$

$$\therefore \frac{1-z}{1+z} \cap 0$$

**Ans. : (C)**

(27) The region of Argand diagram defined by  $|z - i| + |z + i| \leq 4$  is .....

- (A) interior of an ellipse
- (B) exterior of a circle
- (C) interior and boundary of an ellipse
- (D) interior and boundary of a circle.

**Sol. :** A and B are  $0 + i$  and  $0 - i$  in the diagram.

Let  $P(z)$  be any point in the plane.

$$|z - i| + |z + i| \leq 4 \Rightarrow PA + PB \leq 4$$

P lies on the ellipse whose foci are A and B

or it lies in the interior of the ellipse.

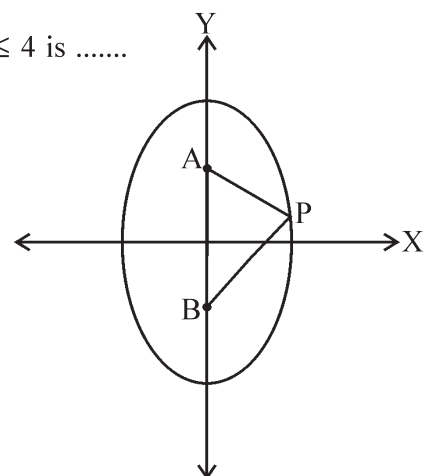


Figure 2.4

**Ans. : (C)**

(28) Among the following conditions which one is not true if the binomial coefficients in the expansion of  $(1+x)^n$  are  $c_0, c_1, c_2, \dots, c_n$ ?

(A)  $c_0 - c_2 + c_4 - c_6 + \dots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$

(B)  $c_1 - c_3 + c_5 - \dots = 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$

(C)  $c_1 + c_5 + c_9 + c_{13} + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right)$

(D)  $c_1 + c_5 + c_9 + c_{13} + \dots = \left( 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right)$

**Sol. :**  $(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$  (n+1) terms (1)

Substituting  $x = i$  in (1)

$(1+i)^n = c_0 + c_1i - c_2 - c_3i + c_4 + \dots$  (n+1) terms (2)

Substituting  $x = -i$  in (1)

$(1-i)^n = c_0 - c_1i - c_2 + c_3i + c_4 + \dots$  (n+1) terms (3)

Adding (2) and (3), we get,

$2(c_0 - c_2 + c_4 - c_6 + \dots) = (1+i)^n + (1-i)^n$

$\therefore c_0 - c_2 + c_4 - c_6 + \dots = \frac{(1+i)^n + (1-i)^n}{2}$

$$= 2^{\frac{n}{2}} \left( \frac{\left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^n + \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)^n}{2} \right)$$

$$= 2^{\frac{n}{2}} \left( \frac{\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} + \cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4}}{2} \right)$$

$\therefore c_0 - c_2 + c_4 - c_6 + \dots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$ . (A) is true. (4)

Subtracting (3) from (2), we get

$2i(c_1 - c_3 + c_5 - \dots) = (1+i)^n - (1-i)^n$

$c_1 - c_3 + c_5 - \dots = 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$  (B) is true. (5)

We know that  $c_1 + c_3 + c_5 + \dots = 2^{n-1}$  (6)

Adding (5) and (6), we get

$2(c_1 + c_3 + c_5 + \dots) = 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$ . (C) is true.

$\therefore c_1 + c_3 + c_5 + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right)$

(D) is not true.

**Ans. : (D)**

(29) If  $|z^2 - 1| = |z|^2 + 1$ , then P(z) lies on .....

(A) a parabola

(B) a circle

(C) the imaginary axis

(D) the real axis

**Sol. :** Let  $z = x + iy$

$|z^2 - 1| = |z|^2 + 1$

$\therefore |(x + iy)^2 - 1| = x^2 + y^2 + 1$

$$\therefore \sqrt{(x^2 - y^2 - 1)^2 + (2xy)^2} = x^2 + y^2 + 1$$

$$\therefore (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$$

$$\therefore -2x^2y^2 + 2y^2 - 2x^2 + 4x^2y^2 = 2x^2y^2 + 2y^2 + 2x^2$$

$$\therefore 4x^2 = 0. \quad \text{So, } x = 0$$

$\therefore P(z)$  lies on the imaginary axis.

**Ans. : (C)**

(30) If  $x = \cos \theta + i \sin \theta$  and  $y = \cos \phi + i \sin \phi$ , then  $x^m y^n + \frac{1}{x^m y^n} = \dots$

(A)  $\cos (m\theta + n\phi)$  (B)  $\cos (m\theta - n\phi)$  (C)  $2\cos (m\theta + n\phi)$  (D)  $2\cos (m\theta - n\phi)$

**Sol. :**  $x = e^{i\theta}, y = e^{i\phi}$

$$\therefore x^m y^n + \frac{1}{x^m y^n} = e^{im\theta} e^{in\phi} + e^{-im\theta} e^{-in\phi}$$

$$= e^{i(m\theta + n\phi)} + e^{-i(m\theta + n\phi)}$$

$$= \cos (m\theta + n\phi) + i \sin (m\theta + n\phi) + \cos (m\theta + n\phi) - i \sin (m\theta + n\phi)$$

$$= 2 \cos (m\theta + n\phi)$$

**Ans. : (C)**

(31) If  $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$ , then  $a_0 + a_3 + a_6 + \dots = \dots$

(A) 1 (B)  $2^n$  (C)  $2^{n-1}$  (D)  $3^{n-1}$

**Sol. :** Given  $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$  (1)

Substitute  $x = 1$  in the equation (1),

$$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + \dots + a_{2n} \quad (2)$$

Substitute  $x = w$  in the equation (1),

$$(1 + w + w^2)^n = a_0 + a_1w + a_2w^2 + a_3w^3 + \dots + a_{2n}w^{2n} \quad (3)$$

Substitute  $x = w^2$  in the equation (1),

$$(1 + w^2 + w^4)^n = a_0 + a_1w^2 + a_2w^4 + a_3w^6 + \dots + a_{2n}w^{4n} \quad (4)$$

Adding (2), (3) and (4) we get,

$$3^n = 3 (a_0 + a_3 + a_6 \dots)$$

$$\therefore a_0 + a_3 + a_6 \dots = 3^{n-1}$$

$$(1 + w + w^2 = 0), (w^3 = 1)$$

**Ans. : (D)**

(32) The roots of the equation  $az^2 + z + 1 = 0$  are purely imaginary numbers, where  $a = \cos \theta + i \sin \theta$ .

Which of the following is true for the function  $f(x) = x^3 - 3x^2 + 3(1 + \cos \theta)x + 5$  ?

(A)  $f(x) = 0$  has three real roots.

(B)  $f(x) = 0$  has one positive real root.

(C)  $f(x) = 0$  has one negative real root.

(D)  $f(x) = 0$  has three distinct real roots.

**Sol. :**  $az^2 + z + 1 = 0$ , where  $a = \cos \theta + i \sin \theta$

Suppose  $z_1$  and  $z_2$  are purely imaginary roots of the equation.

$$\text{Then, } z_1 = -\overline{z_1} \text{ and } z_2 = -\overline{z_2}$$

$$z_1 + z_2 = -\frac{1}{a} \quad (1)$$

$$\therefore \overline{z_1} + \overline{z_2} = \frac{1}{a}$$

$$(\overline{z_1} + \overline{z_2} = -(z_1 + z_2))$$

$$\therefore \overline{\overline{z_1}} + \overline{\overline{z_2}} = \frac{1}{a} \text{ So, } z_1 + z_2 = \frac{1}{a}$$

$$\therefore -\frac{1}{a} = \frac{1}{a} \quad \text{So, } \frac{1}{a} + \frac{1}{a} = 0$$

$$\therefore \cos \theta = 0$$

$$(a = \cos \theta + i \sin \theta)$$

$$\therefore \theta = \frac{\pi}{2}$$

Now  $f(x) = 0$

$$\therefore x^3 - 3x^2 + 3 \left(1 + \cos \frac{\pi}{2}\right) x + 5 = 0$$

$$\therefore x^3 - 3x^2 + 3x + 5 = 0$$

$$\therefore (x-1)^3 + 6 = 0. \quad \text{So, } x = 1 + (-6)^{\frac{1}{3}}$$

$\therefore f(x) = 0$  has one negative real root.

**Ans. : (C)**

(33) If  $z_1 = 24 + 7i$  and  $|z_2| = 6$ , then the maximum and the minimum values of  $|z_1 + z_2|$  are ..... .

(A) 31, 19

(B) 25, 19

(C) 31, 25

(D) 24, 7

**Sol. :**  $|z_2| = 6$  represents a circle.

$|z_1 + z_2| = |z_2 - (-24 - 7i)|$  shows that the distance between any point on the circle  $|z_2| = 6$  and  $(-24, -7)$  is the required distance.

According to the figure, BC is the maximum distance and AC is the minimum distance.

$$OC = \sqrt{24^2 + 7^2} = 25$$

$$OA = 6, OB = 6$$

$$AC = OC - OA = 25 - 6 = 19$$

$$BC = OC + OB = 25 + 6 = 31$$

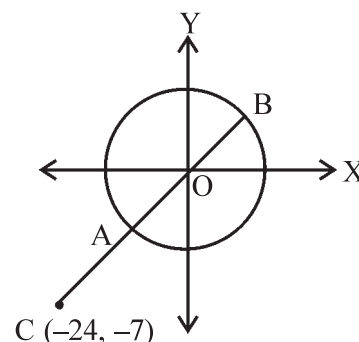


Figure 2.5

**Ans. : (A)**

**Another method :**  $|z_2| = 6 \Rightarrow z_2 = 6e^{i\theta}$  where  $\theta \in \mathbb{R}$

$$|z_1 + z_2|^2 = |24 + 7i + 6 \cos \theta + i 6 \sin \theta|^2$$

$$= (24 + 6 \cos \theta)^2 + (7 + 6 \sin \theta)^2$$

$$= 576 + 36 \cos^2 \theta + 288 \cos \theta + 49 + 36 \sin^2 \theta + 84 \sin \theta$$

$$= 661 + 12 (24 \cos \theta + 7 \sin \theta)$$

$$= 661 + 12 (25) \sin (\theta + \alpha) \quad (\text{where } 7 = 25 \cos \alpha, 24 = 25 \sin \alpha)$$

$$|z_1 + z_2|^2 = 661 + 300 \sin (\theta + \alpha)$$

$\therefore$  The maximum and the minimum values of  $|z_1 + z_2|^2$  are 961 and 361.

$\therefore$  The maximum and the minimum values of  $|z_1 + z_2|$  are 31 and 19.

**Ans. : (A)**

(34) For a complex number  $a$ ,  $|a| = 1$ ; if  $az^2 + z + 1 = 0$  has one purely imaginary root, then .....

$$(A) a = \cos \theta + i \sin \theta, \theta = \cos^{-1} \left( \frac{\sqrt{5}-1}{2} \right) \quad (B) a = \sin \theta + i \cos \theta, \theta = \cos^{-1} \left( \frac{\sqrt{5}-1}{2} \right)$$

$$(C) a = \cos \theta + i \sin \theta, \theta = \cos^{-1} \left( \frac{\sqrt{5}-1}{4} \right) \quad (D) a = \sin \theta + i \cos \theta, \theta = \cos^{-1} \left( \frac{\sqrt{5}-1}{4} \right)$$

**Sol. :**  $az^2 + z + 1 = 0$

Let  $z_1$  be the purely imaginary root of the equation.

$$\therefore az_1^2 + z_1 + 1 = 0 \quad (1)$$

$$\therefore \bar{a} (\bar{z}_1)^2 + \bar{z}_1 + 1 = 0$$

$$\therefore \bar{a} z_1^2 - z_1 + 1 = 0 \quad (\bar{z}_1 = -z_1) \quad (2)$$

By solving (1) and (2),

$$\therefore \frac{z_1^2}{2} = \frac{-z_1}{a - \bar{a}} = \frac{1}{-(a + \bar{a})}$$

$$\therefore z_1^2 = \left( \frac{a - \bar{a}}{a + \bar{a}} \right)^2 = \frac{-2}{a + \bar{a}}$$

$$\therefore (\bar{a} - a)^2 + 2(a + \bar{a}) = 0$$

For  $a = \cos \theta + i \sin \theta$  ( $|a| = 1$ )

$$\therefore (-2i \sin \theta)^2 + 2(2 \cos \theta) = 0$$

$$\therefore -4(1 - \cos^2 \theta) + 4 \cos \theta = 0$$

$$\therefore \cos^2 \theta + \cos \theta - 1 = 0$$

$$\therefore \cos \theta = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\cos \theta = \frac{-1 - \sqrt{5}}{2} \text{ is not possible.} \quad |\cos \theta| \leq 1$$

$$\therefore \cos \theta = \frac{\sqrt{5} - 1}{2}$$

$$\therefore \theta = \cos^{-1} \left( \frac{\sqrt{5} - 1}{2} \right)$$

**Ans. : (A)**

(35) If  $\cos(1 - i) = a + ib$  where  $a, b \in \mathbb{R}$ , then  $a = \dots$   $b = \dots$

(A)  $\frac{1}{2} \left( e - \frac{1}{e} \right) \cos 1, \frac{1}{2} \left( e + \frac{1}{e} \right) \sin 1$

(B)  $\frac{1}{2} \left( e + \frac{1}{e} \right) \cos 1, \frac{1}{2} \left( e - \frac{1}{e} \right) \sin 1$

(C)  $\frac{1}{2} \left( e + \frac{1}{e} \right) \cos 1, \frac{1}{2} \left( e + \frac{1}{e} \right) \sin 1$

(D)  $\frac{1}{2} \left( e - \frac{1}{e} \right) \cos 1, \frac{1}{2} \left( e - \frac{1}{e} \right) \sin 1$

**Sol. :**  $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$\therefore \cos(1 - i) = \frac{e^{i(1-i)} + e^{-i(1-i)}}{2}$$

$$= \frac{e^{1+i} + e^{-1-i}}{2}$$

$$= \frac{e \cdot e^i + \frac{e^{-i}}{e}}{2} = \frac{1}{2e} [e^2 (\cos 1 + i \sin 1) + \cos 1 - i \sin 1]$$

$$\therefore \cos(1 - i) = \frac{1}{2e} (e^2 \cos 1 + \cos 1 + i e^2 \sin 1 - i \sin 1)$$

$$\therefore a = \frac{1}{2e} (e^2 \cos 1 + \cos 1), b = \frac{1}{2e} (e^2 \sin 1 - \sin 1)$$

$$a = \frac{1}{2} \left( e + \frac{1}{e} \right) \cos 1, \quad b = \frac{1}{2} \left( e - \frac{1}{e} \right) \sin 1 \quad \text{Ans. : (B)}$$

(36) If  $z_1, z_2, z_3 \in \mathbb{C}$ ,  $\sum z_1 \operatorname{Im} (\overline{z_2} z_3) = \dots$

$$(A) z_1 + z_2 + z_3 \quad (B) 0 \quad (C) \frac{1}{2i} (z_1 + z_2 + z_3) \quad (D) z_1 z_2 z_3$$

**Sol. :** Let  $z_1 = r_1 e^{i\theta_1}$ ;  $z_2 = r_2 e^{i\theta_2}$ ;  $z_3 = r_3 e^{i\theta_3}$

$$\operatorname{Im} (\overline{z_2} z_3) = \operatorname{Im} (r_2 r_3 e^{i(\theta_3 - \theta_2)}) = r_2 r_3 \sin (\theta_3 - \theta_2)$$

$$\therefore z_1 \operatorname{Im} (\overline{z_2} z_3) = r_1 r_2 r_3 e^{i\theta_1} \sin (\theta_3 - \theta_2)$$

$$\therefore \sum z_1 \operatorname{Im} (\overline{z_2} z_3) = r_1 r_2 r_3 \{e^{i\theta_1} \sin (\theta_3 - \theta_2) + e^{i\theta_2} \sin (\theta_1 - \theta_3) + e^{i\theta_3} \sin (\theta_2 - \theta_1)\}$$

Now, using the formula of  $\sin (\alpha - \beta)$  and  $e^{i\theta} = \cos \theta + i \sin \theta$ , we get

$$\sum z_1 \operatorname{Im} (\overline{z_2} z_3) = 0 \quad \text{Ans. : (B)}$$

$$(37) \quad e^{i(2k \cot^{-1} m)} \left( \frac{mi+1}{mi-1} \right)^k = \dots \quad \text{where } k \in \mathbb{Z}, m > 0$$

$$(A) 0 \quad (B) 1 \quad (C) -1 \quad (D) 2$$

$$\text{Sol. :} \quad e^{i(2k \cot^{-1} m)} \left( \frac{mi+1}{mi-1} \right)^k$$

$$= [\cos (2k \cot^{-1} m) + i \sin (2k \cot^{-1} m)] \left( \frac{mi+1}{mi-1} \right)^k$$

$$= e^{i 2k\theta} \left[ \frac{-(mi+1)^2}{m^2+1} \right]^k \quad \text{where } \cot^{-1} m = \theta$$

$$= e^{i 2k\theta} \left[ \frac{m-i}{\sqrt{m^2+1}} \right]^{2k} = e^{i 2k\theta} [\cos \theta - i \sin \theta]^{2k} = e^{i 2k\theta} e^{-i 2k\theta} = 1 \quad \text{Ans. : (B)}$$

(38) If  $(1 + ir)^3 = s(1 + i)$  where  $r, s \in \mathbb{R}$ , then the sum of the possible values of  $r$  is .....

$$(A) 1 \quad (B) -3 \quad (C) 3 \quad (D) 0$$

**Sol. :** If  $\cos \theta = \frac{1}{\sqrt{1+r^2}}$  and  $\sin \theta = \frac{r}{\sqrt{1+r^2}}$ , then

$$\text{from the given equation, } \left[ \sqrt{1+r^2} (\cos \theta + i \sin \theta) \right]^3 = s(1 + i)$$

$$\therefore \left( \sqrt{1+r^2} \right)^3 (\cos 3\theta + i \sin 3\theta) = s(1 + i)$$

$$(1+r^2)^{\frac{3}{2}} \cos 3\theta = s, (1+r^2)^{\frac{3}{2}} \sin 3\theta = s$$

$$\therefore \cos 3\theta = \sin 3\theta$$

$$\therefore 4 \cos^3 \theta - 3 \cos \theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$\therefore \frac{4}{(\sqrt{1+r^2})^3} - \frac{3}{\sqrt{1+r^2}} = \frac{3r}{\sqrt{1+r^2}} - \frac{4r^3}{(\sqrt{1+r^2})^3}$$

$$\therefore 4 - 3(1+r^2) = 3r(1+r^2) - 4r^3$$

$$\therefore r^3 - 3r^2 - 3r + 1 = 0 \quad (1)$$

$$\therefore \text{The sum of all the possible values of } r = \text{The sum of the roots of the above equation} = 3$$

**Ans. : (C)**

(39) If  $\alpha$  and  $\beta^2$  are the roots of  $8x^2 - 10x + 3 = 0$  with  $\beta^2 > \frac{1}{2}$ , then the equation whose roots are

$(\alpha + i\beta)^{100}$  and  $(\alpha - i\beta)^{100}$  is .....

(A)  $x^2 - x + 1 = 0$  (B)  $x^2 + x + 1 = 0$  (C)  $x^2 - x - 1 = 0$  (D)  $x^2 + x - 1 = 0$

**Sol. :**  $8x^2 - 10x + 3 = 0$

$$\therefore (4x - 3)(2x - 1) = 0$$

$$\therefore x = \frac{3}{4} \text{ or } x = \frac{1}{2}$$

$$\therefore \alpha = \frac{1}{2} \text{ and } \beta = \frac{\pm\sqrt{3}}{2} \quad (\beta^2 > \frac{1}{2})$$

$$\therefore \alpha + i\beta = \frac{1+i\sqrt{3}}{2} = e^{\frac{i\pi}{3}} \text{ and } \alpha - i\beta = \frac{1-i\sqrt{3}}{2} = e^{\frac{-i\pi}{3}}$$

$$\therefore \left(\frac{1+i\sqrt{3}}{2}\right)^{100} + \left(\frac{1-i\sqrt{3}}{2}\right)^{100} = e^{\frac{i100\pi}{3}} + e^{\frac{-i100\pi}{3}} = 2\cos \frac{100\pi}{3} = 2\left(\frac{-1}{2}\right) = -1$$

$$\text{Also, } \left(\frac{1+i\sqrt{3}}{2}\right)^{100} \left(\frac{1-i\sqrt{3}}{2}\right)^{100} = e^{\frac{i100\pi}{3}} \cdot e^{\frac{-i100\pi}{3}} = 1$$

The required equation is  $x^2 + x + 1 = 0$ .

**Ans. : (B)**

(40)  $w$  (where  $\text{Im}(w) \neq 0$ ) is a complex number. The set of points satisfying the equation  $w - \bar{w}z = k(1 - z)$ ,  $k \in \mathbb{R}$ , is .....

(A)  $\{z \mid |z| = 1\}$  (B)  $\{z \mid z = \bar{z}\}$  (C)  $\{z \mid z \neq 1\}$  (D)  $\{z \mid |z| = 1, z \neq 1\}$

**Sol. :** Let  $w = a + ib$ ,  $b \neq 0$  and  $z = x + iy$ ,

$$w - \bar{w}z = k(1 - z)$$

$$\therefore a + ib - (a - ib)(x + iy) = k(1 - x - iy)$$

$$\therefore a + ib - (ax + ayi - bxi + by) = k(1 - x) - iky$$

$$\therefore (a - ax - by) + i(b - ay + bx) = k(1 - x) - kyi$$

Comparing real and imaginary parts, we get

$$k = \frac{a - ax - by}{1 - x}, \quad k = \frac{ay - bx - b}{y}$$

$$\therefore ay - axy - by^2 = ay - bx - b - ayx + bx^2 + bx$$

$$\therefore -by^2 = bx^2 - b$$

$$\therefore x^2 + y^2 = 1 \quad (b \neq 0)$$

$$\text{So, } |z| = 1$$

**Ans. : (A)**

(41) If  $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$ , then  $\theta = \dots$

$$(A) \frac{2k\pi}{n(n+1)}, k \in \mathbb{Z} \quad (B) \frac{k\pi}{n(n+1)}, k \in \mathbb{Z} \quad (C) \frac{4k\pi}{n(n+1)}, n \in \mathbb{Z} \quad (D) \frac{k\pi}{2n(n+1)}, k \in \mathbb{Z}$$

**Sol. :**  $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$

$$\therefore \cos(\theta + 2\theta + \dots + n\theta) + i \sin(\theta + 2\theta + \dots + n\theta) = 1$$

$$\therefore \cos \left\{ \frac{1}{2}n(n+1)\theta \right\} + i \sin \left\{ \frac{1}{2}n(n+1)\theta \right\} = 1$$

$$\therefore \frac{1}{2}n(n+1)\theta = 2k\pi, k \in \mathbb{Z}$$

$$\therefore \theta = \frac{4k\pi}{n(n+1)}, k \in \mathbb{Z} \quad \text{Ans. : (C)}$$

(42) If  $\alpha, \beta$  are the roots of  $u^2 - 2u + 2 = 0$  and  $\cot \theta = x + 1$ , then  $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \dots$

$$(A) \frac{\cos n\theta}{\cos^n \theta} \quad (B) \frac{\sin n\theta}{n \sin \theta} \quad (C) \frac{\sin n\theta}{\cos^n \theta} \quad (D) \frac{\sin n\theta}{\sin^n \theta}$$

**Sol. :**  $u^2 - 2u + 2 = 0$  implies  $u = 1 \pm i$ . Let  $\alpha = 1 + i, \beta = 1 - i$

$$\text{Now, } \frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{[\cot \theta - 1 + 1 + i]^n - [\cot \theta - 1 + 1 - i]^n}{2i}$$

$$= \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{\sin^n \theta \cdot 2i}$$

$$= \frac{2i \sin n\theta}{2i \sin^n \theta} = \frac{\sin n\theta}{\sin^n \theta}$$

**Ans. : (D)**

(43) If  $a, b, c, a_1, b_1, c_1$  are non-zero complex numbers and  $\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} = 1 + i$  and

$$\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} = 0, \text{ then } \frac{a^2}{a_1^2} + \frac{b^2}{b_1^2} + \frac{c^2}{c_1^2} = \dots$$

$$(A) 2i \quad (B) 2 + 2i \quad (C) 2 \quad (D) 1 + 2i$$

**Sol. :** Let  $\frac{a}{a_1} = x$  ;  $\frac{b}{b_1} = y$  ;  $\frac{c}{c_1} = z$

Given equations are  $x + y + z = 1 + i$  and  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ .

$$\therefore (x + y + z)^2 = x^2 + y^2 + z^2 \quad (xy + yz + zx = 0)$$

$$\therefore x^2 + y^2 + z^2 = (1 + i)^2 = 2i$$

**Ans. : (A)**

(44) How many solutions are possible for  $\operatorname{Re}(z)^2 = 0$  and  $|z| = a\sqrt{2}$  ( $a > 0$ ) ?

- (A) 1 (B) 2 (C) 3 (D) 4

**Sol. :** Let  $z = re^{i\theta}$

$$-\pi < \theta \leq \pi$$

$$|z| = r = a\sqrt{2} \text{ and } z^2 = 2a^2 e^{2i\theta}$$

Now,  $\operatorname{Re}(z)^2 = 2a^2 \cos 2\theta = 0$ . So  $\cos 2\theta = 0$  ( $a > 0$ )

$$\therefore 2\theta = n\pi \pm \frac{\pi}{2}; n \in \mathbb{Z}$$

$$\therefore \theta = \frac{n\pi}{2} \pm \frac{\pi}{4}; n \in \mathbb{Z}$$

$$\theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$$

$$\therefore z = \sqrt{2}ae^{\frac{i\pi}{4}}, \sqrt{2}ae^{-\frac{i\pi}{4}}, \sqrt{2}ae^{\frac{i3\pi}{4}}, \sqrt{2}ae^{-\frac{i3\pi}{4}}$$

Hence, there are four solutions.

**Ans. : (D)**

(45)  $z_1, z_2, z_3$  are the vertices of an equilateral triangle lying inside the circle  $|z| = 2$ . If  $z_1 = 1 + \sqrt{3}i$ , then  $z_2, z_3$  are .....

- (A)  $-2, 1 - \sqrt{3}i$  (B)  $-1 + \sqrt{3}i, -2$  (C)  $-2, -1 + \sqrt{3}i$  (D)  $-2, 2 + \sqrt{3}i$

**Sol. :**  $z_1 = 1 + \sqrt{3}i = 2e^{\frac{i\pi}{3}}$

$$m\angle z_1 O z_2 = \frac{2\pi}{3} \text{ and } m\angle z_2 O z_3 = \frac{2\pi}{3}$$

$$\therefore z_2 = z_1 e^{\frac{2\pi i}{3}}, z_3 = z_2 e^{\frac{2\pi i}{3}}$$

$$\text{So, } z_2 = 2e^{i\pi} = -2 \text{ and } z_3 = -2e^{\frac{2\pi i}{3}}$$

$$\therefore z_3 = -2 \left[ \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$$

$$= -2 \left( \frac{-1}{2} + \frac{\sqrt{3}i}{2} \right) = 1 - \sqrt{3}i$$

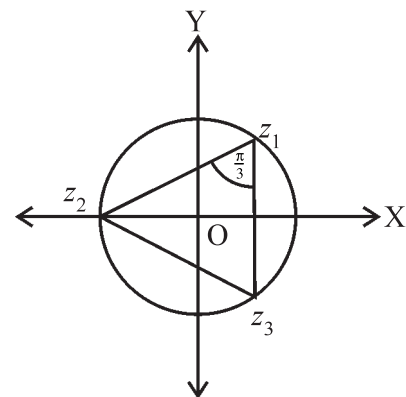


Figure 2.6

**Ans. : (A)**

(46) If  $|z| = 1$  and  $w = \frac{z-1}{z+1}$ , ( $z \neq -1$ ), then  $\operatorname{Re}(w) = \dots\dots\dots$

(IIT 2003)

- (A) 0 (B)  $\frac{1}{|z+1|^2}$  (C)  $\left| \frac{1}{z+1} \right| \frac{1}{|z+1|^2}$  (D)  $\frac{\sqrt{2}}{|z+1|^2}$

**Sol. :**  $w = \frac{z-1}{z+1}$

( $z \neq -1$ )

$$\therefore wz + w = z - 1$$

$$\therefore w + 1 = z(1 - w)$$

$$\therefore z = \frac{w+1}{1-w}$$

$$\therefore |z| = \left| \frac{w+1}{1-w} \right|$$

$$\therefore \left| \frac{w+1}{1-w} \right| = 1$$

**Second method :**

$$|w - 1| = |w + 1|$$

$$\text{Let } w = u + iv$$

$$\therefore |u + iv - 1| = |u + iv + 1|$$

$$\therefore (u - 1)^2 + v^2 = (u + 1)^2 + v^2$$

$$\therefore u = 0$$

$$\therefore \operatorname{Re}(w) = 0$$

$$(|z| = 1)$$

$$\therefore |w + 1| = |w - 1| \text{ (From here see the second method)}$$

$\therefore w$  lies at equidistant from  $(1, 0)$  and  $(-1, 0)$ .

$\therefore w$  lies on perpendicular bisector of  $(1, 0)$  and  $(-1, 0)$ .

$\therefore w$  is a point on the imaginary axis.

$$\therefore \operatorname{Re}(w) = 0$$

**Ans. : (A)**

(47) If  $z_1, z_2$  are non-zero complex numbers such that  $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$ , then  $O(0)$ ,  $P(z_1)$  and  $Q(z_2)$

are .....

(A) on a straight line

(B) the vertices of a right angle triangle

(C) the vertices of an equilateral triangle

(D) on a circle

**Sol. :** Let  $z = \frac{z_1}{z_2}$

$$\therefore z + \frac{1}{z} = 1. \quad \text{So, } z^2 - z + 1 = 0$$

$$\therefore z = \frac{1 \pm \sqrt{3}i}{2}. \quad \text{So, } \frac{z_1}{z_2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \frac{OP}{OQ} = \left| \frac{z_1}{z_2} \right| = \left| \frac{1 \pm \sqrt{3}i}{2} \right| = 1$$

$$\therefore OP = OQ$$

$$\text{Now, } \frac{PQ}{OQ} = \frac{|z_2 - z_1|}{|z_2|} = \left| 1 - \frac{z_1}{z_2} \right| = \left| 1 - \left( \frac{1 \pm \sqrt{3}i}{2} \right) \right| = \left| \frac{1 \mp \sqrt{3}i}{2} \right| = 1$$

$$\therefore PQ = OQ$$

$$\therefore OP = OQ = PQ$$

$\therefore \triangle OPQ$  is an equilateral triangle.

**Ans. : (C)**

(48) If  $z_1, z_2$  and  $z_3$  are vertices of  $\triangle ABC$  in an anticlockwise direction and  $z_0$  is the circumcentre

of  $\triangle ABC$ , then  $\left( \frac{z_0 - z_1}{z_0 - z_2} \right) \frac{\sin 2A}{\sin 2B} + \left( \frac{z_0 - z_3}{z_0 - z_2} \right) \frac{\sin 2C}{\sin 2B} = \dots$

(A) 0

(B) 1

(C) -1

(D) 2

**Sol. :** In  $\triangle ABC$ ,  $O(z_0)$  is the circumcentre. So,  $OA = OB = OC$ .  
 $m\angle AOB = 2C$ ,  $m\angle BOC = 2A$  and  $m\angle AOC = 2B$ .

Taking  $\theta$  rotation,  $\frac{z_0 - z_1}{z_0 - z_2} = \cos 2C - i \sin 2C$

and  $\frac{z_0 - z_3}{z_0 - z_2} = \cos 2A + i \sin 2A$

Now,  $\left(\frac{z_0 - z_1}{z_0 - z_2}\right) \frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right) \frac{\sin 2C}{\sin 2B}$

$$= \frac{\sin 2A \cos 2C - i \sin 2A \sin 2C + \cos 2A \sin 2C + i \sin 2A \sin 2C}{\sin 2B}$$

$$= \frac{\sin(2A + 2C)}{\sin 2B} = -1$$

$$(A + B + C = \pi)$$

**Ans. : (C)**

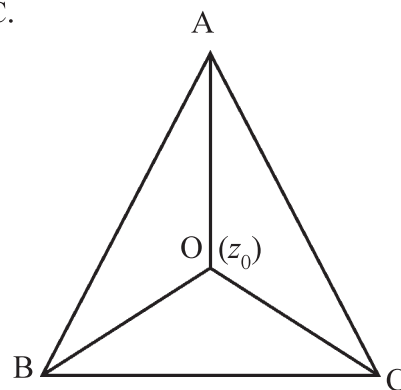


Figure 2.7

(49) If  $e^{\log 2 \frac{|z|^2 - |z| + 4}{|z|^2 + 1}} = \log_{\sqrt{2}}(16)$ , then  $|z| = \dots$

(A)  $\frac{1}{3}$

(B)  $\frac{1}{4}$

(C) 1

(D)  $\frac{1}{2}$

**Sol. :** Let  $|z| = r$

$$e^{\log 2 \frac{|z|^2 - |z| + 4}{|z|^2 + 1}} = \log_{\sqrt{2}} 16$$

$$\therefore 2^{\frac{r^2 - r + 4}{r^2 + 1}} = 8 \log_{\sqrt{2}} \sqrt{2} = 8 = 2^3$$

$$\therefore \frac{r^2 - r + 4}{r^2 + 1} = 3$$

$$\therefore r^2 - r + 4 = 3(r^2 + 1)$$

$$\therefore 2r^2 + r - 1 = 0$$

$$\therefore (2r - 1)(r + 1) = 0$$

$$\therefore r = \frac{1}{2} \text{ or } r = -1. \text{ But } r > 0$$

$$\therefore r = |z| = \frac{1}{2}$$

**Ans. : (D)**

(50) The area of the rectangle formed by the roots of the equation  $z\bar{z}^3 + \bar{z}z^3 = 350$  for  $z = x + iy$ ,  $x, y \in \mathbb{Z}$  is .....

(A) 175

(B) 350

(C) 48

(D) 68

**Sol. :**  $z\bar{z}^3 + \bar{z}z^3 = 350$  is a biquadratic equation. So, there are four complex roots.

$$z\bar{z}^3 + \bar{z}z^3 = 350$$

$$\therefore |z|^2 \bar{z}^2 + |z|^2 z^2 = 350$$

$$(z\bar{z} = |z|^2)$$

$$\therefore |z|^2 (\bar{z}^2 + z^2) = 350$$

$$\therefore (x^2 + y^2) [(x - iy)^2 + (x + iy)^2] = 350$$

$$\therefore (x^2 + y^2) (2x^2 - 2y^2) = 350$$

$$\therefore (x^2 + y^2) (x^2 - y^2) = 175$$

$x$  and  $y$  are integers so that  $x^2 + y^2$  and  $x^2 - y^2$  are also integers. So,  $x^2 + y^2 = 175$ ,  $x^2 - y^2 = 1$  is not possible and  $x^2 + y^2 = 35$ ,  $x^2 - y^2 = 5$  also not possible.

$$175 = 25 \times 7 = (4^2 + 3^2) (4^2 - 3^2)$$

$$\therefore x^2 = 16, y^2 = 9$$

$$\therefore x = \pm 4, y = \pm 3$$

$\therefore$  The vertices of the rectangle are  $(4, 3), (-4, 3), (4, -3), (-4, -3)$ .

Area of this rectangle  $= 8 \times 6 = 48$ .

**Ans. : (C)**

(51) If  $|z| = 1$  and  $\arg(z) = \theta$ , then  $\arg\left(\frac{1+z}{1+\bar{z}}\right) = \dots\dots$

(A)  $-\theta$

(B)  $\frac{\pi}{2} - \theta$

(C)  $\theta$

(D)  $\pi - \theta$

**Sol. :**  $z\bar{z} = |z|^2$

$$z\bar{z} = 1$$

$$(|z| = 1)$$

$$\therefore \bar{z} = \frac{1}{z}$$

$$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta$$

**Ans. : (C)**

(52) If  $|z| = 1$  and  $z \neq \pm 1$ , then for any complex number  $z$ ,  $\frac{z}{1-z^2} \dots\dots$

(IIT : 2007)

(A) lies on a line which does not pass through origin

(B) satisfies  $|z| = \sqrt{2}$

(C) lies on X-axis

(D) lies on Y-axis

**Sol. :**  $|z| = 1$

$$\frac{z}{1-z^2} = \frac{z}{z\bar{z}-z^2} = \frac{z}{z(\bar{z}-z)} = \frac{1}{\bar{z}-z}$$

$$(z\bar{z} = |z|^2 = 1)$$

$z - \bar{z}$  is a purely imaginary number

$\therefore \frac{1}{\bar{z}-z}$  is a purely imaginary number

$\therefore \frac{z}{1-z^2}$  lies on Y-axis.

**Ans. : (D)**

(53)  $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$ , where  $-\pi < \alpha, \beta, \gamma \leq \pi$ .

If  $\sin^n \alpha + \sin^n \beta + \sin^n \gamma = \frac{3}{2}$  and  $\Sigma \cos(\alpha + \beta) = \lambda$ , then  $(n, \lambda) = \dots\dots$

(A)  $(2, 0)$

(B)  $(0, 3)$

(C)  $(2, 3)$

(D)  $(3, 2)$

**Sol. :**  $\Sigma \cos \alpha = 0 = \Sigma \sin \alpha$ , where  $-\pi < \alpha, \beta, \gamma \leq \pi$

Let  $A = e^{i\alpha}$ ,  $B = e^{i\beta}$ ,  $C = e^{i\gamma}$ . Then

$$A + B + C = 0 \text{ and } \frac{1}{A} + \frac{1}{B} + \frac{1}{C} = 0$$

$$\therefore A + B + C = 0 \text{ and } AB + BC + CA = 0$$

$$\therefore A^2 + B^2 + C^2 = 0 \text{ and } AB + BC + CA = 0$$

$$\therefore e^{i2\alpha} + e^{i2\beta} + e^{i2\gamma} = 0 \text{ and } e^{i(\alpha+\beta)} + e^{i(\beta+\gamma)} + e^{i(\gamma+\alpha)} = 0$$

$$\therefore \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0 \text{ and } \cos (\alpha + \beta) + \cos (\beta + \gamma) + \cos (\gamma + \alpha) = 0$$

(Comparing the real parts)

$$\therefore 1 - 2 \sin^2 \alpha + 1 - 2 \sin^2 \beta + 1 - 2 \sin^2 \gamma = 0$$

$$\therefore \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{3}{2} \text{ and } \Sigma \cos (\alpha + \beta) = 0$$

$$\therefore n = 2, \lambda = 0 \text{ implies } (n, \lambda) = (2, 0)$$

**Ans. : (A)**

(54) If  $\alpha = \cos \left( \frac{8\pi}{11} \right) + i \sin \left( \frac{8\pi}{11} \right)$ , then  $\text{Re} (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = \dots$

(A)  $\frac{1}{2}$

(B)  $-\frac{1}{2}$

(C) 0

(D) 1

**Sol. :**  $\text{Re} (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$$= \cos \left( \frac{8\pi}{11} \right) + \cos \left( \frac{16\pi}{11} \right) + \cos \left( \frac{24\pi}{11} \right) + \cos \left( \frac{32\pi}{11} \right) + \cos \left( \frac{40\pi}{11} \right)$$

$$= \cos \left( \frac{8\pi}{11} \right) + \cos \left( \frac{16\pi}{11} \right) + \cos \left( \frac{2\pi}{11} \right) + \cos \left( \frac{10\pi}{11} \right) + \cos \left( \frac{18\pi}{11} \right) \quad (1)$$

$$= \cos \left( \frac{14\pi}{11} \right) + \cos \left( \frac{6\pi}{11} \right) + \cos \left( \frac{20\pi}{11} \right) + \cos \left( \frac{12\pi}{11} \right) + \cos \left( \frac{4\pi}{11} \right) \quad (\cos(2\pi - \theta) = \cos \theta) \quad (2)$$

Adding (1) and (2), we get

$$2 \text{Re} (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = \sum_{m=1}^{10} \cos \left( \frac{2m\pi}{11} \right)$$

We know that  $\sum_{m=0}^{10} \cos \left( \frac{2m\pi}{11} \right) = 0$  (The sum of real parts of 11th roots of 1)

$$\therefore \text{Re} (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = \frac{1}{2} (-1) = -\frac{1}{2}$$

**Ans. : (B)**

(55) If  $f(z) = z^4 - 8z^3 + 4z^2 + 4z + 39$  and  $f(3 + 2i) = a + ib$ , then  $a : b = \dots$

(A) 1 : 8

(B) -1 : 8

(C) 1 : 4

(D) -1 : 4

**Sol. :** Let  $z = 3 + 2i$

$$\therefore (z - 3)^2 = 4i^2 \quad \text{So,} \quad z^2 - 6z + 13 = 0 \quad (1)$$

Dividing  $f(z)$  by  $z^2 - 6z + 13$ ; we get

$$\therefore z^4 - 8z^3 + 4z^2 + 4z + 39 = (z^2 - 6z + 13)(z^2 - 2z - 21) - 96z + 312$$

$$\therefore f(3 + 2i) = 0(z^2 - 2z - 21) - 96(3 + 2i) + 312 = 24 - 192i = a + ib$$

$$\therefore a = 24, b = -192$$

$$\therefore \frac{a}{b} = -\frac{1}{8}$$

**Ans. : (B)**

- (56) If a non-real complex number  $\alpha$  is a root of the equation  $z^{p+q} - z^p - z^q + 1 = 0$ , where  $p, q$  are distinct primes, then .....
- (A)  $1 + \alpha + \alpha^2 + \dots + \alpha^p = 0$
- (B)  $1 + \alpha + \alpha^2 + \dots + \alpha^q = 0$  and  $1 + \alpha + \alpha^2 + \dots + \alpha^{q-1} = 0$
- (C)  $1 + \alpha + \alpha^2 + \dots + \alpha^{p-1} = 0$  and  $1 + \alpha + \alpha^2 + \dots + \alpha^{q-1} = 0$
- (D) both are not possible simultaneously.

**Sol. :**  $z^{p+q} - z^p - z^q + 1 = 0$

$$\therefore (z^p - 1)(z^q - 1) = 0$$

$\alpha$  is a root of the given equation.

$$\alpha^p = 1 \text{ or } \alpha^q = 1$$

$\therefore \alpha$  is a  $p$ th root of 1 or  $\alpha$  is a  $q$ th root of 1. ( $\alpha \neq 1$ )

$$\therefore 1 + \alpha + \alpha^2 + \dots + \alpha^{p-1} = 0 \text{ or } 1 + \alpha + \alpha^2 + \dots + \alpha^{q-1} = 0 \quad (\text{Sum of all } n\text{th roots of 1 is 0.})$$

Suppose  $\alpha$  is a  $p$ th root of 1 and also  $\alpha$  is a  $q$ th root of 1.

$$\therefore \alpha = e^{\frac{2m\pi i}{p}} = e^{\frac{2n\pi i}{q}}, \text{ where } 1 \leq m < p, 1 \leq n < q \quad (\alpha \neq 1)$$

$$\therefore \text{For any positive integer, } m, n, \frac{m}{p} = \frac{n}{q} \text{ is not possible.} \quad (p, q \text{ are primes})$$

**Ans. : (D)**

- (57) If, for any complex number  $z$ ,  $|z - 3 - 2i| \leq 2$ , then the maximum value of  $|2z - 6 + 5i|$  is ..... .  
(IIT : 2011)

- (A) 11                      (B) 12                      (C) 13                      (D) 14

**Sol. :**  $|2z - 6 + 5i| = 2 \left| z - 3 + \frac{5}{2}i \right|$

$$= 2 \left| z - 3 - 2i + \frac{9}{2}i \right| \leq 2 \left( |z - 3 - 2i| + \frac{9}{2} \right) = 2 \left( 2 + \frac{9}{2} \right) = 13$$

If  $z = 3 + 4i$ , then  $|z - 3 - 2i| = 2$  and  $|2z - 6 + 5i| = 13$

$$\therefore \text{Maximum value of } |2z - 6 + 5i| \text{ is } 13. \quad \textbf{Ans. : (C)}$$

**Another method :**  $|z - 3 - 2i| \leq 2$  represents a circle with centre  $(3, 2)$  and radius 2 and its interior.

$$|2z - 6 + 5i| = 2 \left| z - 3 + \frac{5}{2}i \right|$$

$\therefore \left| z - 3 + \frac{5}{2}i \right|$  is the distance of  $z$  from the point  $\left(3, \frac{5}{2}\right)$ .

Maximum distance,  $PA = 2.5 + 4 = 6.5$

$$\therefore \text{Required maximum value} = 2(6.5) = 13$$

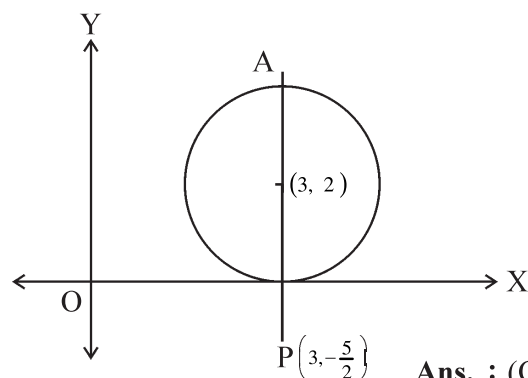


Figure 2.8

**Ans. : (C)**

- (58) If  $z$  and  $w$  are non-zero complex numbers such that  $|zw| = 1$  and  $\arg z - \arg w = \frac{\pi}{2}$ , then  $\bar{z}w = \dots\dots$
- (A) 1 (B) -1 (C)  $i$  (D)  $-i$

**Sol. :**  $|zw| = 1$  implies  $|z| |w| = 1$ . This implies  $|w| = \frac{1}{|z|}$  (1)

Let  $z = re^{i\theta}$  So,  $\bar{z} = re^{-i\theta}$

$\arg w = \arg z - \frac{\pi}{2} = \theta - \frac{\pi}{2}$

$\therefore w = \frac{1}{r} \cdot e^{i(\theta - \frac{\pi}{2})}$  From (1)

$\therefore \bar{z}w = re^{-i\theta} \cdot \frac{1}{r} e^{i(\theta - \frac{\pi}{2})} = e^{-i\frac{\pi}{2}} = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} = -i$  **Ans. : (D)**

- (59)  $z$  and  $w$  are complex numbers. If  $\bar{z} + i\bar{w} = 0$  and  $\arg(zw) = \pi$ , then  $\arg z = \dots\dots$

- (A)  $\frac{\pi}{4}$  (B)  $\frac{\pi}{2}$  (C)  $\frac{-\pi}{4}$  (D)  $\frac{5\pi}{4}$

**Sol. :** Let  $z = re^{i\theta}$

$\arg(zw) = \pi$ . So,  $\arg z + \arg w = \pi$ . So,  $\arg w = \pi - \theta$

$\therefore w = r_1 e^{i(\pi - \theta)}$  where  $r_1 = |w|$

Now,  $\bar{z} + i\bar{w} = 0$

$\therefore re^{-i\theta} + ir_1 e^{i(\theta - \pi)} = 0$

$r(\cos \theta - i \sin \theta) + ir_1 [\cos(\pi - \theta) - i \sin(\pi - \theta)] = 0$

$\therefore r(\cos \theta - i \sin \theta) + r_1 [-i \cos \theta + \sin \theta] = 0$

Comparing real parts and imaginary parts,

$r \cos \theta + r_1 \sin \theta = 0, -r \sin \theta - r_1 \cos \theta = 0$

$\therefore \frac{r}{r_1} = -\tan \theta = \frac{r_1}{r}$  implies  $r^2 = r_1^2$  implies  $r = r_1$

$\therefore \tan \theta = -1$ . So,  $\arg z = \frac{-\pi}{4}$  **Ans. : (C)**

- (60)  $\sum_{k=1}^6 \left( \sin\left(\frac{2k\pi}{7}\right) + i \cos\left(\frac{2k\pi}{7}\right) \right) = \dots\dots$

- (A) -1 (B) 0 (C)  $-i$  (D) 1

**Sol. :**  $z = \sum_{k=1}^6 \left( \sin\left(\frac{2k\pi}{7}\right) + i \cos\left(\frac{2k\pi}{7}\right) \right) = i \sum_{k=1}^6 \left( \cos\left(\frac{2k\pi}{7}\right) - i \sin\left(\frac{2k\pi}{7}\right) \right)$

Let  $\theta = \frac{2\pi}{7}$

$\therefore z = i \sum_{k=1}^6 e^{-ik\theta}$

$= i [e^{-i\theta} + e^{-i2\theta} + \dots + e^{-i6\theta}]$

$= ie^{-i\theta} \left( \frac{e^{-i6\theta} - 1}{e^{-i\theta} - 1} \right) = ie^{-i\theta} \left( \frac{\cos 6\theta - i \sin 6\theta - 1}{\cos \theta - i \sin \theta - 1} \right) = ie^{-i\theta} \left( \frac{1 - \cos 6\theta + i \sin 6\theta}{1 - \cos \theta + i \sin \theta} \right)$

$$\begin{aligned}
&= ie^{-i\theta} \frac{\sin 3\theta}{\sin \frac{\theta}{2}} \left( \frac{\sin 3\theta + i \cos 3\theta}{\sin \frac{\theta}{2} + i \cos \frac{\theta}{2}} \right) \\
&= ie^{-i\theta} \frac{\sin 3\theta}{\sin \frac{\theta}{2}} \left( \frac{\cos 3\theta - i \sin 3\theta}{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2}} \right) \\
&= ie^{-i\theta} \frac{\sin 3\theta}{\sin \frac{\theta}{2}} e^{-3i\theta} e^{\frac{i\theta}{2}} = ie^{\frac{-7i\theta}{2}} \frac{\sin 3\theta}{\sin \frac{\theta}{2}} \\
&= ie^{-i\pi} \frac{\sin \frac{6\pi}{7}}{\sin \frac{\pi}{7}} \quad \left( \theta = \frac{2\pi}{7} \right) \\
&= i (\cos \pi - i \sin \pi) \frac{\sin \left( \pi - \frac{\pi}{7} \right)}{\sin \frac{\pi}{7}} = -i
\end{aligned}$$

**Ans. : (C)**

(61) If  $\arg \left( \frac{z-2}{z+2} \right) = \frac{\pi}{4}$  for any complex number  $z$ , then  $|z - 2i| = \dots\dots$

- (A)  $2\sqrt{2}$                       (B)  $3\sqrt{2}$                       (C)  $4\sqrt{2}$                       (D)  $5\sqrt{2}$

**Sol. :** Let  $z = x + iy$  and  $\arg (z - 2) = \theta_1$  and  $\arg (z + 2) = \theta_2$

$$\arg \left( \frac{z-2}{z+2} \right) = \arg (z - 2) - \arg (z + 2) = \frac{\pi}{4}$$

$$\therefore \theta_1 - \theta_2 = \frac{\pi}{4}$$

$$\therefore \tan^{-1} \left( \frac{y}{x-2} \right) - \tan^{-1} \left( \frac{y}{x+2} \right) = \frac{\pi}{4}$$

$$\therefore \tan^{-1} \frac{\frac{y}{x-2} - \frac{y}{x+2}}{1 + \frac{y^2}{x^2 - 4}} = \frac{\pi}{4}$$

$$\therefore \frac{y(x+2) - y(x-2)}{x^2 - 4 + y^2} = 1$$

$$\therefore x^2 + y^2 - 4y = 4$$

$$\therefore (x - 0)^2 + (y - 2)^2 = 8$$

$$\therefore |x + i(y - 2)|^2 = 8$$

$$\therefore |z - 2i| = 2\sqrt{2}$$

**Ans. : (A)**

$$(62) \sum_{p=1}^{32} (3p+2) \left( \sum_{q=1}^{10} \left( \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right)^p = \dots\dots$$

- (A)  $24(1+i)$                       (B)  $24(1-i)$                       (C)  $48(1+i)$                       (D)  $48(1-i)$

$$\begin{aligned}\text{Sol. : } \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} &= -i \left( \cos \frac{2q\pi}{11} + i \sin \frac{2q\pi}{11} \right) \\ &= -i e^{\frac{i2q\pi}{11}}\end{aligned}$$

$$\begin{aligned}\therefore \sum_{q=1}^{10} \left( \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) &= -i \sum_{q=1}^{10} e^{\frac{i2q\pi}{11}} \\ &= -i \left[ e^{\frac{i2\pi}{11}} + e^{\frac{i4\pi}{11}} + \dots + e^{\frac{i20\pi}{11}} \right] = -i(-1) = i\end{aligned}$$

(Roots of  $z^{11} - 1 = 0$  are  $\cos \frac{2k\pi}{11} + i \sin \frac{2k\pi}{11}$  ( $i = 0, 1, 2, 3, \dots, 10$ ))

$$\begin{aligned}\therefore \sum_{p=1}^{32} (3p+2) \left( \sum_{q=1}^{10} \left( \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right)^p \\ = \sum_{p=1}^{32} (3p+2) i^p = 3 \sum_{p=1}^{32} p i^p + 2 \sum_{p=1}^{32} i^p = 3 S_1 + 2 S_2\end{aligned}$$

$$\text{where } S_1 = \sum_{p=1}^{32} p i^p, S_2 = \sum_{p=1}^{32} i^p$$

$$S_1 = i + 2i^2 + 3i^3 + \dots + 32i^{32}$$

$$i S_1 = i^2 + 2i^3 + 3i^4 + \dots + 32i^{33}$$

$$S_1 (1 - i) = i + i^2 + i^3 + \dots + i^{32} - 32 i^{33} = 8(i + i^2 + i^3 + i^4) - 32 i \cdot i^{32}$$

$$S_1 = \frac{-32i}{1-i} = 16(1-i) \quad [(1-i)^2 = -2i]$$

$$S_2 = 0 \quad (i + i^2 + \dots + i^{32} = 0, i^{33} = i)$$

$$\therefore \text{Required value} = 3 \times 16(1-i) + 0 = 48(1-i)$$

**Ans. : (D)**

### : Questions with More Than One Correct Answers :

(63) If  $z_1, z_2$  are non-zero complex numbers such that,  $|2z_1 - 3z_2|^2 = |2z_1|^2 + |3z_2|^2$ , then

(A)  $z_1 \bar{z}_2 + z_2 \bar{z}_1 = 0$

(B)  $\frac{z_2}{z_1}$  is purely imaginary

(C)  $\frac{z_2}{z_1}$  is purely real

(D)  $\arg \left( \frac{z_2}{z_1} \right) = \pm \frac{\pi}{2}$

$$\text{Sol. : } |2z_1 - 3z_2|^2 = |2z_1|^2 + |3z_2|^2$$

$$\therefore (2z_1 - 3z_2)(2\bar{z}_1 - 3\bar{z}_2) = |2z_1|^2 + |3z_2|^2$$

$$\therefore 4z_1\bar{z}_1 - 6\bar{z}_1z_2 - 6z_2\bar{z}_1 + 9z_2\bar{z}_2 = 4z_1\bar{z}_1 + 9z_2\bar{z}_2$$

$$\therefore \bar{z}_1z_2 + z_1\bar{z}_2 = 0$$

$$\therefore \bar{z}_1 z_2 = -z_1 \bar{z}_2$$

$$\therefore \frac{z_2}{z_1} = -\overline{\left(\frac{z_2}{z_1}\right)}$$

$$\therefore \frac{z_2}{z_1} \text{ is a purely imaginary number.}$$

$$\therefore \arg\left(\frac{z_2}{z_1}\right) = \pm \frac{\pi}{2}$$

**Ans. :** (A), (B), (D)

(64) If  $z_1, z_2, \dots, z_n$  lie on the circle  $|z| = r$  and  $\operatorname{Re}\left(\sum_{j=1}^n \sum_{k=1}^n \frac{z_j}{z_k}\right) = 0$ , then .....

$$(A) \sum_{j=1}^n z_j = 0 \quad (B) \left|\sum_{j=1}^n z_j\right| = 0 \quad (C) \sum_{j=1}^n \bar{z}_j = 0 \quad (D) \sum_{j=1}^n z_j = n$$

**Sol. :** Since  $z_1, z_2, \dots, z_n$  are on  $|z| = r$ ,  $z_i = r(\cos \theta_i + i \sin \theta_i)$

$$\text{Now, } \operatorname{Re}\left(\sum_{j=1}^n \sum_{k=1}^n \frac{z_j}{z_k}\right) = 0$$

$$\therefore \operatorname{Re}\left[(z_1 + z_2 + \dots + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n}\right)\right] = 0$$

$$\therefore \operatorname{Re}[\{(\cos \theta_1 + \dots + \cos \theta_n) + i(\sin \theta_1 + \dots + \sin \theta_n)\} \{(\cos \theta_1 + \dots + \cos \theta_n) - i(\sin \theta_1 + \dots + \sin \theta_n)\}] = 0$$

$$\therefore (\cos \theta_1 + \dots + \cos \theta_n)^2 + (\sin \theta_1 + \dots + \sin \theta_n)^2 = 0$$

$$\therefore \cos \theta_1 + \dots + \cos \theta_n = 0 = \sin \theta_1 + \dots + \sin \theta_n$$

**Ans. :** (A), (B), (C)

(65)  $P(x)$  and  $Q(x)$  are polynomials. If  $x^2 + x + 1$  is a factor of  $f(x) = P(x^3) + xQ(x^3)$ , then .....

(A)  $(x-1)$  is a factor of  $P(x)$  but not a factor of  $Q(x)$ .

(B)  $(x-1)$  is a factor of  $Q(x)$  but not a factor of  $P(x)$ .

(C)  $(x-1)$  is a factor of  $P(x)$  and  $Q(x)$ .

(D)  $(x-1)$  is a factor of  $f(x)$ .

**Sol. :**  $f(x) = P(x^3) + xQ(x^3)$  (1)

Factors of  $1 + x + x^2$  are also factors of  $f(x)$ .

$$\text{Roots of } 1 + x + x^2 \text{ are } \omega, \omega^2, \text{ where } \omega = \frac{-1+i\sqrt{3}}{2}, \quad \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

$$f(\omega) = 0$$

$$\therefore P(\omega^3) + \omega Q(\omega^3) = 0$$

$$\therefore P(1) + \omega Q(1) = 0 \quad (2)$$

Now,  $f(\omega^2) = 0$

$\therefore P(\omega^6) + \omega^2 Q(\omega^6) = 0$

$\therefore P(1) + \omega^2 Q(1) = 0$  (3)

From (2) and (3),

$P(1) = 0$  and  $Q(1) = 0$

$\therefore (x - 1)$  is a factor of  $P(x)$  and  $Q(x)$ .

$\therefore f(1) = P(1) + Q(1) = 0$

$\therefore f(1) = 0$

$\therefore (x-1)$  is a factor of  $f(x)$

**Ans. : (C), (D)**

**: 'Match the Following' Type Questions :**

(66) For  $z_k = \cos\left(\frac{2k\pi}{10}\right) + i \sin\left(\frac{2k\pi}{10}\right), k = 1, 2, \dots, 9$

|                                                                                                        |           |
|--------------------------------------------------------------------------------------------------------|-----------|
| (P) For each $z_k$ , there exists $z_j$ such that<br>$z_k z_j = 1$                                     | (1) true. |
| (Q) There exists $k \in \{1, 2, \dots, 9\}$ such<br>that $z_1 \cdot z = z_k$ has no solution in the C. | (2) false |
| (R) $\frac{ 1-z_1   1-z_2  \dots  1-z_9 }{10}$                                                         | (3) 1     |
| (S) $1 - \sum_{k=1}^9 \cos\left(\frac{2k\pi}{10}\right)$                                               | (4) 2     |

(A) (P)→(1), (Q)→(2), (R)→(4), (S)→(3)

(B) (P)→(2), (Q)→(1), (R)→(3), (S)→(4)

(C) (P)→(1), (Q)→(2), (R)→(3), (S)→(4)

(D) (P)→(2), (Q)→(1), (R)→(4), (S)→(3)

**Sol. :** (P)  $z_k$  is a 10th root of 1.

$\therefore \bar{z}_k$  is also 10th root of 1.

If  $z_j = \bar{z}_k$ , then statement (P) is true. (P)→(1)

(Q)  $z_1 \neq 0$ . By taking  $z = \frac{z_k}{z_1}$ , then statement (Q) is false. (Q)→(2)

(R)  $z^{10} - 1 = (z - 1)(z - z_1) \dots (z - z_9)$

$\therefore \frac{z^{10} - 1}{z - 1} = (z - z_1)(z - z_2) \dots (z - z_9) = 1 + z + z^2 + \dots + z^9$

$(1 - z_1)(1 - z_2) \dots (1 - z_9) = 10$

$$\therefore \frac{|1-z_1| |1-z_2| \dots |1-z_9|}{10} = 1 \quad (R) \rightarrow (3)$$

(S) It is not necessary to solve but required to be understood.

$$1 + z_1 + z_2 + \dots + z_9 = 0$$

$$\therefore \operatorname{Re}(1) + \operatorname{Re}(z_1) + \dots + \operatorname{Re}(z_9) = 0$$

$$\therefore \operatorname{Re}(z_1) + \dots + \operatorname{Re}(z_9) = -1$$

$$\therefore 1 - \sum_{k=1}^9 \cos\left(\frac{2k\pi}{10}\right) = 2$$

**Ans. : (C)**

**Note :** P  $\rightarrow$  (1) and (R)  $\rightarrow$  (3) are enough for the answer.

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                          |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|
| (67) | <p>(P) If <math>\left(\frac{1-i}{1+i}\right)^n</math> is a purely imaginary number and its imaginary part is a positive number, then the minimum value of <math>n</math></p> <p>(Q) <math>\left(\sum_{m=1}^{2n+1} i^{2m}\right)</math></p> <p>(R) For <math>\Delta ABC</math>, <math>1 + e^{iA} \cdot e^{iB} \cdot e^{iC} =</math></p> <p>(S) If <math>z \in \mathbb{C}</math> and <math>\arg(z) = \theta</math></p> <p>The maximum value of <math>\left \frac{z}{ z } - 1\right </math> is .....</p> | <p>(i) 2</p> <p>(ii) -1</p> <p>(iii) 3</p> <p>(iv) 0</p> |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|

|     | (P)   | (Q)   | (R)   | (S)  |
|-----|-------|-------|-------|------|
| (A) | (i)   | (ii)  | (iii) | (iv) |
| (B) | (ii)  | (iii) | (iv)  | (i)  |
| (C) | (iii) | (ii)  | (iv)  | (i)  |
| (D) | (iv)  | (iii) | (ii)  | (i)  |

**Sol. :** (P)  $\frac{1-i}{1+i} = -i$

$$\therefore \left(\frac{1-i}{1+i}\right)^n = (-i)^n. \text{ For } n = 1, 3, 5, \dots \text{ we get purely imaginary number.}$$

For  $n = 3$ ,  $\left(\frac{1-i}{1+i}\right)^3 = (-i)^3 = i$  has positive imaginary part. We get, (P)  $\rightarrow$  (iii)

((P)  $\rightarrow$  (iii) occurs only in answer (C). Hence (C) is the answer.)

(Q)  $\left(\sum_{m=1}^{2n+1} i^{2m}\right)$

$$= \{(-1) + (-1)^2 + (-1)^3 + \dots + (-1)^{2n+1}\} = (-1) \quad (Q) \rightarrow (ii)$$

(R)  $1 + e^{iA} e^{iB} e^{iC}$

$$= 1 + \cos(A+B+C) + i \sin(A+B+C)$$

$$= 1 + \cos \pi + i \sin \pi = 0$$

(R)  $\rightarrow$  (iv)

(S) Let  $z = r(\cos \theta + i \sin \theta)$

$$\frac{z}{|z|} = (\cos \theta + i \sin \theta)$$

$$\begin{aligned}
\therefore \left| \frac{z}{|z|} - 1 \right| &= |(\cos \theta - 1 + i \sin \theta)| \\
&= \left| -2\sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right| \\
&= 2 \left| \sin \frac{\theta}{2} \right|
\end{aligned}$$

$$\therefore \left| \frac{z}{|z|} - 1 \right| \text{ has the maximum value } 2. \quad (\text{S}) \rightarrow (\text{i}) \quad \text{Ans. : (C)}$$

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                          |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|
| (68) | <p>(P) If <math>\left  z + \frac{1}{z} \right  = 2</math>, then minimum value of <math>\frac{ z }{\tan \frac{\pi}{8}}</math> is .....</p> <p>(Q) If <math> z  = 1</math> and <math>z^{2n} + 1 \neq 0</math>, then <math>\frac{z^n}{z^{2n} + 1} - \frac{z^{-n}}{(\bar{z})^{2n} + 1} = \dots</math></p> <p>(R) If <math>8iz^3 + 12z^2 - 18z + 27i = 0</math>, then <math>2 z  = \dots</math></p> <p>(S) If <math>z_1, z_2, z_3, z_4</math> are the roots of the equation <math>z^4 + z^3 + z^2 + z + 1 = 0</math>, then <math>\prod_{i=1}^{i=4} (z_i + 2) = \dots</math></p> | <p>(i) 0</p> <p>(ii) 3</p> <p>(iii) 11</p> <p>(iv) 1</p> |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|

|     | (P)  | (Q)   | (R)   | (S)   |
|-----|------|-------|-------|-------|
| (A) | (i)  | (ii)  | (iii) | (iv)  |
| (B) | (ii) | (ii)  | (iv)  | (iii) |
| (C) | (iv) | (iii) | (ii)  | (i)   |
| (D) | (iv) | (i)   | (ii)  | (iii) |

**Sol. :** (P)  $\left| |z| - \frac{1}{|z|} \right| \leq \left| z + \frac{1}{z} \right| = 2$

$$\therefore -2 \leq |z| - \frac{1}{|z|} \leq 2$$

$$\therefore |z|^2 + 2|z| - 1 \geq 0 \text{ and } |z|^2 - 2|z| - 1 \leq 0$$

$$\therefore (|z| + 1)^2 \geq 2, (|z| - 1)^2 \leq 2 \quad (-\sqrt{2} \leq |z| - 1 \leq \sqrt{2})$$

$$\therefore |z| \geq \sqrt{2} - 1, |z| \leq \sqrt{2} + 1$$

$$\therefore \text{The minimum value of } |z| \text{ is } \sqrt{2} - 1.$$

$$\therefore \text{The minimum value of } \frac{|z|}{\tan \frac{\pi}{8}} \text{ is } \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = 1 \quad (\text{P}) \rightarrow (\text{iv})$$

(Q) Let  $z = \cos \theta + i \sin \theta$

$$\begin{aligned} \frac{z^n}{z^{2n} + 1} - \frac{\bar{z}^{-n}}{(\bar{z})^{2n} + 1} &= \frac{\cos n\theta + i \sin n\theta}{1 + \cos 2n\theta + i \sin 2n\theta} - \frac{\cos n\theta - i \sin n\theta}{1 + \cos 2n\theta - i \sin 2n\theta} \\ &= \frac{\cos n\theta + i \sin n\theta}{2 \cos n\theta (\cos n\theta + i \sin n\theta)} - \frac{\cos n\theta - i \sin n\theta}{2 \cos n\theta (\cos n\theta - i \sin n\theta)} \\ &= \frac{1}{2 \cos n\theta} - \frac{1}{2 \cos n\theta} = 0 \end{aligned} \quad (\text{Q}) \rightarrow (\text{i})$$

(So, answer is D). (You could do it by taking  $\bar{z} = \frac{1}{z}$ )

(R)  $8iz^3 + 12z^2 - 18z + 27i = 0$

$\therefore (2iz + 3)(4z^2 + 9i) = 0$

$\therefore z = \frac{3}{2}i, z^2 = \frac{-9}{4}i$

$\therefore 2|z| = 3$

(R)  $\rightarrow$  (ii)

(S)  $z^4 + z^3 + z^2 + z + 1$

$= (z - z_1)(z - z_2)(z - z_3)(z - z_4)$

Let  $z = -2$ ,

$\prod_{i=1}^4 (z_i + 2) = (-2)^4 + (-2)^3 + (-2)^2 + (-2) + 1 = 11$  (S)  $\rightarrow$  (iii)

**Ans. :** (D)

**Note :** It is easy to calculate (R) and (S). From (R)  $\rightarrow$  (ii), (S)  $\rightarrow$  (iii), we get (D).

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (69) | <p>(P) The area of the triangle formed by the roots of the cubic equation <math>(z + \alpha\beta)^3 = \alpha^3, (\alpha \neq 0)</math></p> <p>(Q) Radius of the circle <math>\left  \frac{z - \alpha}{z - \bar{\alpha}} \right  = 2, \alpha \in \mathbb{C}</math></p> <p>(R) If <math>\arg z = \alpha</math> and <math> z - 1  = 1</math>, then <math>\left  \frac{z - 2}{z} \right  = \dots\dots</math></p> <p>(S) <math>z_1</math> and <math>z_2</math> are the roots of <math>z^2 + pz + q = 0</math>, in Argand diagram. If <math>m\angle AOB = \alpha \neq 0</math> with <math>A(z_1), B(z_2)</math> and <math>OA = OB</math>, then <math>\frac{p^2}{q} = \dots\dots</math></p> | <p>(i) <math> \tan \alpha </math></p> <p>(ii) <math>\frac{3\sqrt{3}}{4}  \alpha ^2</math></p> <p>(iii) <math>\frac{2}{3}  \alpha - \bar{\alpha} </math></p> <p>(iv) <math>4 \cos^2 \frac{\alpha}{2}</math></p> |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

|     | (P)  | (Q)   | (R)   | (S)   |
|-----|------|-------|-------|-------|
| (A) | (i)  | (ii)  | (iii) | (iv)  |
| (B) | (iv) | (iii) | (ii)  | (i)   |
| (C) | (ii) | (iii) | (i)   | (iv)  |
| (D) | (ii) | (i)   | (iv)  | (iii) |

**Sol. :** (P)  $(z + \alpha\beta)^3 = \alpha^3 \Rightarrow z + \alpha\beta = \alpha, \omega\alpha, \omega^2\alpha$  where  $\omega = \frac{-1 \pm i\sqrt{3}}{2}$

Implies,  $z = \alpha - \alpha\beta, \omega\alpha - \alpha\beta, \omega^2\alpha - \alpha\beta$

Let  $z_1 = \alpha - \alpha\beta, z_2 = \omega\alpha - \alpha\beta, z_3 = \omega^2\alpha - \alpha\beta$

Now,  $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = \sqrt{3} |\alpha|$

$\therefore z_1, z_2, z_3$  are vertices of an equilateral triangle.

$$\text{Area} = \frac{\sqrt{3}}{4} |z_1 - z_2|^2 = \frac{3\sqrt{3}}{4} |\alpha|^2 \quad (\text{P}) \rightarrow (\text{ii})$$

$$(\text{Q}) \quad |z - \alpha|^2 = 4 |z - \bar{\alpha}|^2$$

$$\therefore (z - \alpha) (\bar{z} - \bar{\alpha}) = 4 (z - \bar{\alpha}) (\bar{z} - \alpha)$$

$$\therefore z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} = 4 (z\bar{z} - \alpha z - \bar{\alpha}\bar{z} + \alpha\bar{\alpha})$$

$$\therefore 3z\bar{z} + (\alpha - 4\bar{\alpha})\bar{z} + (\bar{\alpha} - 4\alpha)z + 3\alpha\bar{\alpha} = 0$$

$$\therefore z\bar{z} + \frac{\alpha - 4\bar{\alpha}}{3}\bar{z} + \frac{\bar{\alpha} - 4\alpha}{3}z + \alpha\bar{\alpha} = 0 \quad (1)$$

We know that the circle with centre  $a$  and radius  $\sqrt{|a|^2 - b}$  has the equation

$$z\bar{z} - a\bar{z} - \bar{a}z + b = 0 \quad (a \in \mathbb{C}, b \in \mathbb{R}).$$

Radius of circle (1) is,

$$\sqrt{\left|\frac{\alpha - 4\bar{\alpha}}{3}\right|^2 - \alpha\bar{\alpha}} = \sqrt{-\frac{4}{9}(\alpha - \bar{\alpha})^2} = \frac{2}{3} |\alpha - \bar{\alpha}| \quad (\text{Q}) \rightarrow (\text{iii})$$

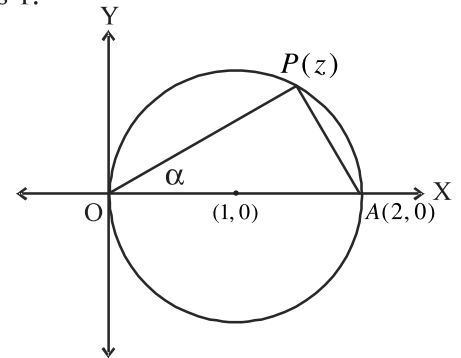
(R) Point  $P(z)$  lies on the circle with centre  $(1, 0)$  and radius 1.

$$m\angle OPA = \pm \frac{\pi}{2}$$

$$\therefore \frac{2 - z}{0 - z} = \left| \frac{2 - z}{z} \right| e^{\pm \frac{i\pi}{2}}$$

$$\therefore \frac{z - 2}{z} = \frac{AP}{OP} (\pm i) = \pm i \tan \alpha$$

$$\therefore \left| \frac{z - 2}{z} \right| = |\tan \alpha|$$



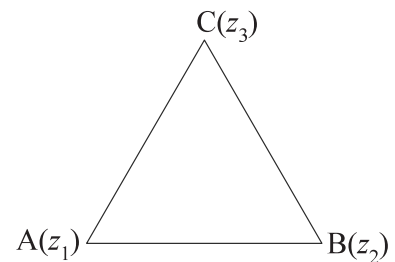
(R)  $\rightarrow$  (i)

(S) In a triangle with vertices  $A(z_1), B(z_2), C(z_3)$ , if  $m\angle BAC = \alpha$ ,

$$\text{then } \frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB} (\cos \alpha + i \sin \alpha)$$

Let  $A = O, C = A$

$$\text{Now, } \frac{z_2}{z_1} = \frac{OA}{OB} (\cos \alpha + i \sin \alpha) = \cos \alpha + i \sin \alpha$$



$$\therefore \frac{z_2 - z_1 \cos \alpha}{z_1} = i \sin \alpha. \quad \text{So, } z_2^2 - 2z_1 z_2 \cos \alpha + z_1^2 \cos^2 \alpha = -z_1^2 \sin^2 \alpha$$

$$\therefore z_1^2 + z_2^2 = 2z_1 z_2 \cos \alpha$$

$$\therefore (z_1 + z_2)^2 = 2z_1 z_2 (1 + \cos \alpha)$$

$$\text{But } z_1 + z_2 = -p, z_1 z_2 = q$$

$$\therefore \frac{p^2}{q} = 4 \cos^2 \frac{\alpha}{2} \quad (\text{S}) \rightarrow (\text{iv})$$

**Note :** Alternative R has different answer in all (A), (B), (C), (D). By solving R  $\rightarrow$  (i) we get answer (C). **Ans. : (C)**

(70) If  $4\arg(z - 5i) = 2\arg(z + 3) = \cos^{-1}(-1)$  for complex number  $z$ , then .....

(A)  $\bar{z} = 2i + 3$

(B)  $z$  lies on  $x + y + 1 = 0$

(C)  $|z|$  is the radius of  $x^2 + y^2 + 2x - 4y - 8 = 0$

(D)  $z + \bar{z} < 0$

**Sol. :**  $4\arg(z - 5i) = 2\arg(z + 3) = \cos^{-1}(-1) = \pi$

$$\therefore \tan^{-1} \left( \frac{y-5}{x} \right) = \frac{\pi}{4}, \quad \tan^{-1} \left( \frac{y}{x+3} \right) = \frac{\pi}{2} \quad (z = x + iy)$$

$$\therefore y - 5 = x, \quad x + 3 = 0 \text{ giving } x = -3, y = 2$$

$$\therefore z = -3 + 2i \quad \text{So, } \bar{z} = -3 - 2i$$

Point  $(-3, 2)$  lies on  $x + y + 1 = 0$ .

$$\therefore z \text{ lies on } x + y + 1 = 0. \quad (\text{B})$$

Radius of the circle  $x^2 + y^2 + 2x - 4y - 8 = 0$  is  $\sqrt{1+4+8} = \sqrt{13} = |z|$

$$\therefore |z| \text{ is the radius of the circle } x^2 + y^2 + 2x - 4y - 8 = 0 \quad (\text{C})$$

$$\text{Now, } z + \bar{z} = -6 < 0 \quad (\text{D})$$

**Ans. : (B), (C), (D)**

(71) If  $f(x) = ax^2 + bx$ , where  $f(1) = 2$ ,  $f(i^2) = 0$  and  $f(z + z^{-1}) = 0$ , then  $|z| = \dots$

(A) 0

(B)  $\frac{1}{2}$

(C) 2

(D) 1

**Sol. :**  $f(x) = ax^2 + bx$

$$f(1) = 2. \quad \text{So, } a + b = 2 \quad (1)$$

$$f(i^2) = f(-1) = 0. \quad \text{So, } a - b = 0 \quad (2)$$

From (1) and (2),  $a = 1, b = 1$

$$\therefore f(x) = x^2 + x$$

$$f(z + z^{-1}) = 0. \quad \text{So, } \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) = 0$$

$$\therefore z^4 + z^3 + 2z^2 + z + 1 = 0$$

$$\therefore (z^2 + 1)(z^2 + z + 1) = 0$$

$$\therefore z = \pm i, \omega, \omega^2$$

$$\therefore |z| = 1$$

**Ans. : (D)**

(72) Match the equation on the left with curve represented on the right :

|     |                                       |                              |
|-----|---------------------------------------|------------------------------|
| (P) | $ z - 3  +  z - i  = 10$              | (i) Circle                   |
| (Q) | $\left  \frac{2z-3}{z-i} \right  = 2$ | (ii) Hyperbola               |
| (R) | $z^2 + \bar{z}^2 = 5$                 | (iii) Line                   |
| (S) | $\left  \frac{z-6}{z-2i} \right  = 3$ | (iv) Ellipse<br>(v) Parabola |

|     | (P)  | (Q)   | (R)   | (S)   |
|-----|------|-------|-------|-------|
| (A) | (iv) | (iii) | (ii)  | (i)   |
| (B) | (iv) | (ii)  | (i)   | (iii) |
| (C) | (iv) | (ii)  | (iii) | (i)   |
| (D) | (i)  | (iv)  | (ii)  | (iii) |

**Sol.** : We know that  $|z - z_1| + |z - z_2| = \lambda$  is the equation of an ellipse with the foci  $z_1, z_2$  and the length of the major axis is  $\lambda$ . ( $\lambda > |z_1 - z_2|$ )

$\therefore$  The equation  $|z - 3| + |z - i| = 10$  represents an ellipse.  $\lambda = 10 > |z_1 - z_2| = \sqrt{10}$  (P)  $\rightarrow$  (iv)

$$(Q) \quad \left| \frac{2z-3}{z-i} \right| = 2. \quad \text{So,} \quad \left| \frac{z-\frac{3}{2}}{z-i} \right| = 1 \quad \Rightarrow \quad \left| z - \frac{3}{2} \right| = |z - i|$$

$\therefore z$  is on the perpendicular bisector of the line segment joining  $\left(\frac{3}{2}, 0\right)$  and  $(0, 1)$ . (Q)  $\rightarrow$  (iii)

(R) Let  $z = x + iy$

$$\therefore z^2 + \bar{z}^2 = 5 \Rightarrow (x + iy)^2 + (x - iy)^2 = 5$$

$$\therefore x^2 - y^2 + x^2 - y^2 = 5$$

$$\therefore x^2 - y^2 = \frac{5}{2} \text{ represents a hyperbola.} \quad (R) \rightarrow (ii)$$

$$(S) \quad \left| \frac{z-6}{z-2i} \right| = 3 \text{ represents a circle.} \quad (S) \rightarrow (i)$$

**Note** : It is easy to calculate (S)  $\rightarrow$  (i), (R)  $\rightarrow$  (ii) from this immediately we get Ans. (A).

**Ans.** : (A)

|      |                                                                                      |                         |
|------|--------------------------------------------------------------------------------------|-------------------------|
| (73) | (P) Centre of the circle $ z - 2 ^2 +  z - 4i ^2 = 20$                               | (i) $1 - i$             |
|      | (Q) Centre of the circle $\left  \frac{z-1}{z+1} \right  = \frac{1}{2}$              | (ii) $\frac{5}{3} + 0i$ |
|      | (R) Centre of the circle<br>$z\bar{z} - (1+i)z - (1+i)\bar{z} + 7 = 0$               | (iii) $-4 - i$          |
|      | (S) Centre of the circle $\arg \left( \frac{z+3+4i}{z+5-2i} \right) = \frac{\pi}{2}$ | (iv) $1 + 2i$           |

- |     |      |       |       |       |
|-----|------|-------|-------|-------|
|     | (P)  | (Q)   | (R)   | (S)   |
| (A) | (iv) | (iii) | (ii)  | (i)   |
| (B) | (iv) | (ii)  | (i)   | (iii) |
| (C) | (iv) | (ii)  | (iii) | (i)   |
| (D) | (i)  | (iv)  | (ii)  | (iii) |

**Sol. :** (P) We know that, if  $k > \frac{1}{2} |z_1 - z_2|^2$ , then  $z_1$  and  $z_2$  are the end points of a diameter of the circle

$$|z - z_1|^2 + |z - z_2|^2 = k.$$

$\therefore |z - 2|^2 + |z - 4i|^2 = 20$  is a circle. Centre of this circle is  $\frac{2+4i}{2} = 1 + 2i$ . (P)  $\rightarrow$  (iv)

(Q)  $4|z - 1|^2 = |z + 1|^2$

$$\therefore 4(z\bar{z} - z - \bar{z} + 1) = z\bar{z} + z + \bar{z} + 1$$

$$\therefore 3z\bar{z} - 5z - 5\bar{z} + 3 = 0$$

$$\therefore z\bar{z} - \frac{5}{3}z - \frac{5}{3}\bar{z} + 1 = 0$$

We know that the centre of the circle  $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$  is  $-a$ .

$\therefore$  The centre of the given circle is  $\frac{5}{3} + 0i$ . (Q)  $\rightarrow$  (ii)

(R) The centre of  $z\bar{z} - (1+i)z - (1-i)\bar{z} + 7 = 0$  is  $1 - i$ . (R)  $\rightarrow$  (i)

(S) We know that,  $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{2}$  is a circle whose end points of a diameter are  $z_1$  and  $z_2$ .

$\therefore$  The centre of the given circle is  $\frac{-3-4i+(-5+2i)}{2} = -4 - i$ . (S)  $\rightarrow$  (iii) **Ans. : (B)**

**Note :** (P)(iv) followed by S(iii) gives answer (B).

### : Assertion and Reason Type Questions :

**Explanation :** Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, options (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

(A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.

(B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is false.

(D) Statement 1 is false, statement 2 is true.

(74) For complex number  $z$ ,  $\arg\left(\frac{z-1}{z-2}\right) = \frac{\pi}{4}$

**Statement 1:** Locus of  $z$  is a circle.

**Statement 2 :** In a plane, the set of all points which are at a constant distance from a fixed point represents a circle.

**Sol. :**  $\arg\left(\frac{z-1}{z-2}\right) = \frac{\pi}{4}$

$$\therefore \arg\left(\frac{(x-1)+iy}{(x-2)+iy}\right) = \frac{\pi}{4} \quad (z = x + iy)$$

$$\therefore \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x-2}\right) = \frac{\pi}{4} \quad \left(\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\right)$$

$$\therefore \tan^{-1} \left\{ \frac{\frac{y}{x-1} - \frac{y}{x-2}}{\frac{(x-1)(x-2)+y^2}{(x-1)(x-2)}} \right\} = \frac{\pi}{4}$$

$$\therefore \frac{y(x-2)-y(x-1)}{(x-1)(x-2)+y^2} = 1$$

$\therefore x^2 + y^2 - 3x + y + 2 = 0$  which represents a circle. Statement 1 is true.

Statement (2) is true, but statement (1) is not explained by it.

**Ans. : (B)**

(75) If  $|z| = 1$  and  $z \neq \pm 1$ , then a possible value of  $\arg(z) - \arg(z+1) - \arg(z-1)$  is .....

(A)  $-\frac{\pi}{6}$

(B)  $\frac{\pi}{3}$

(C)  $-\frac{\pi}{2}$

(D)  $\frac{\pi}{4}$

**Sol. :**  $\arg(z) - \arg(z+1) - \arg(z-1) = \arg\left(\frac{z}{z^2-1}\right)$   
 $= \arg\left(\frac{z}{z^2-z\bar{z}}\right) \quad (\because |z| = 1, z\bar{z} = |z|^2)$   
 $= \arg\left(\frac{1}{z-\bar{z}}\right)$   
 $= \arg\left(\frac{1}{2iy}\right) \quad (z = x + iy)$   
 $= \arg(\text{Purely imaginary number})$

$\therefore$  Alternative (C) is true.

**Ans. : (C)**

(76)  $c$  is a complex number with  $|c| = 1$ . If the roots of  $cz^2 + z + 1 = 0$  are purely imaginary, then  $\tan(\arg c)$  .....

(A)  $\frac{\sqrt{5}-1}{2}$

(B)  $\frac{\sqrt{5}+1}{2}$

(C)  $\sqrt{\frac{\sqrt{5}+1}{2}}$

(D)  $\sqrt{\frac{\sqrt{5}-1}{3-\sqrt{5}}}$

**Sol. :** Suppose,  $z = iy$  is a root of the given equation. Suppose  $c = \cos \alpha + i \sin \alpha$

$$\therefore cz^2 + z + 1 = 0$$

$$\therefore -(\cos \alpha + i \sin \alpha)y^2 + iy + 1 = 0$$

$$\therefore (\cos \alpha \cdot y^2 - 1) + (\sin \alpha \cdot y^2 - y)i = 0$$

$$\therefore y^2 \cos \alpha = 1, \quad y \sin \alpha = 1, \quad (y \neq 0)$$

$$\therefore y^2 \cos \alpha - y^2 \sin^2 \alpha = 0 \Rightarrow y^2 (\sin^2 \alpha - \cos \alpha) = 0$$

$$\therefore 1 - \cos^2 \alpha - \cos \alpha = 0$$

$$\therefore \cos^2 \alpha + \cos \alpha - 1 = 0$$

$$\therefore \cos \alpha = \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore \cos \alpha = \frac{\sqrt{5}-1}{2}$$

$$\left( \cos \alpha \neq \frac{-\sqrt{5}-1}{2} \right)$$

$$\therefore \sec^2 \alpha = \frac{4}{6-2\sqrt{5}} = \frac{2}{3-\sqrt{5}}$$

$$\therefore \tan^2 \alpha = \frac{2}{3-\sqrt{5}} - 1 = \frac{\sqrt{5}-1}{3-\sqrt{5}}$$

$$\therefore \tan \alpha = \sqrt{\frac{\sqrt{5}-1}{3-\sqrt{5}}} = \sqrt{\frac{\sqrt{5}-1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{2\sqrt{5}+2}{4}} = \sqrt{\frac{\sqrt{5}+1}{2}}$$

**Ans. : (C)**

●