Paper Code - 5404

M.Sc. Semester-III (Mathematics) (Old Course)
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MTHE A-3 Number Theory

કુલ સમયઃ– : ૬૦ મીનીટ **કુલ પ્રશ્નો : ૫૦** **વિદ્યાર્થીએ ટીક કરવાના પ્રશ્નો : ૩૫**

નોંધ : વિદ્યાર્થીએ માત્ર ૩૫ પ્રશ્નો જ ટીક કરવાના રહેશે. જો વધારે ટીક કરેલ હશે તો પ્રથમ ૩૫ પ્રશ્નો જ ગણતરીમાં લેવામાં આવશે.

1. For integers a, b , not both zero, determine one of the false statements among the following:
   (A) If $ax + by = 2$ for some integers x, ythen $gcd(a, b) = 2$. (B) If $a\ divides\ b \cdot c$ and $gcd(a, b) = 1$then $a\ divides\ c$.
   (C)The square of an odd integer is of the form $8k + 1$. (D) $cd(a, b) \cdot lcm(a, b) = |a \cdot b|$.
2. Determine one of the false statements among the following:
   (A) If there is no prime divisor $p \leq \sqrt{N}$ of an integer $N > 1$ then N is a prime.
   (B) If $n > 1$ is a prime then $2^n - 1$ is a prime .
   (C)There are infinitely many primes of the form $4k + 3$.
   (D) For integer $N > 2$, the common difference of an Arithmetic Progression of N-primes is divisible by every prime $q < N$.
3. Determine one of the false statements among the following:
   (A) For any integer $n \geq 3$, $\sum_{k=1}^{n} \mu(k!) = 1$.
   (B)For $n > 1$, $\varphi(n) = n - 1$ if and only if n is a prime .
   (C)All of the functions $\tau, \sigma, \mu, \varphi$ are multiplicative functions.
   (D) For a real number x, $[x] + [-x] = 0$.
4. Determine one of the false statements among the following:
   (A) If $a \equiv b(mod\ n)$then $(a, n) = gcd(b, n)$.
   (B)If $a \equiv b(mod\ n)$ then $a \equiv r(mod\ n)$ where r is the least non-negative remainder upon the division of n .
   (C)An integer $n > 1$ is divisible by 11 if and only if the sum of the digits of n (in the decimal representation of n) is divisible by 11 .
   (D) If $a \equiv b(mod\ n)$then$a^k \equiv b^k(mod\ n)$ for any positive integer k .
5. "If $gcd(a, n) = 1$ then _____ " is the false statement.
   (A) $a^{\varphi(n)} \equiv 1(mod\ n)$. (B)$a^n \equiv 1(mod\ n)$.
   (C)$a^{n-1} \equiv 1(mod\ n)$whenever n is prime. . (D)$ab \equiv ac(mod\ n)$implies $b \equiv c\ (mod\ n)$.
6. (i) Euler's Theorem (I) If a prime p with $p \nmid a$ then $a^{p-1} \equiv 1(mod\ p)$
   (ii) Fermat's Little Theorem (II) If a prime p then $(p - 1)! \equiv -1\ (mod\ p)$
   (iii)Wilson's Theorem (III) If$gcd(a, n) = 1$ then $a^{\varphi(n)} \equiv 1(mod\ n)$
   Determine correct Answer.
   (A) $(i) \leftrightarrow (II)$, $(ii) \leftrightarrow (I)$, $(iii) \leftrightarrow (III)$ (B) $(i) \leftrightarrow (III)$, $(ii) \leftrightarrow (I)$, $(iii) \leftrightarrow (II)$
   (C) $(i) \leftrightarrow (II)$, $(ii) \leftrightarrow (III)$, $(iii) \leftrightarrow (I)$ (D) $(i) \leftrightarrow (III)$, $(ii) \leftrightarrow (II)$, $(iii) \leftrightarrow (I)$
7. For the results:
   I. $n = \sum_{d|n} \varphi\ (d)$
   II. If $gcd(n, m) = 1$, then$n^{\varphi(m)} + m^{\varphi(n)} \equiv 1\ (mod\ mn)$
   Determine one of the true statements among the following:
   (A) (I) is true and (II) is true. (B)(I) is true and (II) is false.
   (C)(I) is false and (II) is true. (D)(I) is false and (II) is false.
8. If p is the only prime for which $3p + 1$ is a perfect square then $p =$ _____ .
   (A) 3 (B)7 (C)5 (D)11
9. For each integer $n > 1$, $n^2 + 4$is a _____.
   (A) Perfect square. (B)Composite number. (C)Prime. (D)Cube.
10. If p_n is the n^{th} prime in natural order, the define$p_n^{\#} + 1 = p_1 p_2 p_3 \cdots p_n + 1$. Which of the following is not true.
    (A) $7^{\#} + 1$is a prime. (B)$5^{\#} + 1$is a prime. (C)$11^{\#} + 1$is a prime. (D)$13^{\#} + 1$is a prime.
11. For integer $n > 2$, there are n-consecutive positive integers which are all _____.
    (A) Composite numbers. (B)Primes. (C)Perfect squares. (D)Cubes.
12. $2^n \equiv 3\ (mod\ 4)$ is true if
    (A) $n \equiv 2\ (mod\ 4)$ (B) $n \equiv 1\ (mod\ 4)$ (C) $n \equiv 3\ (mod\ 4)$ (D) $n \equiv 0\ (mod\ 4)$
13. Which the following statement is true.
    (A) If $ac \equiv bc\ (mod\ n)$then $a \equiv b\ (mod\ n)$.
    (B)$2^{20} - 1$is a composite number.
    (C)If $a \equiv b\ (mod\ n)$ and $a \equiv b\ (mod\ m)$then $a \equiv b\ (mod\ nm)$.
    (D)$0, 1^2, 2^2, 3^2, \cdots, 10^2$form a complete set of residues modulo 11.
14. What is the remainder when 4444^{4444} is divided by 9?
    (A) 2 (B)4 (C)8 (C)7
15. Which of the following congruence equation is not solvable?
    (A) $140x \equiv 133\ (mod\ 301)$. (B) $6x \equiv 15\ (mod\ 21)$. (C) $34x \equiv 60\ (mod\ 98)$. (D) $25x \equiv 15\ (mod\ 20)$.
16. If $gcd(a, 30) = 1$, then
    (A) $a^4 \equiv 1\ (mod\ 60)$ (B) $a^4 \equiv 1\ (mod\ 30)$ (C) $a^4 \equiv -1\ (mod\ 60)$ (D) $a^8 \equiv -1\ (mod\ 30)$

17. The congruence equation $x^2 + 1 \equiv 0\ (mod\ p)$, where p is an odd prime, has a solution if and only if _____ .
(A) $p \equiv 3\ (mod\ 4)$ (B) $p \equiv 7\ (mod\ 8)$ (C) $p \equiv 3\ (mod\ 8)$ (D) $p \equiv 1\ (mod\ 4)$

18. Let $ord_n(a) = r$, then exactly one of the following is not necessarily true.
(A) $a \equiv b\ (mod\ n)$ implies $ord_n(a) = r$ (B) $\varphi(n) \equiv 0\ (mod\ r)$
(C) $ord_n(a) = \varphi(n)$ (D) $ord_n(a^k) = \frac{r}{gcd\,(k,r)}$ for positive integer k.

19. Let p be an odd prime and $(a,p) = 1$, which of the following is not true?
(A) $a^{\frac{p-1}{2}} \equiv 1\ (mod\ p)$ if and only if a is not a quadratic residue of p.
(B) $a^{\frac{p-1}{2}} \equiv 1\ (mod\ p)$ if and only if a is a quadratic residue of p.
(C) $a^{\frac{p-1}{2}} \equiv -1\ (mod\ p)$ if and only if a is a quadratic residue of p.
(D) a is a primitive root of p if and only if a is a quadratic residue of p.

20. Let p be an odd prime and $(a,p) = 1$, which of the following is not true?
(A) $a \equiv b\ (mod\ n)$ implies $\left(\frac{a}{p}\right) \neq \left(\frac{b}{p}\right)$ (B) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
(C) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} (mod\ p)$ (D) $\sum_{a=1}^{p-1}\left(\frac{a}{p}\right) = 1$

21. Which of the following is true?
(A) -1 is a quadratic residue of 31 (B)-2 is a quadratic residue of 31
(C)-2 is a quadratic non residue of 31 (D) 3 is a quadratic non residue of 31

22. For an odd prime p, which of the following is not true?
(A) $\left(\frac{3}{p}\right) = \begin{cases} 1 & if\ p \equiv 1\ (mod\ 6) \\ -1 & if\ p \equiv -1\ (mod\ 6) \end{cases}$ (B) $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$
(C) $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$ (D) $\left(\frac{-3}{p}\right) = \begin{cases} 1 & if\ p \equiv 1\ (mod\ 6) \\ -1 & if\ p \equiv -1\ (mod\ 6) \end{cases}$

23. Let p be an odd prime $p \equiv 3\ (mod\ 4)$, which of the following is not true?
(A) p is a sum of five squares. (B) p (is a sum of four squares.
(C) p is a sum of six squares. (D) p is a sum of two squares.

24. Fill the gap in such a way that the statement: "The equation 21x+15y=3 has _____" being correct.
(A) a particular solution which cannot be determined by the Euclidean algorithm.
(B)no positive integral solutions.
(C)infinitely many positive integral solutions.
(D)no integral solutions.

25. Which of the following is not a pythagorean triplet?
(A) For odd integer $n > 2$, $n,\ \frac{1}{2}(n^2-1), \frac{1}{2}(n^2-1)$ (B)For integers $a > 3, d > 0;\ \ a, a+d, a+2d$
(C)For even integer $n > 2$, $n,\ \left(\frac{n^2}{4}-1\right), \left(\frac{n^2}{4}+1\right)$ (D)For integer $t > 0, 2t+1, 2t(t+1),\ 2t(t+1)+1$

26. If x, y, z is a primitive pythagorean triple in which $z - x = 2$ then _____ .
(A) $x = r^2 - 1,\ y = 2r,\ z = r^2 + 1$ for some integer $r > 0$
(B) $x = 98, y = 20, z = 102$
(C) $x = 24, y = 10, z = 26$
(D) $x = r^2 - s^2,\ y = 2rs,\ z = r^2 + s^2$ for some integer $r > s > 0$

27. The primitive solutions of the equation $x^2 + y^2 = z^2$ are given by $x = r^2 - s^2,\ y = 2rs,\ z = r^2 + s^2$ for some integers r, s where
(A) $r > s > 0$, $gcd(r,s) = 1$ and $r \equiv s\ (mod\ 2)$ (B) $r > s > 0$ and $gcd(r,s) = 1$
(C) $r > s > 0$, $gcd(r,s) = 1$ and $r \not\equiv s\ (mod\ 2)$ (D) r, s are relatively prime integers and they are in opposite parity

28. The Diophantine equation $ax + by = c$ is solvable if _____ .
(A) $d|b$ where $gcd(a,c) = d$ (B) $d|c$ where $gcd(a,b) = d$ (C) $d|a$ where $gcd(b,c) = d$ (D) $gcd(a,c) = 1$

29. Which of the following integer is a sum of three squares?
(A) 7 (B) 15 (C) 60 (D) 30

30. Which of the following statement is true?
(A) Every positive integer can be represented as a sum of four squares.
(B) Every odd prime can be represented as a sum of two squares.
(C) Every positive integer can be represented as a sum of three squares.
(D) Every sufficiently large positive can be represented as a sum of four squares of positive integers.

31. The Diophantine equation $x^2 - y^2 = n$ is solvable if and only if _____ .
(A) $n \not\equiv 1\ (mod\ 4)$ (B) $n \not\equiv 2\ (mod\ 4)$ (C) $n \not\equiv 3\ (mod\ 4)$ (D) $n \not\equiv 0\ (mod\ 4)$

32. Which of the following Legendre symbol has the value -1?
(A) $\left(\frac{100}{7}\right)$ (B) $\left(\frac{2}{103}\right)$ (C) $\left(\frac{-4}{101}\right)$ (D) $\left(\frac{10}{7}\right)$

33. Which of the following is not true?
(A) $89|(2^{44} - 1)$ (B) $97|(2^{48} - 1)$ (C) $41|(2^{20} - 1)$ (D) $31|(2^{15} - 1)$

34. Which of the following Legendre symbol has the value 1?
(A) $\left(\frac{2}{83}\right)$ (B) $\left(\frac{-1}{103}\right)$ (C) $\left(\frac{3}{37}\right)$ (D) $\left(\frac{-3}{17}\right)$

35. Which of the following statement is true?
(A) Wilson's Theorem : If p is a prime and $p \nmid a$ then $a^{p-1} \equiv 1 (mod\ p)$
(B) Euler's Theorem : If a and n are relatively prime integers then $a^{\varphi(n)} \equiv 1 (mod\ n)$
(C) Fermat's Theorem : If p is a prime then $(p-1)! \equiv -1 (mod\ p)$
(D) Euler's Criterion: If p is a prime and $p \nmid a$ then $a^{p-1} \equiv 1 (mod\ p)$