

$$(92) \int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx = \dots$$

$$(A) -e^x \tan x + c$$

$$(B) -e^x \cot x + c$$

$$(C) e^x \tan x + c$$

$$(D) e^x \cot x + c$$

$$\begin{aligned} \text{ઉકેલ : } I &= \int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx \\ &= \int e^x \frac{2 - 2 \sin x \cos x}{2 \sin^2 x} dx \\ &= \int e^x (\operatorname{cosec}^2 x - \cot x) dx \\ &= -\int e^x (\cot x - \operatorname{cosec}^2 x) dx \\ &= -e^x \cot x + c \end{aligned}$$

જવાબ : (B)

$$(93) \text{ જો } \int \frac{(2x^2 + 3)}{(x^2 - 1)(x^2 - 4)} dx = \log \left(\frac{x-2}{x+2} \right)^p \left(\frac{x+1}{x-1} \right)^q + c, \text{ તો } p + q = \dots$$

$$(A) \frac{7}{4}$$

$$(B) \frac{1}{12}$$

$$(C) \frac{-1}{12}$$

$$(D) \frac{-7}{4}$$

$$\text{ઉકેલ : } I = \int \frac{(2x^2 + 3)}{(x^2 - 1)(x^2 - 4)} dx$$

$$\text{ધારો કે, } \frac{2x^2 + 3}{(x^2 - 1)(x^2 - 4)} = \frac{A}{x^2 - 1} + \frac{B}{x^2 - 4}$$

$$\therefore 2x^2 + 3 = A(x^2 - 4) + B(x^2 - 1)$$

સરખામણીને આંખે અચળ પદો સરખાવતાં,

$$\therefore A + B = 2, -4A - B = 3$$

$$\text{ઉકેલતાં, } A = \frac{-5}{3}, B = \frac{11}{3}$$

$$\begin{aligned} \therefore I &= \int \frac{\frac{-5}{3}}{(x^2 - 1)} dx + \int \frac{\frac{11}{3}}{x^2 - 4} dx \\ &= \frac{-5}{3} \times \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{11}{3} \times \frac{1}{2(2)} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \frac{-5}{6} \log \left| \frac{x-1}{x+1} \right| + \frac{11}{12} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \frac{5}{6} \log \left| \frac{x+1}{x-1} \right| + \frac{11}{12} \log \left| \frac{x-2}{x+2} \right| + c \\ &= \log \left[\left(\frac{x+1}{x-1} \right)^{\frac{5}{6}} \cdot \left(\frac{x-2}{x+2} \right)^{\frac{11}{12}} \right] + c \end{aligned}$$

$$\therefore p = \frac{11}{12} \text{ અને } q = \frac{5}{6}. \text{ આથી, } p + q = \frac{11}{12} + \frac{10}{12} = \frac{21}{12} = \frac{7}{4}$$

જવાબ : (A)

$$(94) \quad \int \frac{3x+2}{(x-2)^2(x-3)} dx = \dots\dots$$

$$(A) \quad 11 \log \left| \frac{x+3}{x+2} \right| - \frac{8}{x-2} + c$$

$$(B) \quad 11 \log \left| \frac{x-3}{x-2} \right| - \frac{8}{x-2} + c$$

$$(C) \quad 11 \log \left| \frac{x-3}{x-2} \right| + \frac{8}{x-2} + c$$

$$(D) \quad 11 \log \left| \frac{x+3}{x+2} \right| + \frac{8}{x-2} + c$$

$$\text{ઉકેલ : } I = \int \frac{3x+2}{(x-2)^2(x-3)} dx$$

$$\frac{3x+2}{(x-2)^2(x-3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)} \text{ છે.}$$

$$\therefore (3x+2) = A(x-2)(x-3) + B(x-3) + C(x-2)^2$$

$$\text{જો } x = 2, \quad \text{તો } 8 = -B \quad \text{આથી, } B = -8$$

$$\text{જો } x = 3, \quad \text{તો } 11 = C \quad \text{આથી, } C = 11$$

$$\text{જો } x = 0, \quad \text{તો } 2 = 6A - 3B + 4C$$

$$2 = 6A - 3(-8) + 4(11)$$

$$\therefore A = -11, \quad B = -8, \quad C = 11$$

$$\int \frac{3x+2}{(x-2)^2(x-3)} dx = \int \frac{-11}{x-2} dx + \int \frac{-8}{(x-2)^2} dx + \int \frac{11}{x-3} dx$$

$$= -11 \log |x-2| + \frac{8}{x-2} + 11 \log |x-3| + c$$

$$= 11 \log \left| \frac{x-3}{x-2} \right| + \frac{8}{x-2} + c$$

જવાબ : (C)

$$(95) \quad \int \frac{dx}{x(x^5+1)} = \dots\dots \quad (x > 0)$$

$$(A) \quad \frac{1}{5} \log x^4 (x^5+1) + c$$

$$(B) \quad \frac{1}{5} \log \left(\frac{1+x^5}{x^5} \right) + c$$

$$(C) \quad \frac{1}{5} \log \left(\frac{x^4}{(x^5+2)} \right) + c$$

$$(D) \quad \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) + c$$

$$\text{ઉકેલ : } I = \int \frac{dx}{x(x^5+1)} = \int \frac{dx}{x^6 \left(1 + \frac{1}{x^5} \right)}$$

$$\therefore I = \frac{-1}{5} \int \frac{1}{t} dt = \frac{-1}{5} \log |t| + c \quad \left(\text{ધારો કે, } 1 + \frac{1}{x^5} = t, \quad \text{આથી } \frac{-5}{x^6} dx = dt \right)$$

$$= \frac{-1}{5} \log \left(1 + \frac{1}{x^5} \right) + c = \frac{-1}{5} \log \left(\frac{x^5+1}{x^5} \right) + c = \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) + c$$

જવાબ : (D)

$$\text{બીજી રીત : } \int \frac{dx}{x(x^5+1)}$$

$$= \int \frac{x^4 dx}{x^5(x^5+1)}$$

$$= \frac{1}{5} \int \frac{dt}{t^2+t} \quad (x^5 = t, 5x^4 dx = dt)$$

$$= \frac{1}{5} \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{t+\frac{1}{2}-\frac{1}{2}}{t+\frac{1}{2}+\frac{1}{2}} \right| + c$$

$$= \frac{1}{5} \log \left(\frac{x^5}{x^5+1} \right) \quad (x > 0)$$

ત્રીજી રીત :

$$I = \frac{1}{5} \int \frac{dt}{t(t+1)}$$

$$= \frac{1}{5} \left[\int \frac{dt}{t} - \int \frac{dt}{t+1} \right]$$

$$= \frac{1}{5} \log \left| \frac{t}{t+1} \right|$$

$$= \frac{1}{5} \log \left| \frac{x^5}{x^5+1} \right|$$

$$(96) \int \frac{3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx = \dots\dots$$

$$(A) \frac{13}{12} x + \frac{5}{13} \log |3 \cos x + 2 \sin x| + c \quad (B) \frac{12}{13} x + \frac{5}{13} \log |3 \cos x + 2 \sin x| + c$$

$$(C) \frac{12}{13} x - \frac{5}{13} \log |3 \cos x + 2 \sin x| + c \quad (D) \frac{13}{12} x - \frac{5}{13} \log |3 \cos x + 2 \sin x| + c$$

$$\text{ઉકેલ : } I = \int \frac{3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$$

$$\text{ધારો કે } (3 \sin x + 2 \cos x) = l[(3 \cos x + 2 \sin x)] + m \left[\frac{d}{dx} (3 \cos x + 2 \sin x) \right]$$

$$\therefore 3 \sin x + 2 \cos x = l(3 \cos x + 2 \sin x) + m(-3 \sin x + 2 \cos x)$$

$$= (2l - 3m) \sin x + (3l + 2m) \cos x$$

$\sin x$ અને $\cos x$ ની સહગુણકો સરખાવતાં,

$$2l - 3m = 3, \quad 3l + 2m = 2$$

$$\text{ઉકેલતાં, } l = \frac{12}{13}, \quad m = \frac{-5}{13}$$

$$\therefore I = \int \frac{\frac{12}{13} (3 \cos x + 2 \sin x) - \frac{5}{13} (-3 \sin x + 2 \cos x)}{(3 \cos x + 2 \sin x)} dx$$

$$= \frac{12}{13} \int dx - \frac{5}{13} \int \frac{-3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$$

$$= \frac{12}{13} x - \frac{5}{13} \log |3 \cos x + 2 \sin x| + c$$

જવાબ : (C)

$$(97) \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx = \dots\dots$$

$$(A) \sin 2x + c \quad (B) -\frac{1}{2} \sin 2x + c \quad (C) \frac{1}{2} \sin 2x + c \quad (D) -\sin 2x + c$$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx \\
&= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx \\
&= \int (\sin^4 x - \cos^4 x) dx \quad (\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x) \\
&= \int (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) dx \\
&= \int -\cos 2x dx = -\frac{\sin 2x}{2} + c
\end{aligned}$$

જવાબ : (B)

$$(98) \int \frac{x^2}{(a+bx)^2} dx = \dots$$

$$(A) \frac{1}{b^2} \left[x + \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(B) \frac{1}{b^2} \left[x - \frac{2a}{b} \log(a+bx) + \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(C) \frac{1}{b^2} \left[x + \frac{2a}{b} \log(a+bx) + \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

$$(D) \frac{1}{b^2} \left[x + \frac{a}{b} - \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c \quad (\text{IIT : 1979})$$

$$\text{ઉકેલ : } I = \int \frac{x^2}{(a+bx)^2} dx$$

ધારો કે, $a + bx = t$. આથી, $b dx = dt$ આથી, $dx = \frac{1}{b} dt$. એટલે, $x = \frac{t-a}{b}$

$$\therefore I = \int \left(\frac{t-a}{b} \right)^2 \cdot \frac{1}{t^2} \cdot \frac{1}{b} dt$$

$$= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + a^2 t^{-2} \right) dt$$

$$= \frac{1}{b^3} \left[t - 2a \log |t| - \frac{a^2}{t} \right] + c$$

$$= \frac{1}{b^3} \left[(a+bx) - 2a \log |a+bx| - \frac{a^2}{a+bx} \right] + c$$

$$= \frac{1}{b^2} \left[x + \frac{a}{b} - \frac{2a}{b} \log(a+bx) - \frac{a^2}{b} \cdot \frac{1}{a+bx} \right] + c$$

જવાબ : (D)

$$(99) \quad \int \frac{\sqrt{x^2 - a^2}}{x} dx = \dots$$

$$(A) \quad \sqrt{x^2 - a^2} - a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c \quad (B) \quad \sqrt{x^2 - a^2} + a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c$$

$$(C) \quad \sqrt{x^2 - a^2} + a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c \quad (D) \quad \cot^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{ઉકેલ : } I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

$$\text{ધારીએ કે, } x^2 - a^2 = t^2. \quad \text{આથી, } x^2 = a^2 + t^2 \quad (t > 0)$$

$$\therefore 2x dx = 2t dt. \quad \text{આથી, } x dx = t dt$$

$$\begin{aligned} I &= \int \frac{\sqrt{x^2 - a^2}}{x} dx \\ &= \int \frac{\sqrt{x^2 - a^2}}{x^2} x dx \\ &= \int \frac{t}{a^2 + t^2} t dt \\ &= \int \frac{t^2 + a^2 - a^2}{t^2 + a^2} dt \\ &= \left(1 - \frac{a^2}{t^2 + a^2} \right) dt \\ &= t - a^2 \cdot \frac{1}{a} \tan^{-1} \left(\frac{t}{a} \right) + c \\ &= \sqrt{x^2 - a^2} - a \tan^{-1} \left(\frac{\sqrt{x^2 - a^2}}{a} \right) + c \end{aligned}$$

$$\text{બીજી રીત : } I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

$$\text{ધારીએ કે } x = a \sec \theta. \quad \text{આથી, } dx = a \sec \theta \tan \theta d\theta \quad (0 < \theta < \frac{\pi}{2}, a > 0)$$

$$\begin{aligned} &= \int \frac{\sqrt{a^2 \sec^2 \theta - a^2}}{a \sec \theta} a \sec \theta \tan \theta d\theta \\ &= a \int \tan^2 \theta d\theta \\ &= a \int (\sec^2 \theta - 1) d\theta \end{aligned}$$

$$= a \tan \theta - a \theta + c$$

$$= \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a} + c$$

$$= \sqrt{x^2 - a^2} - a \tan^{-1} \frac{\sqrt{x^2 - a^2}}{a} + c$$

જવાબ : (A)

$$(100) \int \tan^3 2x \cdot \sec 2x \, dx = \dots\dots$$

$$(A) \quad \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + c$$

$$(B) \quad \frac{1}{6} \sec^3 2x + \frac{1}{2} \sec 2x + c$$

$$(C) \quad \frac{1}{9} \sec^2 2x - \frac{1}{3} \sec 2x + c$$

$$(D) \quad \frac{1}{9} \sec^2 2x + \frac{1}{3} \sec 2x + c$$

$$\text{ઉકેલ : } I = \int \tan^3 2x \cdot \sec 2x \, dx$$

$$\text{ધારીએ } \sec 2x = t, \text{ તો } \sec 2x \tan 2x \, dx = \frac{1}{2} dt$$

$$I = \frac{1}{2} \int (t^2 - 1) \, dt$$

$$= \frac{1}{2} \cdot \frac{t^3}{3} - \frac{1}{2} t + c$$

$$= \frac{1}{6} \sec^3 2x - \frac{1}{2} \sec 2x + c$$

જવાબ : (A)

$$(101) \int \sqrt{\frac{a-x}{x}} \, dx = \dots\dots$$

$$(A) \quad \sin^{-1} \frac{x}{a} + \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$(B) \quad a \left[\sin^{-1} \sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}} \sqrt{\frac{a-x}{a}} \right] + c$$

$$(C) \quad \sin^{-1} \frac{x}{a} - \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$(D) \quad \sin^{-1} \frac{x}{a} - \frac{x}{a} \sqrt{a^2 - x^2} + c$$

$$\text{ઉકેલ : } I = \int \sqrt{\frac{a-x}{x}} \, dx$$

$$\text{ધારીએ } x = a \sin^2 \theta$$

$$(0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = 2a \sin \theta \cos \theta \, d\theta$$

$$\therefore I = \int \sqrt{\frac{\cos^2 \theta}{\sin^2 \theta}} 2a \sin \theta \cos \theta \, d\theta$$

$$= a \int 2 \cos^2 \theta \, d\theta$$

$$= a \int (1 + \cos 2\theta) \, d\theta$$

$$= a \left[\theta + \frac{\sin 2\theta}{2} \right] + c$$

$$= a [\theta + \sin \theta \cos \theta] + c$$

$$= a \left[\sin^{-1} \sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}} \sqrt{\frac{a-x}{a}} \right] + c$$

જવાબ : (B)

$$(102) \int \left(\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \right) dx = \dots$$

$$(A) \quad \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1}\sqrt{x}] + x + c$$

$$(B) \quad \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1}\sqrt{x}] - x + c$$

$$(C) \quad \frac{2}{\pi} [\sqrt{x-x^2} + (1-2x) \sin^{-1}\sqrt{x}] + x + c$$

$$(D) \quad \frac{2}{\pi} [\sqrt{x-x^2} + (1-2x) \sin^{-1}\sqrt{x}] - x + c$$

(IIT : 1986)

$$\text{Sol : } I = \int \left(\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \right) dx$$

$$= \frac{2}{\pi} \int \left\{ \sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x} \right) \right\} dx \quad (\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2})$$

$$= \frac{2}{\pi} \int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2} \right) dx$$

$$= \frac{4}{\pi} \int \sin^{-1}\sqrt{x} dx - x$$

$$I = \frac{4}{\pi} I_1 - x, \quad \text{where, } I_1 = \int \sin^{-1}\sqrt{x} dx$$

$$\text{Let } x = \sin^2\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\therefore dx = 2\sin\theta \cos\theta d\theta$$

$$\therefore I_1 = \int \theta \cdot \sin 2\theta d\theta$$

$$= \theta \int \sin 2\theta d\theta - \int \left(\frac{d}{d\theta} (\theta) \int \sin 2\theta d\theta \right) d\theta$$

$$= -\theta \cdot \frac{\cos 2\theta}{2} - \int 1 \cdot \left(\frac{-\cos 2\theta}{2} \right) d\theta$$

$$= -\frac{\theta}{2} \cos 2\theta + \frac{1}{4} \sin 2\theta + c$$

$$= -\frac{1}{2} (1-2x) \sin^{-1}\sqrt{x} + \frac{1}{2} \sqrt{x} \sqrt{1-x} + c$$

$$(\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2x, \sin 2\theta = 2\sin\theta \cos\theta = 2\sqrt{x} \sqrt{1-x})$$

$$= -\frac{1}{2} (1-2x) \sin^{-1}\sqrt{x} + \frac{\sqrt{x-x^2}}{2} + c$$

$$\text{Hence, } I = \frac{4}{\pi} I_1 - x$$

$$\begin{aligned}
&= \frac{4}{\pi} \left[-\frac{1}{2} (1-2x) \sin^{-1} \sqrt{x} + \frac{\sqrt{x-x^2}}{2} \right] - x + c \\
&= \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + c
\end{aligned}$$

જવાબ : (B)

$$(103) \int \frac{dx}{(2bx+x^2)^{\frac{3}{2}}} = \dots\dots (b > 0)$$

$$(A) \quad \frac{-1}{b^2} \frac{(x+2b)}{\sqrt{2bx+x^2}} + c$$

$$(B) \quad \frac{-1}{b^2} \frac{(x-b)}{\sqrt{2bx+x^2}} + c$$

$$(C) \quad \frac{-1}{b^2} \frac{2x}{\sqrt{2bx+x^2}} + c$$

$$(D) \quad \frac{-1}{b^2} \frac{(x+b)}{\sqrt{2bx+x^2}} + c$$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{1}{(2bx+x^2)^{\frac{3}{2}}} \cdot dx \\
&= \int \frac{1}{[(x^2+2bx+b^2)-(b)^2]^{\frac{3}{2}}} dx \\
&= \int \frac{dx}{[(x+b)^2-(b)^2]^{\frac{3}{2}}}
\end{aligned}$$

$$\text{હવે, } x+b = b \sec \theta \quad (0 < \theta < \frac{\pi}{2}, b > 0)$$

$$\therefore dx = b \sec \theta \tan \theta d\theta$$

$$\begin{aligned}
I &= \int \frac{b \sec \theta \tan \theta d\theta}{(b^2 \sec^2 \theta - b^2)^{\frac{3}{2}}} \\
&= \frac{1}{b^2} \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} \\
&= \frac{1}{b^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
&= \frac{1}{b^2} \frac{(\sin \theta)^{-1}}{-1} + c \\
&= \frac{-1}{b^2 \sin \theta} + c \\
&= \frac{-1}{b^2} \frac{(x+b)}{\sqrt{x^2+2bx}} + c
\end{aligned}$$

$$\left(\sec \theta = \frac{x+b}{b} \Rightarrow \cos \theta = \frac{b}{x+b} \text{ અને } \sin \theta = \sqrt{1 - \frac{b^2}{(x+b)^2}} = \frac{\sqrt{x^2+2bx}}{x+b} \right) \text{ જવાબ : (D)}$$

$$(104) \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Px + Q \log |9e^{2x} - 4| + c, \text{ તો } \dots$$

$$(A) \quad P = \frac{-3}{2}, Q = \frac{29}{36}$$

$$(B) \quad P = \frac{-3}{2}, Q = \frac{35}{36}$$

$$(C) \quad P = \frac{-1}{2}, Q = \frac{17}{36}$$

$$(D) \quad P = \frac{-1}{2}, Q = \frac{23}{36}$$

(IIT : 1989)

$$\text{ઉકેલ : } I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = \int \left(\frac{4e^{2x} + 6}{9e^{2x} - 4} \right) dx$$

$$\text{ધારો કે, } 4e^{2x} + 6 = m(9e^{2x} - 4) + n \left\{ \frac{d}{dx} (9e^{2x} - 4) \right\}$$

$$\therefore 4e^{2x} + 6 = m(9e^{2x} - 4) + n 18e^{2x}$$

$$4e^{2x} + 6 = (9m + 18n)e^{2x} - 4m$$

સરખાવકો સરખાવતાં $9m + 18n = 4$ અને $-4m = 6$. અર્થાત્, $m = \frac{-3}{2}$, $n = \frac{35}{36}$

$$\therefore I = \int \frac{\frac{-3}{2}(9e^{2x} - 4) + \frac{35}{36} \left\{ \frac{d}{dx} (9e^{2x} - 4) \right\}}{(9e^{2x} - 4)} dx$$

$$= \frac{-3}{2} \int dx + \frac{35}{36} \int \frac{\frac{d}{dx} (9e^{2x} - 4)}{(9e^{2x} - 4)} dx$$

$$= \frac{-3}{2} x + \frac{35}{36} \log (9e^{2x} - 4) + c$$

$$\text{તેથી, } P = \frac{-3}{2}, Q = \frac{35}{36}$$

જવાબ : (B)

બીજી રીત : $e^x = t$ લેતાં $e^x dx = dt$, $tdx = dt$

$$\therefore I = \int \frac{4t^2 + 6}{9t^2 - 4} \left(\frac{dt}{t} \right)$$

$$= \int \frac{(4t^2 + 6) dt}{t(3t + 2)(3t - 2)}$$

$$= \int \frac{-\frac{3}{2} dt}{t} + \int \frac{\frac{35}{12} dt}{3t + 2} + \int \frac{\frac{35}{12} dt}{3t - 2}$$

$$= -\frac{3}{2} \log |t| + \frac{35}{36} \log |(3t + 2)(3t - 2)|$$

$$= -\frac{3}{2} x + \frac{35}{36} \log (9e^{2x} - 4)$$

$$\therefore P = \frac{-3}{2}, Q = \frac{35}{36}$$

$$(105) \int \frac{2 \tan x + 3}{3 \tan x + 4} dx = \dots$$

$$(A) \quad \frac{24}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c \quad (B) \quad \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c$$

$$(C) \quad \frac{14}{25} x + \frac{2}{25} \log |3 \sin x + 4 \cos x| + c \quad (D) \quad \frac{4x}{25} + \frac{2}{25} \log |3 \sin x + 4 \cos x| + c$$

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{2 \tan x + 3}{3 \tan x + 4} dx \\ &= \int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx \end{aligned}$$

$$\text{હવે, ધારો કે } 2 \sin x + 3 \cos x = A(3 \sin x + 4 \cos x) + B \left[\frac{d}{dx} (3 \sin x + 4 \cos x) \right]$$

$$2 \sin x + 3 \cos x = (3A - 4B) \sin x + (4A + 3B) \cos x$$

$$\therefore \quad 3A - 4B = 2, \quad 4A + 3B = 3. \quad \text{આથી, } A = \frac{18}{25}, \quad B = \frac{1}{25}$$

$$\therefore \quad I = \frac{18}{25} \int dx + \frac{1}{25} \int \frac{\frac{d}{dx}(3 \sin x + 4 \cos x)}{3 \sin x + 4 \cos x} dx$$

$$= \frac{18}{25} x + \frac{1}{25} \log |3 \sin x + 4 \cos x| + c$$

જવાબ : (B)

$$(106) \int \sqrt{\tan x} dx = A \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + B \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + c, \text{ તો } \dots$$

$$(A) \quad A = \frac{1}{\sqrt{2}}, \quad B = 2\sqrt{2}$$

$$(B) \quad A = \frac{1}{\sqrt{2}}, \quad B = -\frac{1}{\sqrt{2}}$$

$$(C) \quad A = \frac{1}{\sqrt{2}}, \quad B = \frac{1}{2\sqrt{2}}$$

$$(D) \quad A = \frac{1}{\sqrt{2}}, \quad B = -\frac{1}{2\sqrt{2}}$$

$$\text{ઉકેલ : } I = \int \sqrt{\tan x} dx$$

$$\text{ધારો કે, } \tan x = t^2$$

$$\therefore \quad \sec^2 x dx = 2t dt$$

$$\therefore \quad dx = \frac{2t}{t^4 + 1} dt$$

$$\therefore \quad I = \int \frac{2t^2}{t^4 + 1} dt$$

$$= \int \frac{(t^2 + 1) + (t^2 - 1)}{(t^4 + 1)} dt$$

$$= \int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$$

$$\begin{aligned}
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} dt \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt
\end{aligned}$$

ધારો કે, પ્રથમ સંકલ્ય માટે $t - \frac{1}{t} = u$ અને દ્વિતીય સંકલ્ય માટે $t + \frac{1}{t} = v$

$$\therefore \left(1 + \frac{1}{t^2}\right) dt = du \text{ અને } \left(1 - \frac{1}{t^2}\right) dt = dv$$

$$\begin{aligned}
\therefore I &= \int \frac{du}{u^2 + 2} + \int \frac{dv}{v^2 - 2} \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{2}t} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t} \right| + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + c
\end{aligned}$$

$$\therefore A = \frac{1}{\sqrt{2}}, B = \frac{1}{2\sqrt{2}}$$

જવાબ : (C)

$$(107) \int \frac{(x^2 - 1) dx}{(x^2 + 1)\sqrt{x^4 + 1}} = \dots (x > 0)$$

$$(A) \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}x}{x^2 + 1} \right) + c$$

$$(B) \frac{x}{\sqrt{2}} \sin^{-1} \left(\frac{\sqrt{2}x}{x^2 - 1} \right) + c$$

$$(C) \frac{x}{\sqrt{2}} \sin^{-1} \left(\frac{x - 1}{x + 1} \right) + c$$

$$(D) \frac{1}{\sqrt{2}} \cos^{-1} \left\{ \frac{1}{\sqrt{2}} \left(\frac{x + 1}{x^2 + 1} \right) \right\} + c$$

$$\text{ઉકેલ : } I = \int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}} dx$$

$$= \int \frac{(x^2 - 1) dx}{x(x^2 + 1)\sqrt{x^2 + \frac{1}{x^2}}}$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$$

$$\begin{aligned}
&= \int \frac{dt}{t\sqrt{t^2-2}} \quad \left(\text{ધારો કે, } x + \frac{1}{x} = t \quad \text{અથવા, } \left(1 - \frac{1}{x^2}\right) dx = dt \right) \\
&= \frac{1}{\sqrt{2}} \sec^{-1} \left(\frac{t}{\sqrt{2}} \right) + c \\
&= \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}}{t} \right) + c \\
&= \frac{1}{\sqrt{2}} \cos^{-1} \left(\frac{\sqrt{2}x}{x^2+1} \right) + c \quad \text{જવાબ : (A)}
\end{aligned}$$

$$(108) \int \frac{dx}{\sin^6 x + \cos^6 x} = \tan^{-1}(f(x)) + c, \text{ તો } f(x) = \dots$$

- (A) $\tan x - 1$ (B) $\tan^2 x - 1$ (C) $\tan x - \cot x$ (D) $2\tan^2 x - 3\cot^2 x$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{dx}{\sin^6 x + \cos^6 x} \\
&= \int \frac{dx}{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)} \\
&= \int \frac{dx}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x} \\
&\text{અંશ અને છેદને } \cos^4 x \text{ વડે ભાગતાં,} \\
&= \int \frac{\sec^4 x dx}{\tan^4 x + 1 - \tan^2 x} \\
&= \int \frac{(\tan^2 x + 1) \sec^2 x dx}{\tan^4 x + 1 - \tan^2 x} \\
&= \int \frac{(t^2 + 1) dt}{t^4 + 1 - t^2} \quad \left(\text{ધારો કે, } \tan x = t \quad \text{અથવા, } \sec^2 x dx = dt \right) \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 - 1 + \frac{1}{t^2}\right)} \\
&= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 1} \\
&= \int \frac{du}{u^2 + 1} \quad \left(\text{ધારો કે, } t - \frac{1}{t} = u \quad \text{અથવા, } \left(1 + \frac{1}{t^2}\right) dt = du \right)
\end{aligned}$$

$$\begin{aligned}
&= \tan^{-1}(u) + c \\
&= \tan^{-1}\left(t - \frac{1}{t}\right) + c \\
&= \tan^{-1}(\tan x - \cot x) + c
\end{aligned}$$

$$\therefore f(x) = \tan x - \cot x \quad \text{જવાબ : (C)}$$

$$(109) \quad \int e^{\frac{x}{2}} \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx = \dots$$

$$(A) \quad e^{\frac{x}{2}} \sin \frac{x}{2} + c \quad (B) \quad e^{\frac{x}{2}} \cos \frac{x}{2} + c \quad (C) \quad \sqrt{2} e^{\frac{x}{2}} \cos \frac{x}{2} + c \quad (D) \quad \sqrt{2} e^{\frac{x}{2}} \sin \frac{x}{2} + c$$

$$\text{ઉકેલ : } I = \int e^{\frac{x}{2}} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

$$= 2 \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} - \int \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \frac{1}{2} \cdot 2 e^{\frac{x}{2}} dx$$

$$= 2 \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} - \int \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} dx$$

$$I = 2 \cdot e^{\frac{x}{2}} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \left[2 e^{\frac{x}{2}} \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) + \int \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) e^{\frac{x}{2}} dx \right]$$

$$= 2 e^{\frac{x}{2}} \left\{ \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\} - I$$

$$\therefore 2I = 2 e^{\frac{x}{2}} \left\{ \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) - \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\} + c'$$

$$I = \sqrt{2} e^{\frac{x}{2}} \sin \frac{x}{2} + c \quad \left(\frac{c'}{2} = c\right) \quad \text{જવાબ : (D)}$$

$$(110) \quad \int \frac{x^2}{(x \sin x + \cos x)^2} dx = \dots$$

$$(A) \quad \frac{\sin x + \cos x}{x \sin x + \cos x} \quad (B) \quad \frac{x \sin x - \cos x}{x \sin x + \cos x} \quad (C) \quad \frac{\sin x - x \cos x}{x \sin x + \cos x} \quad (D) \quad \frac{x \sin x + \cos x}{x \sin x + \cos x}$$

$$\text{ઉકેલ : } I = \int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

$$= \int \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot \left(\frac{x}{\cos x}\right) dx$$

$$= \frac{x}{\cos x} \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx - \int \left(\frac{d}{dx} \left(\frac{x}{\cos x} \right) \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx \right) dx$$

$$= \frac{x}{\cos x} \cdot \frac{-1}{(x \sin x + \cos x)} + \int \left(\frac{\cos x + x \sin x}{\cos^2 x} \times \frac{1}{(x \sin x + \cos x)} \right) dx$$

$$\left(\frac{d}{dx} (x \sin x + \cos x) = x \cos x \text{ અને } \int \frac{1}{t^2} dt = \frac{-1}{t} \right)$$

$$\begin{aligned}
&= \frac{-x}{(x \sin x + \cos x) \cos x} + \int \sec^2 x \, dx \\
&= \frac{-x}{(x \sin x + \cos x) \cos x} + \frac{\sin x}{\cos x} + c \\
&= \frac{-x + x \sin^2 x + \sin x \cos x}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\sin x \cos x - x(1 - \sin^2 x)}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\cos x (\sin x - x \cos x)}{(x \sin x + \cos x) \cos x} + c \\
&= \frac{\sin x - x \cos x}{x \sin x + \cos x} + c
\end{aligned}$$

જવાબ : (C)

$$(111) \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} \, dx = \dots + c$$

- (A) $\sin x - 6 \tan^{-1}(\sin x)$ (B) $\sin x - 2(\sin x)^{-1}$
(C) $\sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x)$ (D) $\sin x - 2(\sin x)^{-1} - 5 \tan^{-1}(\sin x)$

(IIT : 1990)

$$\begin{aligned}
\text{ઉદા : } I &= \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} \, dx \\
&= \int \frac{(\cos^2 x + \cos^4 x) \cos x \, dx}{\sin^2 x + \sin^4 x} \\
&= \int \frac{[(1 - \sin^2 x) + (1 - \sin^2 x)^2] \cos x \, dx}{\sin^2 x + \sin^4 x} \\
&\text{ધારો કે, } \sin x = t. \quad \text{તેથી, } \cos x \, dx = dt
\end{aligned}$$

$$\begin{aligned}
\therefore I &= \int \frac{(1 - t^2) + (1 - t^2)^2}{t^2 + t^4} \, dt \\
&= \int \frac{1 - t^2 + 1 - 2t^2 + t^4}{t^4 + t^2} \, dt \\
&= \int \frac{(t^4 + t^2) - 4t^2 + 2}{(t^4 + t^2)} \, dt \\
&= \int 1 \, dt - \int \frac{4t^2 - 2}{t^2(t^2 + 1)} \, dt
\end{aligned}$$

અહીંથી નીચે પ્રમાણે પણ આગળ વધી શકાય :

$$= t - \int \left(\frac{-2}{t^2} + \frac{6}{t^2+1} \right) dt \quad (\text{અંશિક અપૂર્ણાંક})$$

$$= t - \frac{2}{t} - 6 \tan^{-1} t$$

$$= \sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$$

$$= \int 1 dt - 4 \int \frac{1}{t^2+1} dt + 2 \int \frac{1}{t^2(t^2+1)} dt$$

$$= \int 1 dt - 4 \int \frac{1}{t^2+1} dt + 2 \int \frac{(t^2+1) - (t^2)}{t^2(t^2+1)} dt$$

$$= \int 1 dt - 4 \int \frac{1}{t^2+1} dt + 2 \int \frac{1}{t^2} dt - 2 \int \frac{1}{t^2+1} dt$$

$$= t - 4 \tan^{-1} t + 2 \frac{t^{-1}}{-1} - 2 \tan^{-1} t + c$$

$$= \sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$$

જવાબ : (C)

$$(112) \int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}} dx = \dots + c.$$

$$(A) \frac{2}{a-b} \sqrt{\frac{x-a}{x-b}}$$

$$(B) \frac{-2}{a-b} \sqrt{\frac{x-b}{x-a}}$$

$$(C) \frac{1}{\sqrt{(x-a)(x-b)}}$$

$$(D) \frac{2}{a-b} \sqrt{\frac{x-b}{x-a}}$$

(IIT : 1996)

$$\text{ઉકેલ : } I = \int \frac{dx}{(x-a)\sqrt{(x-a)(x-b)}} dx$$

$$= \int \frac{dx}{(x-a)^{\frac{3}{2}}(x-b)^{\frac{1}{2}}}$$

$$= \int \frac{dx}{(x-a)^2 \sqrt{\frac{x-b}{x-a}}}$$

$$\text{ધારો કે, } \sqrt{\frac{x-b}{x-a}} = t, \quad \text{અર્થાત્, } \frac{x-b}{x-a} = t^2$$

$$\therefore \frac{(x-a) - (x-b)}{(x-a)^2} dx = 2t dt$$

$$\frac{b-a}{(x-a)^2} dx = 2t dt$$

$$\frac{1}{(x-a)^2} dx = \frac{2t}{b-a} dt$$

$$\begin{aligned}
\therefore I &= \frac{1}{b-a} \int \frac{2t}{t} dt \\
&= \frac{2t}{b-a} + c \\
&= \frac{-2}{a-b} \sqrt{\frac{x-b}{x-a}} + c
\end{aligned}$$

જવાબ : (B)

$$(113) \int \frac{x+1}{x(1+xe^x)^2} dx = \dots + c.$$

(IIT : 1996)

$$(A) \log \left| \frac{xe^x}{xe^x + 1} \right|$$

$$(B) \log \left| \frac{xe^x}{xe^x + 1} \right| - \frac{1}{xe^x + 1}$$

$$(C) \log \left| \frac{xe^x}{xe^x + 1} \right| + \frac{1}{xe^x + 1}$$

$$(D) \frac{1}{1+xe^x} + \log |xe^x + 1|$$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int \frac{x+1}{x(1+xe^x)^2} dx \\
&= \int \frac{(x+1)e^x}{xe^x(1+xe^x)^2} dx
\end{aligned}$$

$$\text{ધારો કે, } xe^x = t \quad \text{આથી, } (1+x)e^x dx = dt$$

$$\begin{aligned}
\therefore I &= \int \frac{dt}{t(t+1)^2} = \int \frac{(t+1) - (t)}{t(t+1)^2} dt \\
&= \int \frac{1}{t(t+1)} dt - \int \frac{1}{(t+1)^2} dt \\
&= \int \frac{(t+1) - (t)}{t(t+1)} dt - \int (t+1)^{-2} dt \\
&= \int \frac{1}{t} dt - \int \frac{1}{t+1} dt - \int (t+1)^{-2} dt \\
&= \log |t| - \log |t+1| - \frac{(t+1)^{-1}}{-1} + c \\
&= \log \left| \frac{t}{t+1} \right| + \frac{1}{t+1} + c \\
&= \log \left| \frac{xe^x}{xe^x + 1} \right| + \frac{1}{xe^x + 1}
\end{aligned}$$

જવાબ : (C)

$$(114) \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}} = \dots + c. \quad (x > \frac{7}{2})$$

$$(A) \quad 2 \sec^{-1} (2x-7)$$

$$(B) \quad \sec^{-1} (2x-7)$$

$$(C) \quad \frac{1}{2} \sec^{-1} (2x-7)$$

$$(D) \quad -2 \sec^{-1} (2x-7) + c$$

(IIT : 1997)

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}} \\ &= 2 \int \frac{dx}{(2x-7)\sqrt{4x^2-28x+48}} \\ &= 2 \int \frac{dx}{(2x-7)\sqrt{(2x-7)^2-(1)^2}} \\ &= 2 \cdot \frac{1}{2} \sec^{-1} (2x-7) + c \\ &= \sec^{-1} (2x-7) + c \end{aligned}$$

જવાબ : (B)

$$(115) \int \frac{\cos x - \sin x}{\cos x + \sin x} (2 + 2\sin 2x) dx = \dots + c.$$

$$(A) \quad \sin 2x$$

$$(B) \quad \cos 2x$$

$$(C) \quad \tan 2x$$

$$(D) \quad 2\sin 2x$$

(IIT : 1997)

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{\cos x - \sin x}{\cos x + \sin x} (2 + 2\sin 2x) dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} 2(1 + \sin 2x) dx \\ &= 2 \int \frac{(\cos x - \sin x)(\cos x + \sin x)^2}{(\cos x + \sin x)} dx \\ &= 2 \int (\cos^2 x - \sin^2 x) dx \\ &= 2 \int (\cos 2x) dx \\ &= 2 \times \frac{1}{2} (\sin 2x) + c \\ &= \sin 2x + c \end{aligned}$$

જવાબ : (A)

$$(116) \int \frac{\cos x + 3\sin x - 5}{3 - 3\sin x - \cos x} dx = \dots + c.$$

$$(A) \quad x - 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right|$$

$$(B) \quad -x - \frac{1}{2} \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} + 1} \right|$$

$$(C) \quad -x - 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right|$$

$$(D) \quad -x + 2 \log \left| \frac{2\tan \frac{x}{2} - 2}{2\tan \frac{x}{2} - 1} \right|$$

ઉકેલ : $I = \int \frac{\cos x + 3 \sin x - 5}{3 - 3 \sin x - \cos x} dx$

ધારો કે $(\cos x + 3 \sin x - 5) = P(3 - 3 \sin x - \cos x) + Q \left[\frac{d}{dx}(3 - 3 \sin x - \cos x) \right] + R$

$$= P(3 - 3 \sin x - \cos x) + Q (-3 \cos x + \sin x) + R$$

$$\cos x + 3 \sin x - 5 = (3P + R) + (-3P + Q) \sin x + (-P - 3Q) \cos x$$

હવે, સહગુણકો સરખાવતાં, $3P + R = -5$, $-3P + Q = 3$, $-P - 3Q = 1$

ઉકેલતાં, $P = -1$, $Q = 0$, $R = -2$

$$\therefore I = \int \frac{-(3 - 3 \sin x - \cos x) - 2}{3 - 3 \sin x - \cos x} dx$$

$$= - \int dx - 2 \int \frac{dx}{3 - 3 \sin x - \cos x} = -x - 2I_1$$

જ્યાં, $I_1 = \int \frac{dx}{3 - 3 \sin x - \cos x}$

ધારો કે, $\tan \frac{x}{2} = t$. આથી, $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore I_1 = \int \frac{1}{3 - 3 \left(\frac{2t}{1+t^2} \right) - \left(\frac{1-t^2}{1+t^2} \right)} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{3 + 3t^2 - 6t - 1 + t^2}$$

$$= 2 \int \frac{dt}{4t^2 - 6t + 2}$$

$$= \int \frac{dt}{2t^2 - 3t + 1}$$

$$= \frac{1}{2} \int \frac{dt}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{1}{4}\right)} \log \left| \frac{t - \frac{3}{4} - \frac{1}{4}}{t - \frac{3}{4} + \frac{1}{4}} \right| + c$$

$$= \log \left| \frac{t-1}{t-\frac{1}{2}} \right| + c$$

$$I_1 = \log \left| \frac{2t-2}{2t-1} \right| + c$$

$$I = -x - 2 \log \left| \frac{2t-2}{2t-1} \right| + c$$

$$= -x - 2 \log \left| \frac{2 \tan \frac{x}{2} - 2}{2 \tan \frac{x}{2} - 1} \right| + c$$

જવાબ : (C)

$$(117) \int \frac{(3 \cos x + 2)}{\sin x + 2 \cos x + 3} dx = \dots + c$$

$$(A) \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{16}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right)$$

$$(B) \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{8}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right)$$

$$(C) \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{6}{5} \log \left| \frac{\tan \frac{x}{2} + 3}{\tan \frac{x}{2} - 1} \right|$$

$$(D) \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{4}{5} \log \left| \frac{\tan \frac{x}{2} - 1}{\tan \frac{x}{2} + 3} \right|$$

$$\text{ઉકેલ : } I = \int \frac{(3 \cos x + 2)}{\sin x + 2 \cos x + 3} dx$$

$$\text{ધારો કે } 3 \cos x + 2 = P(\sin x + 2 \cos x + 3) + Q \left[\frac{d}{dx}(\sin x + 2 \cos x + 3) \right] + R$$

$$\therefore 3 \cos x + 2 = P(\sin x + 2 \cos x + 3) + Q(\cos x - 2 \sin x) + R$$

$$\therefore 3 \cos x + 2 = (P - 2Q) \sin x + (2P + Q) \cos x + (3P + R)$$

$$\text{સહગુણકો સરખાવતાં, } P - 2Q = 0, 2P + Q = 3, 3P + R = 2$$

$$\text{સમીકરણો ઉકેલતાં, } P = \frac{6}{5}, Q = \frac{3}{5}, R = \frac{-8}{5}$$

$$\therefore I = \int \frac{\frac{6}{5}(\sin x + 2 \cos x + 3) + \frac{3}{5}(\cos x - 2 \sin x) - \frac{8}{5}}{\sin x + 2 \cos x + 3} dx$$

$$= \frac{6}{5} \int dx + \frac{3}{5} \int \frac{\cos x - 2 \sin x}{\sin x + 2 \cos x + 3} dx - \frac{8}{5} \int \frac{dx}{\sin x + 2 \cos x + 3}$$

$$I = \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2 \cos x + 3| - \frac{8}{5} I_1 \quad \dots(1)$$

$$\text{જ્યાં, } I_1 = \int \frac{dx}{\sin x + 2 \cos x + 3}$$

$$\text{ધારો કે, } \tan \frac{x}{2} = t. \text{ આથી, } dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\begin{aligned}
\therefore I_1 &= \int \frac{1}{\frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right) + 3} \times \frac{2dt}{1+t^2} \\
&= \int \frac{2dt}{2t + 2 - 2t^2 + 3 + 3t^2} \\
&= \int \frac{2dt}{t^2 + 2t + 1 + 4} \\
&= \int \frac{2dt}{(t+1)^2 + (2)^2} = 2 \times \frac{1}{2} \tan^{-1} \left(\frac{t+1}{2} \right) + c = \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) + c
\end{aligned}$$

સમીકરણ (1)માં મૂકતાં,

$$I = \frac{6}{5}x + \frac{3}{5} \log |\sin x + 2\cos x + 3| - \frac{8}{5} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{2} \right) \quad \text{જવાબ : (B)}$$

$$(118) \int \frac{\sqrt{\cos x}}{\sin x} dx = \dots + c. \quad (0 < x < \frac{\pi}{2})$$

$$(A) \quad \frac{1}{2} \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1}(\sqrt{\cos x}) \quad (B) \quad \frac{1}{2} \log \left| \frac{\sqrt{\cos x} + 1}{\sqrt{\cos x} - 1} \right| + \tan^{-1}(\sqrt{\cos x})$$

$$(C) \quad \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1}(\sqrt{\cos x}) \quad (D) \quad \log \left| \frac{\sqrt{\cos x} + 1}{\sqrt{\cos x} - 1} \right| + \tan^{-1}(\sqrt{\cos x})$$

$$\text{ઉકેલ : } I = \int \frac{\sqrt{\cos x}}{\sin x} dx \quad (0 < x < \frac{\pi}{2})$$

$$\text{ધારો કે, } \cos x = u^2. \quad \text{અર્થાત્, } -\sin x dx = 2u du \quad (u > 0)$$

$$\therefore \frac{dx}{\sin x} = - \frac{2u du}{1 - \cos^2 x}$$

$$\therefore \frac{dx}{\sin x} = - \frac{2u du}{1 - u^4}$$

$$\begin{aligned}
\therefore I &= \int u \left(\frac{-2u}{1-u^4} \right) du \\
&= \int \frac{2u^2}{u^4-1} du = \int \frac{(u^2+1) + (u^2-1)}{(u^2-1)(u^2+1)} du \\
&= \int \frac{du}{u^2-1} + \int \frac{du}{u^2+1}
\end{aligned}$$

$$= \frac{1}{2} \log \left| \frac{u-1}{u+1} \right| + \tan^{-1}(u) + c$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{\cos x} - 1}{\sqrt{\cos x} + 1} \right| + \tan^{-1}(\sqrt{\cos x}) + c$$

જવાબ : (A)

$$(119) \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx = \dots + c.$$

$$(A) \quad \frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + c$$

$$(B) \quad \frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + c$$

$$(C) \quad -\frac{1}{3} \log | \tan x + 1 | + \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + c$$

$$(D) \quad \frac{1}{3} \log | \tan x + 1 | - \frac{1}{6} \log | \tan^2 x - \tan x + 1 | + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + c$$

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx \\ &= \int \frac{\tan x (1 + \tan^2 x)}{1 + \tan^3 x} dx \\ &= \int \frac{\tan x \cdot \sec^2 x}{1 + \tan^3 x} dx \end{aligned}$$

$$\text{ધારો કે, } \tan x = t. \text{ તો, } \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{t dt}{1 + t^3} \\ &= \int \frac{t dt}{(1+t)(t^2 - t + 1)} \end{aligned}$$

$$\text{ધારો કે, } \frac{t}{(1+t)(t^2 - t + 1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2 - t + 1}$$

$$\therefore t = A(t^2 - t + 1) + (Bt + C)(t + 1)$$

$$t = -1 \quad \Rightarrow \quad -1 = 3A \quad \text{તો } A = -\frac{1}{3}$$

$$t = 0 \quad \Rightarrow \quad A + C = 0 \Rightarrow C = -A \quad \text{તો } C = \frac{1}{3}$$

$$t = 1 \quad \Rightarrow \quad A + 2B + 2C = 1$$

$$-\frac{1}{3} + 2B + \frac{2}{3} = 1 \quad \text{તો } B = \frac{1}{3}$$

$$\therefore I = \int \frac{-\frac{1}{3}}{t+1} dt + \int \frac{\frac{1}{3}t + \frac{1}{3}}{t^2 - t + 1} dt$$

$$\begin{aligned}
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{3} \int \frac{\frac{1}{2}(2t-1) + \frac{3}{2}}{(t^2-t+1)} dt \\
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2} \int \frac{dt}{t^2-t+1} \\
&= -\frac{1}{3} \int \frac{1}{t+1} dt + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2} \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= -\frac{1}{3} \log |t+1| + \frac{1}{6} \log |t^2-t+1| + \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c \\
&= -\frac{1}{3} \log |\tan x + 1| + \frac{1}{6} \log |\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2\tan x - 1}{\sqrt{3}} \right) + c
\end{aligned}$$

જવાબ : (C)

$$(120) \int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx = \dots + c.$$

$$(A) \quad \frac{1}{27} \log \left| \frac{2 + \sin x}{1 - \sin x} \right| + \frac{1}{9(\sin x - 1)} + \frac{1}{6(\sin x - 1)^2} + c$$

$$(B) \quad \frac{1}{27} \log \left| \frac{2 + \sin x}{1 - \sin x} \right| - \frac{1}{9(1 - \sin x)} - \frac{1}{6(1 - \sin x)^2} + c$$

$$(C) \quad \frac{1}{27} \log \left| \frac{1 - \sin x}{2 + \sin x} \right| - \frac{1}{9(1 - \sin x)} - \frac{1}{6(1 - 2\sin x)^2} + c$$

$$(D) \quad \frac{1}{27} \log \left| \frac{1 - \sin x}{2 + \sin x} \right| + \frac{1}{9(\sin x - 1)^2} - \frac{1}{6(\sin x - 1)^2} + c$$

$$\text{ઉકેલ : } I = \int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx$$

$$\text{ધારો કે, } \sin x = t. \text{ આથી, } \cos x dx = dt$$

$$= \int \frac{dt}{(2+t)(1-t)^3}$$

$$\text{ધારો કે, } \frac{1}{(2+t)(1-t)^3} = \frac{A}{2+t} + \frac{B}{1-t} + \frac{C}{(1-t)^2} + \frac{D}{(1-t)^3}$$

$$1 = A(1-t)^3 + B(2+t)(1-t)^2 + C(2+t)(1-t) + D(2+t)$$

$$\text{જો } t = -2, \text{ તો } 1 = 27A. \text{ આથી, } A = \frac{1}{27}$$

$$\text{જો } t = 1, \text{ તો } 1 = 3D. \text{ આથી, } D = \frac{1}{3}$$

$$\text{હવે, } 1 = \frac{1}{27} (1 - t^3 - 3t + 3t^2) + B(2 - 3t + t^3) + C(2 - t - t^2) + \frac{1}{3}(2 + t)$$

હવે, t^3 ની સહગુણકને સરખાવતાં, $-\frac{1}{27} + B = 0$. અર્થાત્, $B = \frac{1}{27}$

t^2 ની સહગુણકને સરખાવતાં, $\frac{1}{9} - C = 0$. અર્થાત્, $C = \frac{1}{9}$

$$\begin{aligned}\therefore \int \frac{dt}{(2+t)(1-t)^3} &= \frac{1}{27} \int \frac{dt}{t+2} + \frac{1}{27} \int \frac{dt}{1-t} + \frac{1}{9} \int (1-t)^{-2} dt + \frac{1}{3} \int (1-t)^{-3} dt \\ &= \frac{1}{27} \log |t+2| - \frac{1}{27} \log |1-t| + \frac{1}{9} \frac{(1-t)^{-1}}{-1} + \frac{1}{3} \frac{(1-t)^{-2}}{-2} + c \\ &= \frac{1}{27} \log \left| \frac{2+\sin x}{1-\sin x} \right| - \frac{1}{9(1-\sin x)} - \frac{1}{6(1-\sin x)^2} + c \quad \text{જવાબ : (B)}\end{aligned}$$

$$(121) \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \dots + c.$$

$$(A) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) \quad (B) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + 3 \tan^{-1} \left(\frac{x}{2} \right)$$

$$(C) \quad x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) - 3 \tan^{-1} \left(\frac{x}{2} \right) \quad (D) \quad x - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + 3 \tan^{-1} \left(\frac{x}{2} \right)$$

$$\begin{aligned}\text{ઉકેલ : } I &= \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx \\ &= \int \frac{x^4+3x^2+2}{x^4+7x^2+12} dx \\ &= \int \frac{(x^4+7x^2+12)-4x^2-10}{x^4+7x^2+12} dx\end{aligned}$$

$$I = \int 1 dx - 2 \int \frac{2x^2+5}{(x^2+3)(x^2+4)} dx$$

$$I = x - 2I_1, \text{ જ્યાં } I_1 = \int \frac{2x^2+5}{(x^2+3)(x^2+4)} dx \quad (1)$$

આંશિક અપૂર્ણક લેવા $x^2 = t$ લેતાં,

$$\therefore \frac{2t+5}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4}$$

$$\therefore I_1 = \int \frac{-1}{x^2+3} dx + 3 \int \frac{dx}{x^2+4}$$

$$I_1 = \frac{-1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) + c_1$$

સમીકરણ (1)માં મૂકતાં,

$$I = x - 2 \left[\frac{-1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{3}{2} \tan^{-1} \left(\frac{x}{2} \right) + c_1 \right]$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \left(\frac{x}{2} \right) + c \quad (-2c_1 = c) \quad \text{જવાબ : (C)}$$

$$(122) \int \frac{\sqrt{x^2+1} (\log(x^2+1) - 2 \log x)}{x^4} dx = \dots + c$$

$$(A) \quad \frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] \quad (B) \quad -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right]$$

$$(C) \quad -\frac{1}{3} \left(1 - \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 - \frac{1}{x^2}\right) - \frac{2}{3} \right] \quad (D) \quad -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right]$$

$$\text{ଉତ୍ତର : I} = \int \frac{\sqrt{x^2+1} (\log(x^2+1) - 2 \log x)}{x^4} dx$$

$$= \int x \sqrt{1 + \frac{1}{x^2}} \cdot \log \left(\frac{x^2+1}{x^2} \right) \cdot \frac{dx}{x^4}$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \cdot \frac{dx}{x^3}$$

$$\text{ଧରିବା, } 1 + \frac{1}{x^2} = t$$

$$\therefore \frac{-2}{x^3} dx = dt. \quad \text{ଅର୍ଥାତ୍, } \frac{1}{x^3} dx = \frac{-1}{2} dt$$

$$I = \frac{-1}{2} \int \sqrt{t} \log t \, dt$$

$$= \frac{-1}{2} \left[\log t \int \sqrt{t} \, dt - \int \left(\frac{d}{dt} (\log t) \int \sqrt{t} \, dt \right) dt \right]$$

$$= \frac{-1}{2} \left[(\log t) \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \left(\frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) dt \right]$$

$$= \frac{-1}{2} \left[\frac{2}{3} \cdot t^{\frac{3}{2}} \log t - \frac{2}{3} \int t^{\frac{1}{2}} dt \right]$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \log t + \frac{1}{3} \times \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \log t + \frac{2}{9} t^{\frac{3}{2}} + c$$

$$= \frac{-1}{3} t^{\frac{3}{2}} \left(\log t - \frac{2}{3} \right) + c$$

$$= -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log \left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] + c$$

ଞ୍ଜାଳ : (B)

$$(123) \int \frac{dx}{x(x^n + 1)} dx = + c. \quad (x > 0)$$

$$(A) \quad \frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(B) \quad n \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(C) \quad \log \left(\frac{x^n}{x^n + 1} \right)$$

$$(D) \quad \text{અભિર્થેલ એક પણ નહિ.} \quad (\text{AIEEE : 2002})$$

$$\text{ઉકેલ : } I = \int \frac{dx}{x(x^n + 1)}$$

$$\text{ધારીએ } x^n + 1 = t. \text{ અર્થે, } nx^{n-1} dx = dt$$

$$I = \frac{1}{n} \int \frac{n \cdot x^{n-1} dx}{x^n \cdot (x^n + 1)}$$

$$= \frac{1}{n} \int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n} \left(\frac{1}{t-1} - \frac{1}{t} \right) dt$$

$$= \frac{1}{n} [\log(t-1) - \log(t)] + c$$

$$= \frac{1}{n} \log \left(\frac{t-1}{t} \right) + c = \frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right) + c$$

જવાબ : (A)

$$(124) \int \frac{dx}{\cos x - \sin x} =$$

$$(A) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{3\pi}{8} \right) \right| + c$$

$$(B) \quad \frac{1}{\sqrt{2}} \log \left| \cot \frac{x}{2} \right| + c$$

$$(C) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right| + c$$

$$(D) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + c$$

(AIEEE : 2003)

$$\text{ઉકેલ : } I = \int \frac{dx}{\cos x - \sin x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right)}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos\left(x + \frac{\pi}{4}\right)} \\
&= \frac{1}{\sqrt{2}} \int \sec\left(x + \frac{\pi}{4}\right) dx \\
&= \frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{\pi}{4} + \left(\frac{x}{2} + \frac{\pi}{8}\right)\right) \right| + c = \frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{x}{2} + \frac{3\pi}{8}\right) \right| + c \quad \text{જવાબ : (D)}
\end{aligned}$$

$$(125) \int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log |\sin(x - \alpha)| + c, \text{ તો } (A, B) = \dots$$

- (A) $(-\sin \alpha, \cos \alpha)$ (B) $(\cos \alpha, \sin \alpha)$ (C) $(\sin \alpha, \cos \alpha)$ (D) $(-\cos \alpha, \sin \alpha)$

(AIEEE-2004)

$$\begin{aligned}
\text{ઉત્તર : I} &= \int \frac{\sin x}{\sin(x - \alpha)} dx \\
&= \int \frac{\sin[(x - \alpha) + \alpha]}{\sin(x - \alpha)} dx \\
&= \int \frac{\sin(x - \alpha) \cos \alpha + \cos(x - \alpha) \sin \alpha}{\sin(x - \alpha)} dx \\
&= \int (\cos \alpha + \sin \alpha \cdot \cot(x - \alpha)) dx \\
&= \cos \alpha \cdot x + \sin \alpha \log |\sin(x - \alpha)| + c
\end{aligned}$$

$$\therefore A = \cos \alpha, B = \sin \alpha. \text{ આથી, } (A, B) = (\cos \alpha, \sin \alpha)$$

જવાબ : (B)

$$(126) \text{ જો } f(x) = \int \left(\frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx \text{ અને } f(0) = 0, \text{ તો } f(1) = \dots \quad [\text{Online May 19, 2012}]$$

- (A) $\tan 1 - \frac{\pi}{4}$ (B) $\tan 1 + 1$ (C) $\frac{\pi}{4}$ (D) $1 - \frac{\pi}{4}$

$$\begin{aligned}
\text{ઉત્તર : } f(x) &= \int \left(\frac{x^2 + \sin^2 x}{1 + x^2} \right) \sec^2 x dx \\
&= \int \frac{x^2 \sec^2 x + \tan^2 x}{1 + x^2} dx \\
&= \int \frac{x^2(1 + \tan^2 x) + \tan^2 x}{1 + x^2} dx \\
&= \int \frac{x^2}{1 + x^2} dx + \int \tan^2 x dx \\
&= \int \frac{1 + x^2 - 1}{1 + x^2} dx + \int (\sec^2 x - 1) dx
\end{aligned}$$

$$= \int 1 \, dx - \int \frac{1}{1+x^2} \, dx + \int \sec^2 x \, dx - \int 1 \, dx$$

$$= -\tan^{-1}x + \tan x + c$$

$$\text{දିଅ, } f(0) = 0, \text{ \↑ } f(0) = -\tan^{-1}0 + \tan 0 + c$$

$$\therefore c = 0$$

$$\therefore f(x) = -\tan^{-1}x + \tan x$$

$$\text{එබැවින් } f(1) = -\tan^{-1}1 + \tan 1 = \tan 1 - \frac{\pi}{4}$$

ඉතිරි : (A)

$$(127) \int \frac{dx}{\cos x + \sqrt{3} \sin x} = \dots\dots$$

$$(A) \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(B) \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

$$(C) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

$$(D) \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) \right| + c$$

(AIEEE : 2007)

$$\text{ප්‍රශ්න : I} = \int \frac{dx}{\cos x + \sqrt{3} \sin x}$$

$$\therefore \text{ I} = \frac{1}{2} \int \frac{dx}{\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x}$$

$$= \frac{1}{2} \int \frac{dx}{\sin \left(x + \frac{\pi}{6} \right)}$$

$$= \frac{1}{2} \int \operatorname{cosec} \left(x + \frac{\pi}{6} \right) dx$$

$$= \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

ඉතිරි : (C)

$$(128) \sqrt{2} \int \frac{\sin x \, dx}{\sin \left(x - \frac{\pi}{4} \right)} = \dots\dots + c .$$

$$(A) x + \log \left| \sin \left(x - \frac{\pi}{4} \right) \right|$$

$$(B) x - \log \left| \cos \left(x - \frac{\pi}{4} \right) \right|$$

$$(C) x + \log \left| \cos \left(x - \frac{\pi}{4} \right) \right|$$

$$(D) x - \log \left| \sin \left(x - \frac{\pi}{4} \right) \right|$$

(AIEEE : 2008)

$$\text{ප්‍රශ්න : I} = \sqrt{2} \int \frac{\sin x \, dx}{\sin \left(x - \frac{\pi}{4} \right)}$$

$$\begin{aligned}
&= \sqrt{2} \int \frac{\sin \left[\left(x - \frac{\pi}{4} \right) + \frac{\pi}{4} \right]}{\sin \left(x - \frac{\pi}{4} \right)} dx \\
&= \sqrt{2} \int \frac{\sin \left(x - \frac{\pi}{4} \right) \cos \frac{\pi}{4} + \cos \left(x - \frac{\pi}{4} \right) \sin \frac{\pi}{4}}{\sin \left(x - \frac{\pi}{4} \right)} dx \\
&= \sqrt{2} \int \cos \frac{\pi}{4} dx + \sqrt{2} \sin \frac{\pi}{4} \int \cot \left(x - \frac{\pi}{4} \right) dx \\
&= x + \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c
\end{aligned}$$

જવાબ : (A)

$$(129) \int \frac{5 \tan x}{\tan x - 2} dx = x + a \log | \sin x - 2 \cos x | + c \text{ તો } a = \dots$$

- (A) 1 (B) 2 (C) -1 (D) -2 (AIEEE : 2009)

$$\text{ઉકેલ : } I = \int \frac{5 \tan x}{\tan x - 2} dx = x + a \log | \sin x - 2 \cos x | + c$$

અંતે બાજુ x ને સાપેક્ષ વિકલન કરતાં,

$$\frac{5 \tan x}{\tan x - 2} = 1 + \frac{a(\cos x + 2 \sin x)}{\sin x - 2 \cos x}$$

$$\therefore \frac{5 \sin x}{\sin x - 2 \cos x} = \frac{\sin x - 2 \cos x + a \cos x + 2a \sin x}{\sin x - 2 \cos x}$$

$$\therefore 5 \sin x = (1 + 2a) \sin x + (a - 2) \cos x$$

$$\therefore 5 = 1 + 2a, \quad a - 2 = 0$$

$$\therefore a = 2$$

જવાબ : (B)

$$(130) \int \frac{5 \tan x}{\tan x - 2} dx = x + k \log | \sin x - 2 \cos x | + c, \text{ તો } k = \dots$$

- (A) -2 (B) 1 (C) 2 (D) -1 (AIEEE : 2012)

$$\text{ઉકેલ : } I = \int \frac{5 \tan x}{\tan x - 2} dx$$

$$= \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$\text{હવે, } 5 \sin x = P(\sin x - 2 \cos x) + Q \left[\frac{d}{dx} (\sin x - 2 \cos x) \right] \text{ લેતાં,}$$

$$\therefore 5 \sin x = P(\sin x - 2 \cos x) + Q(\cos x + 2 \sin x)$$

$$\therefore 5 \sin x = (P + 2Q) \sin x + (Q - 2P) \cos x$$

$\sin x$ અને $\cos x$ ની સહગુણકો સરખાવતાં,

$$P + 2Q = 5, \quad Q - 2P = 0. \text{ ઉકેલતાં } P = 1 \text{ અને } Q = 2$$

$$\begin{aligned}
 I &= \int \frac{1(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)} dx \\
 &= \int 1 dx + 2 \int \frac{\frac{d}{dx}(\sin x - 2\cos x)}{(\sin x - 2\cos x)} dx \\
 &= x + 2 \log |\sin x - 2\cos x| + c
 \end{aligned}$$

$\therefore k = 2$ (ત્રીજા : પ્રશ્ન (129) ની બીજી રીત) જવાબ : (C)

$$(131) \int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1}x} dx = A(x) \cdot e^{\cot^{-1}x} + c, \text{ તો } A(x) = \dots$$

- (A) $-x$ (B) x (C) $\sqrt{1-x}$ (D) $\sqrt{1+x}$

(JEE : 2013)

$$\text{ઉકેલ : } I = \int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1}x} dx,$$

$$\text{ધારો કે, } \cot^{-1}x = t. \text{ તો, } x = \cot t$$

$$\therefore \frac{-1}{1+x^2} dx = dt. \text{ આથી } \frac{dx}{1+x^2} = -dt$$

$$\begin{aligned}
 \therefore I &= - \int (1 + \cot^2 t - \cot t) e^t dt \\
 &= \int (-\operatorname{cosec}^2 t + \cot t) e^t dt \\
 &= e^t \cot t + c \\
 &= x \cdot e^{\cot^{-1}x} + c \\
 \therefore A(x) &= x
 \end{aligned}$$

જવાબ : (B)

$$(132) \int \frac{x dx}{2 - x^2 + \sqrt{2 - x^2}} = \dots$$

- (A) $\log |1 + \sqrt{2 + x^2}| + c$ (B) $-\log |1 + \sqrt{2 - x^2}| + c$
 (C) $-x \log |1 - \sqrt{2 - x^2}| + c$ (D) $x \log |1 - \sqrt{2 + x^2}| + c$

(JEE : 2013)

$$\text{ઉકેલ : } I = \int \frac{x dx}{2 - x^2 + \sqrt{2 - x^2}} \quad (|x| < \sqrt{2})$$

$$\text{ધારો કે, } 2 - x^2 = t^2, t > 0. \text{ તો, } -2x dx = 2t dt. \text{ આથી, } x dx = -t dt$$

$$\begin{aligned}
 \therefore I &= \int \frac{-t dt}{t^2 + t} \\
 &= - \int \frac{dt}{t + 1} \\
 &= - \log |t + 1| + c = - \log |\sqrt{2 - x^2} + 1| + c
 \end{aligned}$$

જવાબ : (B)

બીજી રીત :

$$I = \int \frac{x dx}{\sqrt{2-x^2} (\sqrt{2-x^2} + 1)}$$

$$\sqrt{2-x^2} = t \text{ એટલે } \frac{-2x dx}{2\sqrt{2-x^2}} = dt$$

$$\therefore I = \int \frac{-dt}{t+1}$$

$$= -\log |1 + \sqrt{2-x^2}| + c$$

જવાબ : (B)

$$(133) \text{ જો } \int \frac{dx}{x+x^7} = P(x), \text{ તો } \int \frac{x^6 dx}{x+x^7} = \dots\dots$$

$$(A) \log |x| - P(x) + c$$

$$(B) \log |x| + P(x) + c$$

$$(C) x - P(x) + c$$

$$(D) x + P(x) + c$$

(JEE : 2013)

$$\text{ઉકેલ : એટલે, } P(x) = \int \frac{1}{x+x^7} dx. \text{ ધારેલ છે, } Q(x) = \int \frac{x^6}{x+x^7} dx$$

$$\text{તેથી, } P(x) + Q(x) = \int \frac{1+x^6}{x(1+x^6)} dx$$

$$= \int \frac{1}{x} dx$$

$$= \log |x| + c$$

$$Q(x) = \log |x| - P(x) + c$$

જવાબ : (A)

$$(134) \text{ જો } \int f(x) dx = \psi(x), \text{ તો } \int x^5 f(x^3) dx = \dots\dots$$

$$(A) \frac{1}{3} x^3 \psi(x^3) - 3 \int x^3 \psi(x^3) dx + c$$

$$(B) \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$$

$$(C) \frac{1}{3} [x^3 \psi(x^3) - \int x^5 \psi(x^3) dx] + c$$

$$(D) \frac{1}{3} [x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + c$$

(JEE : 2014)

$$\text{ઉકેલ : } I = \int x^5 f(x^3) dx$$

$$= \frac{1}{3} \int t \cdot f(t) dt$$

$$(\text{ધારેલ છે, } x^3 = t, \text{ આથી } 3x^2 dx = dt)$$

$$= \frac{1}{3} \left[t \int f(t) dt - \int \left(\frac{d}{dt} (t) \right) \int f(t) dt \right] dt$$

$$= \frac{1}{3} \left[t \cdot \psi(t) - \int 1 \cdot \psi(t) dt \right]$$

$$= \frac{1}{3} t \cdot \psi(t) - \frac{1}{3} \int \psi(t) dt$$

$$= \frac{1}{3} x^3 \psi(x^3) - \frac{1}{3} \int \psi(x^3) 3x^2 dx$$

$$= \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$$

જવાબ : (B)

$$(135) \int \left(1 + x - \frac{1}{x}\right) \cdot e^{\left(x + \frac{1}{x}\right)} dx = \dots$$

$$(A) (x - 1) e^{x + \frac{1}{x}} + c$$

$$(B) x \cdot e^{x + \frac{1}{x}} + c$$

$$(C) (x + 1) e^{x + \frac{1}{x}} + c$$

$$(D) -x \cdot e^{x + \frac{1}{x}} + c$$

(JEE : 2014)

$$\text{ଓଢ଼େଇ} : I = \int \left(1 + x - \frac{1}{x}\right) \cdot e^{\left(x + \frac{1}{x}\right)} dx$$

$$= \int \left[e^{x + \frac{1}{x}} + \left(x - \frac{1}{x}\right) e^{x + \frac{1}{x}} \right] dx$$

$$= \int \left\{ e^{x + \frac{1}{x}} + x \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}} \right\} dx$$

$$\text{ଓଢ଼େଇ, ଧାରଣା କର} \quad e^{x + \frac{1}{x}} = f(x) \quad \text{ଏବଂ} \quad f'(x) = e^{x + \frac{1}{x}} \left(1 - \frac{1}{x^2}\right)$$

$$\therefore I = \int [f(x) + x \cdot f'(x)] dx$$

$$= x \cdot f(x) + c$$

$$= x \cdot e^{x + \frac{1}{x}} + c$$

$$\left(\int (f(x) + x \cdot f'(x)) dx = x \cdot f(x) + c \right)$$

ଞ୍ଜାଣ : (B)

$$(136) \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx = \dots$$

$$(A) \frac{1}{2} \sin 2x + c$$

$$(B) -\frac{1}{2} \sin 2x + c$$

$$(C) -\frac{1}{2} \sin x + c$$

$$(D) -\sin^2 x + c$$

(JEE : 2014)

$$\text{ଓଢ଼େଇ} : I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^4 x + \cos^4 x)} dx$$

$$= \int (\sin^4 x - \cos^4 x) dx$$

$$= \int (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) dx$$

$$= -\int \cos 2x dx = -\frac{\sin 2x}{2} + c$$

ଞ୍ଜାଣ : (B)

$$(137) \int x \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) dx = \dots \quad (x > 0).$$

$$(A) -x + (1 + x^2) \tan^{-1} x + c$$

$$(B) x - (1 + x^2) \cot^{-1} x + c$$

$$(C) -x + (1 + x^2) \cot^{-1} x + c$$

$$(D) x - (1 + x^2) \tan^{-1} x + c$$

(JEE : 2014)

$$\text{ઉકેલ : } I = \int x \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) dx$$

$$\text{હીરેં ડે } x = \tan \theta. \quad \text{અથલ, } dx = \sec^2 \theta d\theta \quad (0 < \theta < \frac{\pi}{2})$$

$$\therefore I = \int \tan \theta \cdot \cos^{-1}(\cos 2\theta) \cdot \sec^2 \theta d\theta \quad (0 < 2\theta < \pi)$$

$$= 2 \int \theta \cdot \tan \theta \sec^2 \theta d\theta$$

$$= 2 \left[\theta \int \tan \theta \sec^2 \theta d\theta - \int \left(\frac{d}{d\theta} (\theta) \int \tan \theta \sec^2 \theta d\theta \right) d\theta \right]$$

$$= 2 \left[\theta \cdot \frac{\tan^2 \theta}{2} - \int \left(1 \cdot \frac{\tan^2 \theta}{2} \right) d\theta \right]$$

$$= \theta \cdot \tan^2 \theta - \int (\sec^2 \theta - 1) d\theta$$

$$= \theta \cdot \tan^2 \theta - \tan \theta + \theta + c$$

$$= x^2 \cdot \tan^{-1} x - x + \tan^{-1} x + c$$

$$= -x + (1 + x^2) \tan^{-1} x + c$$

જવાબ : (A)

$$(138) \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \dots (x > 0).$$

$$(A) \frac{1}{1 + \cot^3 x} + c \quad (B) \frac{-1}{3(1 + \tan^3 x)} + c \quad (C) \frac{\sin^3 x}{1 + \cos^3 x} + c \quad (D) \frac{\cos^3 x}{3(1 + \sin^3 x)} + c$$

(JEE : 2014)

$$\text{ઉકેલ : } I = \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

$$= \int \frac{\sin^2 x \cos^2 x}{\cos^6 x (\tan^3 x + 1)^2} dx$$

$$= \int \frac{\sec^2 x \tan^2 x}{(\tan^3 x + 1)^2} dx,$$

$$= \frac{1}{3} \int \frac{dt}{(t+1)^2} \quad \text{હીરેં ડે } \tan^3 x = t. \quad \text{અથલ, } 3 \tan^2 x \sec^2 x dx = dt$$

$$= \frac{1}{3} \left[\frac{-1}{t+1} \right] + c$$

$$= \frac{-1}{3(1 + \tan^3 x)} + c$$

જવાબ : (B)

$$(139) \text{ જો } m \text{ શૂન્યેતર હોય અંતે } \int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx = f(x) + c, \text{ તો } f(x) = \dots$$

$$(A) \frac{x^{5m}}{2m(x^{2m} + x^m + 1)^2}$$

$$(B) \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

$$(C) \frac{2m(x^{5m} + x^{4m})}{(x^{2m} + x^m + 1)}$$

$$(D) \frac{x^{5m} + x^{4m}}{2m(x^{2m} + x^m + 1)^2}$$

(JEE : 2014)

$$\begin{aligned}
 \text{ઉકેલ : } I &= \int \frac{x^{5m-1} + 2x^{4m-1}}{(x^{2m} + x^m + 1)^3} dx \\
 &= \int \frac{x^{5m-1} + 2x^{4m-1}}{x^{6m} \left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^3} dx \\
 &= \int \frac{(x^{-m-1} + 2x^{-2m-1})}{\left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^3} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{ધારીએ } 1 + x^{-m} + x^{-2m} &= t \\
 (-m \cdot x^{-m-1} - 2m \cdot x^{-2m-1})dx &= dt \\
 -m(x^{-m-1} + 2x^{-2m-1})dx &= dt
 \end{aligned}$$

$$\begin{aligned}
 \therefore I &= \frac{-1}{m} \int \frac{dt}{t^3} \\
 &= \frac{-1}{m} \int t^{-3} dt = \frac{-1}{m} \cdot \frac{t^{-2}}{-2} + c = \frac{1}{2m(t)^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore I &= \frac{1}{2m \left(1 + \frac{1}{x^m} + \frac{1}{x^{2m}}\right)^2} + c \\
 &= \frac{x^{4m}}{2m(x^{2m} + x^m + 1)^2} + c
 \end{aligned}$$

જવાબ : (B)

$$(140) \int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}} = \dots$$

$$\text{(A) } \left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c \quad \text{(B) } (x^4 + 1)^{\frac{1}{4}} + c \quad \text{(C) } -(x^4 + 1)^{\frac{1}{4}} + c \quad \text{(D) } -\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c$$

(JEE : 2015)

$$\text{ઉકેલ : } I = \int \frac{dx}{x^2 (x^4 + 1)^{\frac{3}{4}}} = \int \frac{dx}{x^5 \left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}}}$$

$$\text{ધારીએ, } 1 + \frac{1}{x^4} = t, \quad \text{તેથી, } -\frac{4}{x^5} dx = dt$$

$$I = \frac{-1}{4} \int \frac{dt}{t^{\frac{3}{4}}}$$

$$= \frac{-1}{4} \left[\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} \right] + c = \frac{-1}{4} \left[\frac{t^{\frac{1}{4}}}{\frac{1}{4}} \right] + c = -t^{\frac{1}{4}} + c = -\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + c$$

જવાબ : (D)

(141) $\sin 6x$ નો પૂર્વગ

(A) $\frac{1}{3}(\sin^6 x - \sin^3 x) + c$

(B) $\frac{1}{3} \sin^2 3x + c$

(C) $-\frac{1}{3} \cos^2 3x + c$

(D) $\frac{1}{3} \sin \left(3x + \frac{\pi}{7} \right) \sin \left(3x - \frac{\pi}{7} \right) + c$

ઉકેલ : $\int \sin 6x \, dx = -\frac{1}{6} \cos 6x + c'$

$$= -\frac{1}{6} (1 - 2\sin^2 3x) + c'$$

$$= -\frac{1}{6} + \frac{1}{3} \sin^2 3x + c'$$

$$= \frac{1}{3} \sin^2 3x + c \quad (c' - \frac{1}{6} = c) \quad (B)$$

હવે, $-\frac{1}{6} \cos 6x + c' = -\frac{1}{6} (2\cos^2 3x - 1) + c'$

$$= -\frac{1}{3} \cos^2 3x + \frac{1}{6} + c'$$

$$= -\frac{1}{3} \cos^2 3x + c \quad (\frac{1}{6} + c' = c) \quad (C)$$

વળી, $\frac{d}{dx} \left[\frac{1}{3} \sin \left(3x + \frac{\pi}{7} \right) \sin \left(3x - \frac{\pi}{7} \right) \right]$

$$= \frac{1}{6} \frac{d}{dx} \left[2 \sin \left(3x + \frac{\pi}{7} \right) \sin \left(3x - \frac{\pi}{7} \right) \right]$$

$$= \frac{1}{6} \frac{d}{dx} \left(\cos \frac{2\pi}{7} - \cos 6x \right)$$

$$= \frac{1}{6} (0 - (-\sin 6x) 6)$$

$$= \sin 6x$$

$$\therefore \int \sin 6x \, dx = \frac{1}{3} \sin \left(3x + \frac{\pi}{7} \right) \sin \left(3x - \frac{\pi}{7} \right) \quad (D) \quad \text{જવાબ : (B, C, D)}$$

(142) જો $P(x) = \lim_{n \rightarrow \infty} e^{x \tan \left(\frac{1}{n} \right) \cdot \log \left(\frac{1}{n} \right)}$ અને $\int \frac{P(x)}{\sqrt[3]{\sin^{11} x \cos x}} \, dx = Q(x) + c$, તો

(A) $Q\left(\frac{\pi}{4}\right) = \frac{3}{2}$

(B) $Q(x)$ પ્રત્યેક x માટે સતત છે.

(C) $Q(x)$ એ x ની અસંખ્ય કિંમતો માટે વિકલનીય નથી. (D) $Q\left(\frac{\pi}{4}\right) = \frac{-15}{8}$

ઉકેલ : $\lim_{n \rightarrow \infty} \tan \left(\frac{1}{n} \right) \cdot \log \left(\frac{1}{n} \right)$

$$= \lim_{n \rightarrow \infty} \frac{\tan \left(\frac{1}{n} \right)}{\left(\frac{1}{n} \right)} \cdot \frac{\log n^{-1}}{n}$$

$$= \lim_{n \rightarrow \infty} -\frac{\tan \left(\frac{1}{n} \right)}{\left(\frac{1}{n} \right)} \cdot \frac{\log n}{n} = -1 \times \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n}}{\frac{1}{1}} \right) = -1 \times 0 = 0$$

તેથી $P(x) = e^0 = 1$

$$\begin{aligned}\therefore \int \frac{P(x)}{\sqrt[3]{\sin^{11} x \cos x}} dx &= \int \frac{1}{\sin^{\frac{11}{3}} x \cos^{\frac{1}{3}} x} dx \\ &= \int \sin^{\frac{-11}{3}} x \cdot \cos^{\frac{-1}{3}} x dx \\ &= \int (\tan x)^{\frac{-11}{3}} \cdot \cos^{-4} x dx \\ &= \int (\tan x)^{\frac{-11}{3}} \cdot \sec^4 x dx \\ &= \int (\tan x)^{\frac{-11}{3}} (1 + \tan^2 x) \cdot \sec^2 x dx \\ &= \int (\tan x)^{\frac{-11}{3}} \sec^2 x dx + \int (\tan x)^{\frac{-5}{3}} \sec^2 x dx \\ &= \frac{(\tan x)^{\frac{-11}{3}+1}}{\frac{-11}{3}+1} + \frac{(\tan x)^{\frac{-5}{3}+1}}{\frac{-5}{3}+1} + c\end{aligned}$$

$$\therefore Q(x) = \frac{-3}{8}(\tan x)^{\frac{-8}{3}} - \frac{3}{2}(\tan x)^{\frac{-2}{3}}$$

$$\therefore Q\left(\frac{\pi}{4}\right) = \frac{-3}{8} - \frac{3}{2} = \frac{-15}{8} \quad (D)$$

હવે, $Q(x)$ એ જ્યારે $\tan x = 0$ હોય ત્યારે વિકલનીય નથી.

$$\therefore x = n\pi + \frac{\pi}{2} \quad n \in Z \text{ માટે } Q(x) \text{ વિકલનીય નથી.} \quad (C)$$

$Q(x)$ એ $x = n\pi + \frac{\pi}{2}$ માટે સતત નથી.

જવાબ : (C, D)

$$(143) \int \sin(\log x) dx = P(x) ((\sin Q(x) - \cos R(x))) + c, \text{ તો}$$

$$(A) \lim_{x \rightarrow 2} P(x) = 2 \quad (B) \lim_{x \rightarrow 1} \frac{Q(x)}{R(x)} = 1 \quad (C) Q(e^3) = 3 \quad (D) R(e^5) = 5$$

$$\text{ઉકેલ : } I = \int \sin(\log x) dx$$

$$\text{ધારો કે, } \log x = t.$$

$$\therefore x = e^t. \text{ આથી, } dx = e^t dt$$

$$\therefore I = \int e^t \sin t dt$$

$$= \frac{e^t}{2} (\sin t - \cos t) + c$$

$$= \frac{x}{2} [\sin(\log x) - \cos(\log x)] + c$$

$$= P(x) [\sin(Q(x)) - \cos(R(x))] + c$$

$$P(x) = \frac{x}{2}, \quad Q(x) = \log x, \quad R(x) = \log x$$

$$\text{વિકલ્પ A : } \lim_{x \rightarrow 2} P(x) = \lim_{x \rightarrow 2} \frac{x}{2} = \frac{2}{2} = 1 \neq 2. \quad A \text{ સત્ય નથી.}$$

$$\text{વિકલ્પ B : } \lim_{x \rightarrow 1} \frac{Q(x)}{R(x)} = \lim_{x \rightarrow 1} \frac{\log x}{\log x} = 1. \quad \text{તેથી B સત્ય છે.}$$

વિકલ્પ C : $Q(e^3) = \log e^3 = 3$

સત્ય છે.

વિકલ્પ D : $R(e^5) = \log e^5 = 5$

સત્ય છે.

જવાબ : વિકલ્પ (B, C, D)

(144) $\int (\sin 2x - \cos 2x) dx = \dots + c$

(A) $-\frac{\cos 2x}{2} - \frac{\sin 2x}{2}$

(B) $-\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right)$

(C) $\frac{1}{\sqrt{2}} \sin\left(2x + \frac{9\pi}{4}\right)$

(D) $\frac{1}{\sqrt{2}} \sin\left(2x + \frac{5\pi}{4}\right)$

ઉકેલ : $I = \int (\sin 2x - \cos 2x) dx$

$$= -\frac{\cos 2x}{2} - \frac{\sin 2x}{2} + c$$

જવાબ : (A)

વિકલ્પ B : $-\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \left(\sin 2x \cos \frac{\pi}{4} + \cos 2x \sin \frac{\pi}{4}\right)$

$$= -\frac{1}{\sqrt{2}} \left(\sin 2x \cdot \frac{1}{\sqrt{2}} + \cos 2x \cdot \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x$$

જવાબ : (B)

વિકલ્પ C : $\frac{1}{\sqrt{2}} \sin\left(2x + \frac{9\pi}{4}\right) = \frac{1}{\sqrt{2}} \sin\left(2\pi + \left(2x + \frac{\pi}{4}\right)\right)$

$$= \frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right) \neq -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right)$$

વિકલ્પ D : $\frac{1}{\sqrt{2}} \sin\left(2x + \frac{5\pi}{4}\right)$

$$= \frac{1}{\sqrt{2}} \sin\left(\pi + \left(2x + \frac{\pi}{4}\right)\right)$$

$$= -\frac{1}{\sqrt{2}} \sin\left(2x + \frac{\pi}{4}\right)$$

જવાબ : વિકલ્પ (A, B, D)

(145) $\int \sin^{\frac{-3}{2}} x \sin^{\frac{-1}{2}} (x + \theta) dx = \dots + c$

(A) $\frac{-2 \sin x}{\sin \theta \sin (x + \theta)}$

(B) $\frac{-2}{\sqrt{\sin \theta}} (\cos \theta + \cot x)^{\frac{1}{2}}$

(C) $\frac{-2}{\sin \theta} \left(\frac{\sin (\theta - x)}{\sin x}\right)^{\frac{1}{2}}$

(D) $\frac{-2}{\sin \theta} \left(\frac{\sin (\theta + x)}{\sin x}\right)^{\frac{1}{2}}$

ઉકેલ : $I = \int \sin^{\frac{-3}{2}} x \sin^{\frac{-1}{2}} (x + \theta) dx$

$$= \int \frac{1}{\sqrt{\sin^3 x \sin (x + \theta)}} dx$$

$$= \int \frac{dx}{\sqrt{\sin^3 x (\sin x \cos \theta + \cos x \sin \theta)}}$$

$$= \int \frac{\operatorname{cosec}^2 x dx}{\sqrt{\cos \theta + \cot x \sin \theta}}$$

$$= \frac{1}{\sqrt{\sin \theta}} \int \frac{\operatorname{cosec}^2 x \, dx}{\sqrt{\cot \theta + \cot x}}$$

$$\text{ધારી } \xi, \cot x = t$$

$$\therefore -\operatorname{cosec}^2 x \, dx = dt. \text{ આથી, } \operatorname{cosec}^2 x \, dx = -dt$$

$$= \frac{-1}{\sqrt{\sin \theta}} \int (\cot \theta + t)^{\frac{-1}{2}} dt$$

$$= \frac{-1}{\sqrt{\sin \theta}} \times \frac{(\cot \theta + t)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= \frac{-2}{\sqrt{\sin \theta}} (\cot \theta + \cot x)^{\frac{1}{2}} + c \quad (\text{B})$$

$$= \frac{-2}{\sin \theta} (\cos \theta + \sin \theta \cdot \cot x)^{\frac{1}{2}} + c$$

$$= \frac{-2}{\sin \theta} \frac{(\sin x \cdot \cos \theta + \cos x \cdot \sin \theta)^{\frac{1}{2}}}{\sqrt{\sin x}} + c$$

$$= \frac{-2}{\sin \theta} \left(\frac{\sin (\theta + x)}{\sin x} \right)^{\frac{1}{2}} + c \quad (\text{D})$$

જવાબ : (B, D)

$$(146) \int (\cos^{-1} x + \cos^{-1} \sqrt{1-x^2}) \, dx = Ax + P(x) \sin^{-1} x - 2\sqrt{1-x^2} + c, \forall x \in [-1, 0]$$

$$(A) \quad P(x) = x$$

$$(B) \quad P(x) = -2x$$

$$(C) \quad A = \frac{\pi}{4}$$

$$(D) \quad A = \frac{\pi}{2}$$

$$\text{ઉંડા : } x \in [-1, 0] \Rightarrow -1 \leq x < 0$$

$$\therefore \cos^{-1} \sqrt{1-x^2} = -\sin^{-1} x$$

$$I = \int (\cos^{-1} x + \cos^{-1} \sqrt{1-x^2}) \, dx$$

$$= \int (\cos^{-1} x - \sin^{-1} x) \, dx$$

$$= \int \left(\frac{\pi}{2} - 2\sin^{-1} x \right) dx$$

$$= \int \frac{\pi}{2} dx - 2 \left\{ \sin^{-1} x \int 1 \, dx - \int \left(\frac{d}{dx} \sin^{-1} x \int 1 \, dx \right) dx \right\}$$

$$= \frac{\pi}{2} x - 2 \left\{ x \cdot \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \right\}$$

$$= \frac{\pi}{2} x - 2x \sin^{-1} x + 2(-\sqrt{1-x^2}) + c$$

$$= \frac{\pi}{2} x - 2x \sin^{-1} x - 2\sqrt{1-x^2} + c$$

$$\therefore A = \frac{\pi}{2} \text{ અને } P(x) = -2x$$

જવાબ : (B, D)

$$(147) \int \frac{dx}{\sin x + \cos x}$$

$$(A) \quad \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 - \sqrt{2}}{\tan \frac{x}{2} - 1 + \sqrt{2}} \right| + c$$

$$(B) \quad \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| + c$$

$$(C) \quad \frac{1}{\sqrt{2}} \log \left(\tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right)$$

$$(D) \quad \frac{1}{\sqrt{2}} \log \left| \tan \left(x + \frac{\pi}{4} \right) \right| + c$$

$$\text{ଓଡ଼ିଆ} : I = \int \frac{dx}{\sin x + \cos x}$$

$$\text{ଦିଅଁ } t, \tan \frac{x}{2} = t. \text{ ଆଉ, } dx = \frac{2dt}{1+t^2}, \text{ ଏବଂ } \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore I = \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2dt}{2t + 1 - t^2}$$

$$= 2 \int \frac{dt}{2 - (t^2 - 2t + 1)}$$

$$= 2 \int \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= 2 \times \frac{1}{2\sqrt{2}} \log \left| \frac{t-1+\sqrt{2}}{t-1-\sqrt{2}} \right| + c$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| + c$$

(B)

$$\text{ଓଡ଼ିଆ, } I = \int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int \operatorname{cosec} \left(x + \frac{\pi}{4} \right) dx$$

$$= \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right| + c$$

(C)

ଞ୍ଜାଳ : (B, C)

નિષ્કર્ષ અને કારક (Assertion and Reason) પ્રકારના પ્રશ્નો

સમજૂતી : આ પ્રકારના પ્રશ્નોમાં બે વિધાન હોય છે. વિધાન 1 (નિષ્કર્ષ) અને વિધાન 2 (કારક). આ પ્રકારના પ્રશ્નોમાં ચાર વિકલ્પો હોય છે. વિકલ્પ (A), (B), (C) અને (D), જે નીચે મુજબ છે :

- (A) વિધાન 1 અને વિધાન 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.
 (B) વિધાન 1 અને વિધાન 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની સમજૂતી માટે પર્યાપ્ત નથી.
 (C) વિધાન 1 સત્ય છે અને વિધાન 2 અસત્ય છે.
 (D) વિધાન 1 અસત્ય છે અને વિધાન 2 સત્ય છે.

(148) **વિધાન 1 :** $\int \tan 5x \tan 3x \tan 2x dx = \frac{1}{5} \log | \sec 5x | - \frac{1}{3} \log | \sec 3x | - \frac{1}{2} \log | \sec 2x | + c$

વિધાન 2 : $\tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$

ઉકેલ : $5x = 3x + 2x$

$$\therefore \tan 5x = \tan (3x + 2x) = \frac{\tan 3x + \tan 2x}{1 - \tan 3x \tan 2x}$$

$$\therefore \tan 5x - \tan 5x \tan 3x \tan 2x = \tan 3x + \tan 2x$$

$$\therefore \tan 5x - \tan 3x - \tan 2x = \tan 5x \tan 3x \tan 2x$$

$$\therefore \text{વિધાન 2 સત્ય છે.}$$

$$\begin{aligned} \text{હવે, } \int \tan 5x \tan 3x \tan 2x dx &= \int (\tan 5x - \tan 3x - \tan 2x) dx \\ &= \frac{1}{5} \log | \sec 5x | - \frac{1}{3} \log | \sec 3x | - \frac{1}{2} \log | \sec 2x | + c \end{aligned}$$

$$\therefore \text{વિધાન 1 સત્ય છે.}$$

$$\therefore \text{વિધાન 1 અને વિધાન 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.}$$

જવાબ : (A)

(149) **વિધાન 1 :** $F(x) = \int \sin^2 x dx$. $\forall x \in \mathbb{R}$ માટે વિધાન $F(x + \pi) = F(x)$ નું સમાધાન કરે છે.

વિધાન 2 : $\sin^2 (\pi + x) = \sin^2 x$, $\forall x \in \mathbb{R}$

ઉકેલ : $F(x) = \int \sin^2 x dx$

$$\begin{aligned} &= \int \left(\frac{1 - \cos 2x}{2} \right) dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c' \\ &= \frac{1}{4} [2x - \sin 2x] + c' \end{aligned}$$

હવે, $F(x + \pi) = \frac{1}{4} [2(x + \pi) - \sin (2x + 2\pi)] + c'$

$$= \frac{1}{4} [2x + 2\pi - \sin 2x] + c'$$

$$= \frac{1}{4} [2x - \sin 2x] + \frac{\pi}{2} + c'$$

$$= \frac{1}{4} [2x - \sin 2x] + c \quad c = \left(\frac{\pi}{2} + c' \right)$$

બીજી રીત :

વિધાન 2 સત્ય છે, કારણ કે $\sin^2 x$ નું મૂલ્ય આવર્તમાન π છે.

$$\begin{aligned} F(x + \pi) &= \int \sin^2 (x + \pi) dx \\ &= \int \sin^2 x dx \\ &= F(x) \end{aligned}$$

$\therefore$ વિધાન 1 પણ સત્ય છે.

જવાબ : (A)

$$F(x + \pi) = F(x)$$

∴ વિધાન 1 સત્ય છે.

વળી, $\sin^2 x (\pi + x) = \sin^2 x$, $\forall x \in \mathbb{R}$ કારણ કે, $\sin^2 x$ નું આવર્તમાન π છે.

∴ વિધાન 2 સત્ય છે.

∴ વિધાન 1 અને 2 બંને સત્ય છે અને વિધાન 2 એ વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.

જવાબ : (A)

$$(150) \quad \text{વિધાન 1 : } \int \frac{(3-2x) dx}{\sqrt{4+2x-x^2}} = 2\sqrt{4+2x-x^2} + \sin^{-1} \left(\frac{x-1}{\sqrt{5}} \right) + c$$

$$\text{વિધાન 2 : } \int \frac{1}{\sqrt{a^2-x^2}} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$$

$$\begin{aligned} \text{ઉકેલ : } F(x) &= \int \frac{(3-2x) dx}{\sqrt{4+2x-x^2}} \\ &= \int \frac{(2-2x)}{\sqrt{4+2x-x^2}} dx + \int \frac{1}{\sqrt{4+2x-x^2}} dx \\ &= \int \frac{(2-2x)}{\sqrt{4+2x-x^2}} dx + \int \frac{1}{\sqrt{(\sqrt{5})^2 - (x-1)^2}} dx \\ &= 2\sqrt{4+2x-x^2} + \sin^{-1} \left(\frac{x-1}{\sqrt{5}} \right) + c \end{aligned}$$

∴ વિધાન 1 સત્ય છે.

$$\text{હવે, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$$

∴ વિધાન 2 અસત્ય છે.

જવાબ : (C)

$$(151) \quad \text{વિધાન 1 : } \int \frac{e^x(1+\sin x)}{1+\cos x} dx = e^x \tan \frac{x}{2} + c$$

$$\text{વિધાન 2 : } \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

$$\begin{aligned} \text{ઉકેલ : } I &= \int \frac{e^x(1+\sin x)}{1+\cos x} dx \\ &= \int \frac{e^x(1+2\sin \frac{x}{2} \cos \frac{x}{2})}{2\cos^2 \frac{x}{2}} dx \\ &= \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx \\ &= \int e^x \left[\tan \frac{x}{2} + \frac{d}{dx} \left(\tan \frac{x}{2} \right) \right] dx \\ &= e^x \tan \frac{x}{2} \end{aligned}$$

∴ વિધાન 1 સત્ય છે તથા વિધાન 2 સત્ય છે અને તે વિધાન 1ની યોગ્ય સમજૂતી માટે પર્યાપ્ત છે.

જવાબ : (A)

(152) વિધાન 1 : જો $f(n) = \int \cot^n x \, dx$, તો $5[f(6) + f(4)] = -\cot^5 x$

વિધાન 2 : જો $f(n) = \int \cot^n x \, dx$, તો $f(n) = \frac{\cot^{n-1} x}{n-1} - f(n-2)$, જ્યાં $n \geq 2$.

ઉકેલ : $f(n) = \int \cot^n x \, dx$
 $= \int \cot^{n-2} x \cot^2 x \, dx$
 $= \int \cot^{n-2} x \cdot (\operatorname{cosec}^2 x - 1) \, dx$
 $= \int (\cot x)^{n-2} \cdot \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$

$\therefore f(n) = -\frac{(\cot x)^{n-1}}{n-1} - f(n-2)$

$\therefore$ વિધાન 2 અસત્ય છે.

હવે, $n = 6$ લેતાં,

$f(6) = -\frac{\cot^5 x}{5} - f(4)$

$5(f(6) + f(4)) = -\cot^5 x$

$\therefore$ વિધાન 1 સત્ય છે.

જવાબ : (C)

(153) વિધાન 1 : જો $\int \frac{1}{f(x)} \, dx = \log (f(x))^2 + c$, તો $f(x) = \frac{x}{2} + c$

વિધાન 2 : જ્યારે $f(x) = \frac{x}{2}$ હોય, ત્યારે $\int \frac{1}{f(x)} \, dx = \frac{1}{2} \log |x| + c$.

ઉકેલ : $\int \frac{1}{f(x)} \, dx = \log (f(x))^2 + c$

બંને બાજુ વિકલન કરતાં,

$\frac{1}{f(x)} = \frac{1}{(f(x))^2} \cdot 2f(x) \cdot f'(x) + 0$

$\therefore f'(x) = \frac{1}{2}$

$\therefore f(x) = \frac{x}{2} + c$

$\therefore$ વિધાન 1 સત્ય છે.

હવે, $\int \frac{1}{f(x)} \, dx = \int \frac{2}{x} \, dx = 2 \int \frac{1}{x} \, dx = 2 \log |x| + c$.

$\therefore$ વિધાન 2 પણ અસત્ય છે.

જવાબ : (C)

(154) વિધાન 1 : $\int \frac{x^4 + 1}{x^6 + 1} \, dx = \tan^{-1} x + \frac{1}{3} \tan^{-1} x^3 + c$

વિધાન 2 : $\int \frac{x \sin x + \cos x}{x(x + \cos x)} \, dx = \log (x^2 + x \cos x) + c$

ઉકેલ : $\int \frac{x^4 + 1}{x^6 + 1} \, dx = \int \frac{(x^4 - x^2 + 1) + x^2}{(x^2 + 1)(x^4 - x^2 + 1)} \, dx$

$$\begin{aligned}
&= \int \frac{1}{x^2 + 1} dx + \int \frac{x^2}{x^6 + 1} dx \\
&= \int \frac{1}{x^2 + 1^2} dx + \int \frac{x^2}{(x^3)^2 + 1} dx \\
&= \tan^{-1}x + \frac{1}{3} \tan^{-1}x^3 + c
\end{aligned}$$

∴ વિધાન 1 સત્ય છે.

$$\begin{aligned}
\text{હવે, } \int \frac{x \sin x + \cos x}{x(x + \cos x)} dx &= \int \frac{(\cos x + x) - x(1 - \sin x)}{x(x + \cos x)} dx \\
&= \int \frac{1}{x} dx - \int \frac{1 - \sin x}{x + \cos x} dx \\
&= \log |x| - \log |x + \cos x| + c \\
&= \log \left| \frac{x}{x + \cos x} \right| + c
\end{aligned}$$

∴ વિધાન 2 અસત્ય છે.

જવાબ : (C)

$$(155) \quad \text{વિધાન 1 : જો } \int \frac{2^x}{\sqrt{1-4^x}} dx = k \sin^{-1}(2^x) + c, \text{ જ્યાં } k = \frac{1}{\log 2}$$

$$\text{વિધાન 2 : } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \quad (a > 0)$$

$$\text{ઉકેલ : } I = \int \frac{2^x}{\sqrt{1-4^x}} dx = \int \frac{2^x}{\sqrt{1-(2^x)^2}} dx$$

$$\text{ધારો કે, } 2^x = t. \text{ આથી, } 2^x \log 2 dx = dt. \text{ આથી, } 2^x dx = \frac{1}{\log_e 2} dt$$

$$I = \int \frac{dt}{\log 2 (\sqrt{1-t^2})} = \frac{1}{\log 2} \sin^{-1} t + c = \frac{1}{\log 2} \sin^{-1} (2^x) + c$$

$$\therefore k = \frac{1}{\log 2}$$

∴ વિધાન 1 સત્ય છે અને વિધાન 2 સત્ય છે અને તે વિધાન 1ની યોગ્ય સમજૂતી આપે છે.

જવાબ : (A)

$$(156) \quad \text{વિધાન 1 : } \int \frac{ax^{a-1} + a^x \log a}{a^x + x^a} dx = \log |a^x + x^a| + c$$

$$\text{વિધાન 2 : } \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$$

$$\text{ઉકેલ : } I = \int \frac{ax^{a-1} + a^x \log a}{a^x + x^a} dx$$

$$\text{ધારો કે, } a^x + x^a = t. \text{ આથી, } (a^x \log a + a \cdot x^{a-1}) dx = dt$$

$$\therefore I = \int \frac{1}{t} dt = \log |t| + c = \log |a^x + x^a| + c$$

∴ વિધાન 1 સત્ય છે અને વિધાન 2 પણ સત્ય છે અને તે વિધાન 1ની યોગ્ય સમજૂતી આપે છે.

જવાબ : (A)



કેટલાંક ઉપયોગી પરિણામો

સરવાળાના લક્ષ તરીકે નિયત સંકલન :

ધારો કે $f : [a, b] \rightarrow \mathbb{R}$ સતત વિધેય છે. કોઈ ધન પૂર્ણાંક n માટે $h = \frac{b-a}{n}$ લઈએ, તો $[a, b]$ નું n એકરૂપ ઉપઅંતરાલોમાં વિભાજન કરતાં બિંદુઓ $a, a+h, a+2h, \dots, a+nh = b$ થશે.

$$\begin{aligned} \therefore \int_a^b f(x) &= \lim_{n \rightarrow \infty} \left(\frac{b-a}{n} \right) \sum_{i=1}^n f(a+ih), \quad h = \frac{b-a}{n} \\ &= \lim_{h \rightarrow 0} h \sum_{i=1}^n f(a+ih) \end{aligned}$$

નિયત સંકલનનો મૂળભૂત સિદ્ધાંત :

ધારો કે વિધેય f એ અંતરાલ $[a, b]$ પર સતત છે. જો F એ x નું એવું વિકલનીય વિધેય હોય કે જેથી, $F'(x) = f(x), \forall x \in [a, b]$ થાય તો,

$$\int_a^b f(x) dx = F(b) - F(a)$$

અહીં, $F(x)$ એ $f(x)$ નું પ્રતિવિકલિત છે. $F(b) - F(a)$ ને સંકેત $[F(x)]_a^b$ વડે દર્શાવીશું. આમ,

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

$$(1) \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$(2) \text{ જો } a > b \text{ તો } \int_a^b f(x) dx = -\int_b^a f(x) dx$$

$$(3) \text{ જો } a = b \text{ તો } \int_a^b f(x) dx = \int_a^a f(x) dx = 0$$

નિયત સંકલનના કાર્ય નિયમો :

જો વિધેય f અને g એ અંતરાલ $[a, b]$ પર સતત હોય, તો

$$(1) \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$(2) \int_a^b k \cdot f(x) dx = k \cdot \int_a^b f(x) dx, k \in \mathbb{R}$$

$$(3) \text{ જો } a < c < b \text{ હોય, તો } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

નિયત સંકલન માટે આદેશની રીત :

$f : [a, b] \rightarrow \mathbb{R}$ સતત વિધેય છે અને $g : [\alpha, \beta] \rightarrow [a, b]$ એ વધતું અથવા ઘટતું વિધેય છે. $x = g(t)$ એ $[\alpha, \beta]$ માં સતત અને (α, β) પર વિકલનીય વિધેય છે. $g'(t)$ એ (α, β) માં સતત છે. $g'(t) \neq 0$. $\forall t \in (\alpha, \beta)$ તથા $a = g(\alpha)$ અને $b = g(\beta)$, તો,

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) \cdot g'(t) dt$$

નિયત સંકલન માટે ખંડશઃ સંકલનની રીત :

$f(x)$, $g(x)$, $f'(x)$ અને $g'(x)$ બધાં જ $[a, b]$ પર સતત હોય, તો

$$\int_a^b f(x) g'(x) dx = [f(x) g(x)]_a^b - \int_a^b f'(x) g(x) dx$$

$$\therefore \int_a^b f(x) g'(x) dx = [f(b) g(b) - f(a) g(a)] - \int_a^b f'(x) g(x) dx$$

જો $f(x) = u$ અને $g'(x) = v$ તો

$$\therefore \int_a^b uv dx = \left[u \int v dx \right]_a^b - \int_a^b \left[\frac{du}{dx} \int v dx \right] dx$$

નિયત સંકલન માટે કેટલાંક ઉપયોગી પરિણામો

$$(1) \text{ જો } f \text{ એ } [0, a] \text{ માં સતત હોય, તો } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$(2) \text{ જો } f \text{ એ } [a, b] \text{ માં સતત હોય, તો } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$(3) \text{ જો વિધેય } f \text{ એ } [0, 2a] \text{ પર સતત હોય, તો}$$

$$\int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx$$

$$\int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{જો } f(2a-x) = f(x), \\ 0, & \text{જો } f(2a-x) = -f(x) \end{cases}$$

$$(4) \text{ જો વિધેય } f \text{ એ } [-a, a] \text{ પર સતત હોય, તો}$$

$$\int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx$$

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{જો } f \text{ યુગ્મ વિધેય હોય,} \\ 0, & \text{જો } f \text{ અયુગ્મ વિધેય હોય,} \end{cases}$$

(5) જો વિધેય $f(x)$ આવર્તી વિધેય હોય તથા તેનું આવર્તમાન T હોય, તો

$$\int_{\alpha}^{\alpha+nT} f(x) dx = n \int_0^T f(x) dx, \quad n \text{ પૂર્ણાંક}$$

(1) જો $\alpha = 0$, તો $\int_{\alpha}^{\alpha+nT} f(x) dx = \int_0^{nT} f(x) dx = n \int_0^T f(x) dx$

(2) જો $n = 1$ હોય, તો $\int_{\alpha}^{\alpha+nT} f(x) dx = \int_{\alpha}^{\alpha+T} f(x) dx = \int_0^T f(x) dx$

(3) $\int_{\alpha+nT}^{\beta+nT} f(x) dx = \int_{\alpha}^{\beta} f(x) dx$

(6) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$

બહુવિકલ્પી પ્રશ્નો

(1) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx = \dots\dots$

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) $3\frac{\pi}{2}$

(D) π

ઉકેલ : $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx \quad (1)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cot\left(\frac{\pi}{2} - x\right)} + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \quad (2)$$

પરિણામ (1) અને (2)નો સરવાળો કરતાં,

$$2I = \int_0^{\frac{\pi}{2}} 1 dx = [x]_0^{\frac{\pi}{2}}$$

$\therefore 2I = \frac{\pi}{2}$. અર્થાત્, $I = \frac{\pi}{4}$

જવાબ : (B)

$$(2) \int_{-4}^4 |x+2| dx = \dots$$

(A) 30

(B) 20

(C) 50

(D) 10

$$\text{ઉકેલ : } |x+2| = \begin{cases} x+2 & \text{જ્યાં } x \geq -2 \\ -(x+2) & \text{જ્યાં } x < -2 \end{cases}$$

$$\begin{aligned} \therefore I &= \int_{-4}^4 |x+2| dx = \int_{-4}^{-2} |x+2| dx + \int_{-2}^4 |x+2| dx \\ &= -\int_{-4}^{-2} (x+2) dx + \int_{-2}^4 (x+2) dx \\ &= -\left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^4 \\ &= -[(2-4) - (8-8)] + [(8+8) - (2-4)] \\ &= -(-2) + (16+2) = 20 \end{aligned}$$

જવાબ : (B)

$$(3) \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx = \dots$$

(A) $\frac{-\pi a}{2}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi a}{4}$

(D) πa

$$\text{ઉકેલ : } I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$$

$$\text{ધારો કે, } x = a \cos 2\theta. \quad \text{આથી, } dx = -2a \sin 2\theta d\theta \quad (0 \leq \theta \leq \frac{\pi}{2})$$

$$\text{હવે, } x = -a \Rightarrow -a = a \cos 2\theta \Rightarrow \cos 2\theta = -1$$

$$\therefore 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$$

$$x = a \Rightarrow a = a \cos 2\theta \Rightarrow \cos 2\theta = 1. \quad \text{આથી, } \theta = 0$$

$$\therefore I = \int_{\frac{\pi}{2}}^0 \sqrt{\frac{a-a \cos 2\theta}{a+a \cos 2\theta}} (-2a \sin 2\theta) d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} \tan \theta \cdot \sin 2\theta d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta} \cdot 2 \sin \theta \cdot \cos \theta d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} 2 \sin^2 \theta \cdot d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= 2a \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 2a \left[\left(\frac{\pi}{2} - 0 \right) \right] = \pi a$$

જવાબ : (D)

$$(4) \int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{\sin x + \cos x} = \dots\dots$$

(A) $\sqrt{2} \log (\sqrt{2} - 1)$ (B) $\sqrt{2} \log (\sqrt{2} + 1)$ (C) $\frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$ (D) $\frac{1}{\sqrt{2}} \log \frac{(\sqrt{2} - 1)}{(\sqrt{2} + 1)}$

ઉકેલ : I = $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{\sin x + \cos x}$ (1)

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx$$
 (2)

પરિણામ (1) અને (2) ને સરવાળે કરતાં,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{\cos \left(x - \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec \left(x - \frac{\pi}{4} \right) dx$$

$$= \frac{1}{\sqrt{2}} \left[\log \left| \sec \left(x - \frac{\pi}{4} \right) + \tan \left(x - \frac{\pi}{4} \right) \right| \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}} \left[\log (\sqrt{2} + 1) - \log (\sqrt{2} - 1) \right]$$

$$= \frac{1}{\sqrt{2}} \log \left[\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right]$$

$$\therefore 2I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)^2$$

$$\therefore 2I = \frac{2}{\sqrt{2}} \log (\sqrt{2} + 1)$$

$$\therefore I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$$

જવાબ : (C)

$$(5) \int_0^1 \left| \sin \frac{3\pi x}{2} \right| dx = \dots\dots$$

$$(A) \frac{3\pi}{2}$$

$$(B) \frac{2\pi}{3}$$

$$(C) \frac{2}{\pi}$$

$$(D) \frac{\pi}{3}$$

$$\text{ઉકેલ : } I = \int_0^1 \left| \sin \frac{3\pi x}{2} \right| dx$$

$$\text{હવે, } \left| \sin \frac{3\pi x}{2} \right| = \sin \frac{3\pi x}{2}, \text{ જ્યારે, } 0 \leq \frac{3\pi x}{2} \leq \pi, \text{ i.e. } x \leq \frac{2}{3} \quad \text{i.e. } 0 \leq x \leq \frac{2}{3}$$

$$\text{અને } \left| \sin \frac{3\pi x}{2} \right| = -\sin \frac{3\pi x}{2}, \text{ જ્યારે, } \frac{3\pi x}{2} > \pi, \text{ i.e. } x > \frac{2}{3} \quad \text{i.e. } \frac{2}{3} \leq x \leq 1$$

$$\therefore I = \int_0^{\frac{2}{3}} \sin \frac{3\pi x}{2} dx + \int_{\frac{2}{3}}^1 \left(-\sin \frac{3\pi x}{2} \right) dx$$

$$= \frac{2}{3\pi} \left[-\cos \frac{3\pi x}{2} \right]_0^{\frac{2}{3}} + \frac{2}{3\pi} \left[\cos \frac{3\pi x}{2} \right]_{\frac{2}{3}}^1$$

$$= \frac{2}{3\pi} [1 - (-1)] + \frac{2}{3\pi} [0 - (-1)] = \frac{2}{3\pi} (2 + 1) = \frac{2}{\pi}$$

જવાબ : (C)

$$(6) \int_{-1}^1 (x - [x]) dx = \dots\dots$$

$$(A) 0$$

$$(B) 1$$

$$(C) 2$$

$$(D) \frac{1}{2}$$

(IIT : 1998)

$$\text{ઉકેલ : } I = \int_{-1}^1 (x - [x]) dx$$

$$\therefore I = \int_{-1}^1 x dx - \int_{-1}^1 [x] dx$$

$$= 0 - \left\{ \int_{-1}^0 [x] dx + \int_0^1 [x] dx \right\}$$

($f(x) = x$ અયુગ્મ વિધેય)

$$= -[-1 + 0]$$

$$I = 1$$

$$(-1 < x < 0, [x] = -1, 0 < x < 1, [x] = 0)$$

જવાબ : (B)

$$(7) \int_0^{\frac{\pi}{2}} \log \cos x \, dx = \dots\dots$$

$$(A) \frac{-\pi}{2} \log 2$$

$$(B) \frac{\pi}{2} \log 2$$

$$(C) \pi \log \frac{1}{2}$$

$$(D) -\pi \log \frac{1}{2}$$

$$\text{ઉકેલ : } I = \int_0^{\frac{\pi}{2}} \log \cos x \, dx \quad (1)$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\cos \left(\frac{\pi}{2} - x \right) \right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \log (\sin x) \, dx \quad (2)$$

(1) તથા (2) ને સરવાળે કરતાં,

$$2I = \int_0^{\frac{\pi}{2}} (\log \cos x + \log \sin x) \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{(2 \sin x \cos x)}{2} \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

$$2x = t \text{ લેતાં, } 2dx = dt$$

$$x = 0 \Rightarrow t = 0 \text{ અને } x = \frac{\pi}{2} \Rightarrow t = \pi$$

$$\therefore 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \log 2 [x]_0^{\frac{\pi}{2}}$$

$$2I = \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \cdot \log 2$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2$$

$$\therefore 2I = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2$$

જવાબ : (A)

નોંધ : બરેબર તો આ ‘અનુચિત’ (Improper) સંકલન છે.

(8) જો $n \in \mathbb{N}$ તો $\int_0^{2\pi} \frac{x \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = \dots$ (IIT : 1996)

(A) 0 (B) $\frac{\pi^2}{2}$ (C) π^2 (D) $2\pi^2$

ઉકેલ : $I = \int_0^{2\pi} \frac{x \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x}$ (1)

$\therefore I = \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n}(2\pi - x) \, dx}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}$

$= \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x}$

$= 2\pi \int_0^{2\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} - I$

$\therefore 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = 4\pi \int_0^{\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = 8\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x}$ (2)

$\therefore 2I = 8\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\left(\frac{\pi}{2} - x\right) \, dx}{\sin^{2n}\left(\frac{\pi}{2} - x\right) + \cos^{2n}\left(\frac{\pi}{2} - x\right)}$

$2I = 8\pi \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} x \, dx}{\cos^{2n} x + \sin^{2n} x}$ (3)

(2) તથા (3) નો સરવાળો કરતાં,

$4I = 8\pi \int_0^{\frac{\pi}{2}} 1 \, dx = 8\pi \left[x\right]_0^{\frac{\pi}{2}} = 4\pi^2$ એથી, $I = \pi^2$ જવાબ : (C)

(9) $\int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta = \dots$

(A) $\frac{\pi}{4} \log 2$ (B) $-\frac{\pi}{8} \log \frac{1}{2}$ (C) $\frac{\pi}{8} \log \frac{1}{2}$ (D) $\frac{\pi}{4} \log \frac{1}{2}$

ઉકેલ : $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) \, d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan \theta}{1 + \tan \theta}\right) \, d\theta$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \log \left(\frac{2}{1+\tan\theta} \right) d\theta \\
&= \int_0^{\frac{\pi}{4}} \log 2 \, d\theta - \int_0^{\frac{\pi}{4}} \log (1+\tan\theta) \, d\theta
\end{aligned}$$

$$I = \log 2 \int_0^{\frac{\pi}{4}} d\theta - I$$

$$\therefore 2I = \log 2 \left(\frac{\pi}{4} \right). \text{ અર્થાત્, } I = \frac{\pi}{8} \log 2 = -\frac{\pi}{8} \log \left(\frac{1}{2} \right) \quad \text{જવાબ : (B)}$$

$$(10) \int_{e^{-1}}^{e^2} \left| \frac{\log t}{t} \right| dt = \dots\dots$$

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) 5 (D) 3

$$\text{ઉકેલ : } I = \int_{e^{-1}}^{e^2} \left| \frac{\log t}{t} \right| dt$$

$$= - \int_{e^{-1}}^1 \frac{\log t}{t} dt + \int_1^{e^2} \frac{\log t}{t} dt$$

$$= -\frac{1}{2} [(\log t)^2]_{e^{-1}}^1 + \frac{1}{2} [(\log t)^2]_1^{e^2} = -\frac{1}{2} (0 - 1) + \frac{1}{2} (4 - 0) = \frac{1}{2} + 2 = \frac{5}{2} \quad \text{જવાબ : (B)}$$

$$(11) \text{ જો } f(t) + f\left(\frac{1}{t}\right) = F(t), \text{ જ્યાં } f(t) = \int_1^t \frac{\log x}{1+x} dx, \text{ તો } F(e) = \dots\dots$$

(A) $\frac{1}{2}$ (B) 1 (C) 0 (D) 2 (IIT : 2000)

$$\text{ઉકેલ : } F(e) = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log x}{1+x} dx + \int_1^{\frac{1}{e}} \frac{\log x}{1+x} dx$$

$$\text{બીજા સંકલનમાં, } x = \frac{1}{u} \text{ લેતાં, } dx = -\frac{1}{u^2} du \quad \left\{ \begin{array}{l} x=1 \Rightarrow u=1 \\ x=\frac{1}{e} \Rightarrow u=e \end{array} \right\}$$

$$\begin{aligned}
\therefore F(e) &= \int_1^e \frac{\log x \, dx}{1+x} + \int_1^e \frac{\log u \, du}{u(1+u)} \\
&= \int_1^e \frac{\log x \, dx}{1+x} + \int_1^e \frac{\log x \, dx}{x(1+x)} \\
&= \int_1^e \frac{\log x}{1+x} \left(1 + \frac{1}{x}\right) dx \\
&= \int_1^e \frac{\log x}{x} dx = \frac{1}{2} [(\log x)^2]_1^e = \frac{1}{2}
\end{aligned}$$

જવાબ : (A)

$$(12) \int_0^{\frac{\pi}{2}} (x - [\sin x]) \, dx = \dots\dots$$

(A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{8} - 2$ (C) $\frac{\pi^2}{8} - 1$ (D) $\frac{\pi^2}{4} - 1$

$$\begin{aligned}
\text{ઉકેલ : } I &= \int_0^{\frac{\pi}{2}} (x - [\sin x]) \, dx \\
&= \int_0^{\frac{\pi}{2}} x \, dx - \int_0^{\frac{\pi}{2}} [\sin x] \, dx \\
&= \left[\frac{x^2}{2} \right]_0^{\frac{\pi}{2}} - 0 \quad \left[0 < x < \frac{\pi}{2}; \quad 0 < \sin x < 1; \quad \text{અર્થાત્ } [\sin x] = 0 \right] \\
&= \frac{\pi^2}{8}
\end{aligned}$$

જવાબ : (A)

$$(13) \quad \text{જો } \int_{\log 2}^x \frac{du}{(e^u - 1)^{\frac{1}{2}}} = \frac{\pi}{6}, \text{ તો } e^x = \dots\dots$$

(A) 1 (B) 2 (C) 4 (D) 5

$$\text{ઉકેલ : } \int_{\log 2}^x \frac{du}{(e^u - 1)^{\frac{1}{2}}} = \frac{\pi}{6}$$

ધારો કે, $e^u - 1 = t^2$. ($t > 0$) આથી, $e^u du = 2t dt$

$$\therefore e^u = t^2 + 1. \text{ આથી, } du = \frac{2t}{t^2 + 1} dt$$

$$\text{હવે, } u = \log 2 \Rightarrow t^2 = 2 - 1 = 1$$

$$u = x \Rightarrow t^2 = e^x - 1 \Rightarrow t = \sqrt{e^x - 1}$$

$$\therefore \int_1^{\sqrt{e^x - 1}} \frac{2t}{t(t^2 + 1)} dt = \frac{\pi}{6}$$

$$\therefore 2 [\tan^{-1} t]_1^{\sqrt{e^x - 1}} = \frac{\pi}{6}$$

$$\therefore \tan^{-1} \sqrt{e^x - 1} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$\therefore \tan^{-1} \sqrt{e^x - 1} = \frac{\pi}{3}$$

$$\therefore \sqrt{e^x - 1} = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\therefore e^x - 1 = 3$$

$$\therefore e^x = 4$$

જવાબ : (C)

$$(14) \text{ જો } \frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}, \quad x > 0 \text{ હોય, તો } \int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1), \text{ તો } k \text{ ની શક્ય}$$

વિધિ

(A) 15

(B) 16

(C) 63

(D) 64

$$\text{ઉકેલ : } \frac{d}{dx} F(x) = \frac{e^{\sin x}}{x} \Rightarrow F(x) = \int \frac{e^{\sin x}}{x} dx$$

$$\text{હવે, } I = \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$$

$$\text{ધારો કે, } x^3 = t. \text{ આથી, } 3x^2 dx = dt$$

$$x = 1 \Rightarrow t = 1 \text{ તથા } x = 4 \Rightarrow t = 64$$

$$\therefore I = \int_1^{64} \frac{e^{\sin t}}{t} dt = [F(t)]_1^{64} = F(64) - F(1)$$

$$\therefore k = 64$$

જવાબ : (D)

$$(15) \quad I_n = \int_0^1 e^x (x-1)^n dx, \quad \text{જ્યાં, } n \leq 5, n \in \mathbb{N} \cup \{0\} \text{ તો, } I_3 = \dots$$

$$(A) \quad 2e - 5$$

$$(B) \quad 8e - 16$$

$$(C) \quad 2 - e$$

$$(D) \quad 16 - 6e$$

$$\text{ઉકેલ : } I_n = \int_0^1 e^x (x-1)^n dx$$

$$I_n = [(x-1)^n \cdot e^x]_0^1 - n \int_0^1 (x-1)^{n-1} \cdot e^x dx$$

$$\therefore I_n = (-1)^{n+1} - n \cdot I_{n-1}$$

$$\therefore I_0 = \int_0^1 e^x dx = e - 1$$

$$\text{જો } n = 1, \text{ તો } I_1 = (-1)^2 - 1I_0 = 1 - (e - 1) = 2 - e$$

$$\text{જો } n = 2, \text{ તો } I_2 = (-1)^3 - 2I_1 = -1 - 2(2 - e) = 2e - 5$$

$$\text{જો } n = 3, \text{ તો } I_3 = 1 - 3I_2 = 1 - 3(2e - 5) = 16 - 6e$$

જવાબ : (D)

$$(16) \quad \int_0^{\pi} \ln(1 + \cos x) dx = \dots \quad (\ln = \text{natural logarithm} = \log_e)$$

$$(A) \quad \frac{\pi}{2} \ln 2$$

$$(B) \quad \pi \ln 2$$

$$(C) \quad -\frac{\pi}{2} \ln 2$$

$$(D) \quad -\pi \ln 2$$

$$\text{ઉકેલ : } I = \int_0^{\pi} \ln(1 + \cos x) dx \quad (1)$$

$$= \int_0^{\pi} \ln(1 + \cos(\pi - x)) dx = \int_0^{\pi} \ln(1 - \cos x) dx \quad (2)$$

(1) અને (2) ને સરવાળે કરતાં,

$$2I = \int_0^{\pi} \ln(1 + \cos x)(1 - \cos x) dx = \int_0^{\pi} \ln \sin^2 x dx$$

$$\therefore 2I = 2 \int_0^{\pi} \ln(\sin x) dx$$

$$= 4 \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = 4 \left(-\frac{\pi}{2} \ln 2 \right)$$

$$\therefore 2I = -2\pi \ln 2$$

$$\therefore I = -\pi \ln 2$$

જવાબ : (D)

$$(17) \int_0^1 \tan^{-1}(1 - x + x^2) dx = \dots\dots$$

$$(A) \frac{\pi}{2}$$

$$(B) \frac{\pi}{4}$$

$$(C) \ln 2$$

$$(D) \frac{\pi}{2} - \ln 2$$

$$\text{ઉકેલ : } I = \int_0^1 \tan^{-1}(1 - x + x^2) dx$$

$$= \int_0^1 \left[\frac{\pi}{2} - \cot^{-1}(1 - x + x^2) \right] dx$$

$$= \int_0^1 \frac{\pi}{2} dx - \int_0^1 \tan^{-1} \left(\frac{1}{1 - x + x^2} \right) dx \quad (1 - x + x^2 > 0)$$

$$= \frac{\pi}{2} [x]_0^1 - \int_0^1 \tan^{-1} \left(\frac{(x) + (1 - x)}{1 - x(1 - x)} \right) dx$$

$$= \frac{\pi}{2} - \int_0^1 (\tan^{-1}x + \tan^{-1}(1 - x)) dx \quad (0 < x < 1, 0 < 1 - x < 1; x(1 - x) < 1)$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(1 - x) dx$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(1 - (1 - x)) dx$$

$$= \frac{\pi}{2} - 2 \int_0^1 \tan^{-1}x dx$$

$$= \frac{\pi}{2} - 2 \left\{ [x \tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx \right\}$$

$$= \frac{\pi}{2} - 2 \left(1 \cdot \frac{\pi}{4} - 0 \right) + \int_0^1 \frac{2x}{1 + x^2} dx = \frac{\pi}{2} - \frac{\pi}{2} + [\ln |1 + x^2|]_0^1 = \ln 2$$

જવાબ : (C)

$$(18) \text{ જો } g(1) = g(2) \text{ તો } \int_1^2 [f\{g(x)\}]^{-1} \cdot f'(g(x)) \cdot g'(x) dx = \dots\dots$$

$$(A) 1$$

$$(B) 2$$

$$(C) 0$$

$$(D) -1$$

$$\text{ઉકેલ : } I = \int_1^2 [f\{g(x)\}]^{-1} \cdot f'(g(x)) \cdot g'(x) dx$$

$$\text{ધારો કે, } f(g(x)) = t. \quad \text{આથી } f'(g(x)) g'(x) dx = dt$$

જ્યારે $x = 1 \Rightarrow t = f(g(1))$ તથા $x = 2 \Rightarrow t = f(g(2))$

$$\therefore I = \int_{f(g(1))}^{f(g(2))} \frac{1}{t} dt = [\log t]_{f(g(1))}^{f(g(2))}$$

$$= \log |f(g(2))| - \log |f(g(1))|$$

$$= 0$$

$$(g(1) = g(2))$$

જવાબ : (C)

(19) જો $I_1 = \int_0^1 (1 - x^{50})^{100} dx$ અને $I_2 = \int_0^1 (1 - x^{50})^{101} dx$ તો $\frac{I_1}{I_2} = \dots\dots$

(A) $\frac{5051}{5050}$

(B) $\frac{5051}{5049}$

(C) $\frac{51}{50}$

(D) $\frac{101}{100}$

ઉકેલ : $I_2 = \int_0^1 (1 - x^{50})^{101} dx$

$$= \int_0^1 (1 - x^{50})(1 - x^{50})^{100} dx$$

$$= \int_0^1 (1 - x^{50})^{100} dx - \int_0^1 x^{50} (1 - x^{50})^{100} dx$$

$$I_2 = I_1 - \int_0^1 x(1 - x^{50})^{100} \cdot x^{49} dx$$

$$I_2 = I_1 + \left[\frac{x(1 - x^{50})^{101}}{5050} \right]_0^1 - \frac{1}{5050} \int_0^1 (1 - x^{50})^{101} dx$$

$$I_2 = I_1 + 0 - \frac{1}{5050} I_2$$

$$\therefore I_1 = I_2 \left(1 + \frac{1}{5050} \right)$$

$$\therefore I_1 = \frac{5051}{5050} I_2$$

$$\therefore \frac{I_1}{I_2} = \frac{5051}{5050}$$

જવાબ : (A)

(20) $\int_0^{1000} e^{x - [x]} dx = \dots\dots$

(A) $e^{1000} - 1$

(B) $\frac{e^{1000} - 1}{e - 1}$

(C) $1000(e - 1)$

(D) $\frac{e - 1}{1000}$

ઉકેલ : $I = \int_0^{1000} e^{x - [x]} dx$

$e^{x - [x]}$ એ આવર્તિય વિધેય છે. તેનું આવર્તમાન 1 છે.

$$\therefore I = \int_0^{1000} e^{x - [x]} dx$$

$$= 1000 \int_0^1 e^{x - [x]} dx; \text{ ઇન્કે, } 0 < x < 1 \text{ અર્થ, } [x] = 0$$

$$= 1000 \int_0^1 e^x dx$$

$$= 1000 [e^x]_0^1$$

$$= 1000 [e - 1]$$

જવાબ : (C)

$$(21) \lim_{n \rightarrow \infty} \left[\frac{n}{n^2 + 1} + \frac{n}{n^2 + 4} + \frac{n}{n^2 + 9} + \dots + \frac{1}{2n} \right] = \dots$$

(A) 1

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$

(D) 0

$$\text{ઉદા : } \lim_{n \rightarrow \infty} \left[\frac{n}{n^2 + 1} + \frac{n}{n^2 + 4} + \frac{n}{n^2 + 9} + \dots + \frac{1}{2n} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 + r^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n^2} \left[\frac{n}{1 + \left(\frac{r}{n}\right)^2} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\frac{1}{1 + \left(\frac{r}{n}\right)^2} \right]$$

$$= \int_0^1 \frac{1}{1 + x^2} dx$$

$$= [\tan^{-1} x]_0^1$$

$$= \frac{\pi}{4}$$

જવાબ : (C)

$$(22) \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx = \dots$$

(A) 3

(B) 2

(C) 1

(D) 0

$$\text{ઉદા : } I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x + \cos x) dx \quad (0 < \theta < \frac{\pi}{2})$$

$$= [-\cos x + \sin x]_0^{\frac{\pi}{2}} = (0 + 1) - (-1 + 0) = 2 \quad \text{જવાબ : (B)}$$

(23) જો $f(y) = e^y$, $g(y) = y$, $y > 0$ અને $F(t) = \int_0^t f(t-y) g(y) dy$ તો $F(t) = \dots$

- (A) $-t - 1$ (B) $t \cdot e^{-t}$ (C) $e^t - (1 + t)$ (D) $t \cdot e^{-t} + (1 - t)$
(AIEEE : 2004)

ઉત્તર : $F(t) = \int_0^t f(t-y) g(y) dy$

$$= \int_0^t e^{t-y} \cdot y dy$$

$$= e^t \int_0^t e^{-y} \cdot y dy$$

$$= e^t \left\{ [-y e^{-y}]_0^t - \int_0^t 1 \cdot (-e^{-y}) dy \right\}$$

$$= e^t \{ -t \cdot e^{-t} + [-e^{-y}]_0^t \}$$

$$= e^t \{ -t \cdot e^{-t} - e^{-t} + 1 \}$$

$$= -t e^0 - e^0 + e^t$$

$$= -t - 1 + e^t$$

$$= e^t - (t + 1)$$

જવાબ : (C)

(24) $\int_{\frac{1}{2}}^2 \frac{1}{x} \sin \left(x - \frac{1}{x} \right) dx = \dots$

- (A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) π

ઉત્તર : $I = \int_{\frac{1}{2}}^2 \frac{1}{x} \sin \left(x - \frac{1}{x} \right) dx$

ધારો કે, $\frac{1}{x} = t$ આથી, $-\frac{1}{x^2} dx = dt$

$\therefore dx = -x^2 dt$ આથી, $dx = -\frac{1}{t^2} dt$

$x = \frac{1}{2} \Rightarrow t = 2$ તથા $x = 2 \Rightarrow t = \frac{1}{2}$

$$\therefore I = \int_{\frac{1}{2}}^2 t \cdot \sin\left(\frac{1}{t} - t\right) \cdot \frac{1}{t^2} dt$$

$$= \int_{\frac{1}{2}}^2 \left\{ -\frac{1}{t} \sin\left(t - \frac{1}{t}\right) \right\} dt$$

$$= - \int_{\frac{1}{2}}^2 \frac{1}{x} \sin\left(x - \frac{1}{x}\right) dx$$

$\therefore I = -I$

$\therefore 2I = 0$. આથી, $I = 0$

જવાબ : (A)

(25) $\int_0^{\infty} \frac{dx}{(x + \sqrt{1+x^2})^3} = \dots\dots$

(A) $-\frac{1}{4}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{2}$

(D) $\frac{3}{8}$

ઉકેલ : $I = \int_0^{\infty} \frac{dx}{(x + \sqrt{1+x^2})^3}$

ધારો કે, $x = \tan\theta$. આથી, $dx = \sec^2\theta d\theta$

$\therefore x = 0 \Rightarrow \theta = 0$ તથા $x = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta d\theta}{(\tan\theta + \sec\theta)^3}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1 + \sin\theta)^3} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 + \sin \theta)^{-3} (\cos \theta) d\theta$$

$$= \left[\frac{(1 + \sin \theta)^{-2}}{-2} \right]_0^{\frac{\pi}{2}} = -\frac{1}{2} \left[\frac{1}{4} - 1 \right] = -\frac{1}{2} \left(\frac{-3}{4} \right) = \frac{3}{8}$$

જવાબ : (D)

(26) $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \in [a, b]$ હોય, તો $[a, b] = \dots$ જ્યાં $[a, b]$ નાનામાં નાનો શક્ય અંતરાલ છે.

(A) $\left[\frac{1}{2}, 1\right]$

(B) $\left[\frac{1}{\sqrt{2}}, 1\right]$

(C) $[0, 1]$

(D) $\left[\frac{3}{4}, 1\right]$

ઉકેલ : $0 \leq x \leq 1 \Rightarrow 0 \leq x^4 \leq 1$

$$\Rightarrow 1 \leq 1 + x^4 \leq 2$$

$$\Rightarrow 1 \leq \sqrt{1 + x^4} \leq \sqrt{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1 + x^4}} \leq 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \int_0^1 \frac{1}{\sqrt{1 + x^4}} dx \leq 1$$

$$\therefore [a, b] = \left[\frac{1}{\sqrt{2}}, 1\right]$$

જવાબ : (B)

(27) $\int_{-2}^0 (x^3 + 3x^2 + 3x + 3 + (x + 1) \cos (x + 1)) dx = \dots$

(A) -4

(B) 0

(C) 4

(D) 6

(IIT : 2005)

ઉકેલ : $I = \int_{-2}^0 (x^3 + 3x^2 + 3x + 3 + (x + 1) \cos (x + 1)) dx$

$$= \int_{-2}^0 ((x + 1)^3 + 2 + (x + 1) \cos (x + 1)) dx$$

ધારો કે, $x + 1 = t$. અર્થાત્, $dx = dt$. અર્થાત્, $x = -2 \Rightarrow t = -1$ અને $x = 0 \Rightarrow t = 1$

$$\therefore I = \int_{-1}^1 (t^3 + 2 + t \cos t) dt$$

$$= \int_{-1}^1 (t^3 + t \cos t) dt + 2 \int_{-1}^1 dt \quad (t^3 + t \cos t = \text{અચુગ્મ વિધેય})$$

$$= 0 + 2[t]_{-1}^1 = 2(1 + 1) = 4$$

જવાબ : (C)

(28) $\int_0^{\frac{3}{2}} [x^2] dx = \dots []$ એ મહત્તમ પૂર્ણિક વિધેય દર્શાવે છે.

(A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$ (C) $-2 - \sqrt{2}$ (D) $-2 + \sqrt{2}$

ઉકેલ : $I = \int_0^{\frac{3}{2}} [x^2] dx$

$0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0$

$1 < x < \sqrt{2} \Rightarrow 1 < x^2 < 2 \Rightarrow [x^2] = 1$

$\sqrt{2} < x < \frac{3}{2} \Rightarrow 2 < x^2 < 2.25 \Rightarrow [x^2] = 2$

$\therefore I = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\frac{3}{2}} [x^2] dx$

$= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\frac{3}{2}} 2 dx$

$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\frac{3}{2}}$

$= (\sqrt{2} - 1) + (3 - 2\sqrt{2})$

$= 2 - \sqrt{2}$

જવાબ : (B)

(29) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2 \sin x] dx = \dots$, જ્યાં $[]$ એ મહત્તમ પૂર્ણિક વિધેય દર્શાવે છે.

(A) $-\pi$ (B) 0 (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$ (IIT : 1999)

ઉકેલ : $I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2 \sin x] dx$

ધારો કે $x = \frac{\pi}{2} + t$. અર્થાત્, $dx = dt$. $x = \frac{\pi}{2} \Rightarrow t = 0$ તથા $x = \frac{3\pi}{2} \Rightarrow t = \pi$

$\therefore I = \int_0^{\pi} [2 \cos t] dt$

$0 < t < \frac{\pi}{3} \Rightarrow 1 < 2 \cos t < 2 \Rightarrow [2 \cos t] = 1$

$\frac{\pi}{3} < t < \frac{\pi}{2} \Rightarrow 0 < 2 \cos t < 1 \Rightarrow [2 \cos t] = 0$

$\frac{\pi}{2} < t < \frac{2\pi}{3} \Rightarrow -1 < 2 \cos t < 0 \Rightarrow [2 \cos t] = -1$

$\frac{2\pi}{3} < t < \pi \Rightarrow -2 < 2 \cos t < -1 \Rightarrow [2 \cos t] = -2$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{3}} 1 \, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 0 \, dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} -1 \, dx + \int_{\frac{2\pi}{3}}^{\pi} (-2) \, dx \\
&= \frac{\pi}{3} - \frac{\pi}{6} - \frac{2\pi}{3} = -\frac{\pi}{2}
\end{aligned}$$

જવાબ : (C)

$$(30) \int_1^5 (|x-3| + |x-1|) \, dx = \dots\dots$$

- (A) $\frac{5}{6}$ (B) 10 (C) 12 (D) 21

$$\begin{aligned}
\text{ઉકેલ : } I &= \int_1^5 (|x-3| + |x-1|) \, dx \\
&= \int_1^3 |x-3| \, dx + \int_3^5 |x-3| \, dx + \int_1^5 |x-1| \, dx \\
&= \int_1^3 (-x+3) \, dx + \int_3^5 (x-3) \, dx + \int_1^5 (x-1) \, dx \\
&= \left[\frac{-x^2}{2} + 3x \right]_1^3 + \left[\frac{x^2}{2} - 3x \right]_3^5 + \left[\frac{x^2}{2} - x \right]_1^5 \\
&= \left[\left(\frac{-9}{2} + 9 \right) - \left(\frac{-1}{2} + 3 \right) \right] + \left[\left(\frac{25}{2} - 15 \right) - \left(\frac{9}{2} - 9 \right) \right] + \left[\left(-5 + \frac{25}{2} \right) - \left(-1 + \frac{1}{2} \right) \right] \\
&= \frac{9}{2} - \frac{5}{2} - \frac{5}{2} + \frac{9}{2} + \frac{15}{2} + \frac{1}{2} = 12
\end{aligned}$$

જવાબ : (C)

$$(31) \int_{-3}^3 \log |x + \sqrt{x^2+1}| \, dx = \dots\dots$$

- (A) 0 (B) $\log 3$ (C) $\log (19 + 6\sqrt{10})$ (D) $\log \frac{1}{3}$

$$\text{ઉકેલ : } \text{ધારો કે } f(x) = \log |x + \sqrt{x^2+1}|$$

$$\therefore f(-x) = \log |-x + \sqrt{x^2+1}|$$

$$= \log \left(\left(\sqrt{x^2+1} - x \right) \times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} \right)$$

$$= \log \left(\frac{x^2+1-x^2}{\sqrt{x^2+1} + x} \right)$$

$$= \log \frac{1}{\sqrt{x^2+1} + x} = -\log (\sqrt{x^2+1} + x) = -f(x)$$

$$\therefore I = 0 \quad (f(x) \text{ એ અયુગ્મ વિધેય છે.})$$

જવાબ : (A)

$$(32) \int_0^1 \log (\sqrt{1-x} + \sqrt{1+x}) dx = \dots$$

$$(A) \frac{1}{2} \left[\log 2 - \frac{\pi}{4} + 1 \right]$$

$$(B) \frac{1}{2} \left[\log 2 + \frac{\pi}{2} - 1 \right]$$

$$(C) \frac{1}{2} \left[\log 2 - \frac{3\pi}{2} + 1 \right]$$

$$(D) \frac{1}{2} \left[\log 2 + \frac{\pi}{3} - 1 \right]$$

$$\text{જણાવો : } I = \int_0^1 \log (\sqrt{1-x} + \sqrt{1+x}) dx$$

$$= \left[x \log (\sqrt{1-x} + \sqrt{1+x}) \right]_0^1 - \int_0^1 \left(\frac{1}{\sqrt{1-x} + \sqrt{1+x}} \right) \left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}} \right) \cdot x dx$$

$$= \log \sqrt{2} + \frac{1}{2} \int_0^1 \frac{(\sqrt{1+x} - \sqrt{1-x}) \cdot x}{(\sqrt{1+x} + \sqrt{1-x}) \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{(\sqrt{1+x} - \sqrt{1-x})^2 \cdot x}{(\sqrt{1+x} + \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x}) \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{(2 - 2\sqrt{1-x^2}) \cdot x}{2x \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int_0^1 dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} [\sin^{-1} x]_0^1 - \frac{1}{2} [x]_0^1$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2}$$

$$= \frac{1}{2} \left[\log 2 + \frac{\pi}{2} - 1 \right]$$

જવાબ : (B)

$$(33) \int_1^{e^{37}} \frac{\pi \sin(\pi \log x)}{x} dx = \dots$$

$$(A) 0$$

$$(B) 2$$

$$(C) 37$$

$$(D) 74$$

$$\text{ઉકેલ : I} = \int_1^{e^{37}} \frac{\pi \sin(\pi \log x)}{x} dx$$

$$\text{ધારો કે, } \pi \log x = t. \quad \text{આથી, } \frac{\pi}{x} dx = dt.$$

$$x = 1 \Rightarrow t = 0 \quad \text{તથા } x = e^{37} \Rightarrow t = 37\pi$$

$$= \int_0^{37\pi} \sin t \, dt$$

$$= [-\cos t]_0^{37\pi} = -[\cos 37\pi - \cos 0] = -(-1 - 1) = 2$$

જવાબ : (B)

$$(34) \int_0^1 \cos \left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx = \dots$$

$$(A) \quad \frac{-1}{2}$$

$$(B) \quad -1$$

$$(C) \quad \frac{-3}{4}$$

$$(D) \quad \frac{-3}{8}$$

$$\text{ઉકેલ : I} = \int_0^1 \cos \left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx$$

$$\text{ધારો કે, } x = \cos \theta. \quad \text{આથી, } dx = -\sin \theta \, d\theta$$

$$x = 0 \Rightarrow \theta = \frac{\pi}{2} \quad \text{તથા } x = 1 \Rightarrow \theta = 0$$

$$\therefore I = - \int_{\frac{\pi}{2}}^0 \cos \left(2 \cot^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \right) \sin \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos \left(2 \cot^{-1} \left(\tan \frac{\theta}{2} \right) \right) \sin \theta \, d\theta \quad \left(0 < \frac{\theta}{2} < \frac{\pi}{4} \right)$$

$$= \int_0^{\frac{\pi}{2}} \cos \left\{ 2 \cot^{-1} \left(\cot \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right) \right\} \sin \theta \, d\theta \quad \left(0 < \frac{\pi}{4} < \frac{\pi}{2} - \frac{\theta}{2} < \frac{\pi}{2} < \pi \right)$$

$$= \int_0^{\frac{\pi}{2}} \cos (\pi - \theta) \sin \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\cos\theta \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} -\sin 2\theta \, d\theta$$

$$= \frac{1}{2} \left[\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} [\cos\pi - \cos 0] = \frac{1}{4} [-1 - 1] = -\frac{1}{2}$$

જવાબ : (A)

$$(35) \quad \int_{\frac{1}{e}}^{\tan x} \frac{t}{1+t^2} \, dt + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)} = \dots\dots$$

(A) 1

(B) 0

(C) $\frac{1}{e}$

(D) e

$$\text{ઉકેલ : } I = \int_{\frac{1}{e}}^{\tan x} \frac{t \, dt}{1+t^2} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{હરેલે છે, } I = I_1 + I_2$$

$$\text{જ્યાં, } I_1 = \int_{\frac{1}{e}}^{\tan x} \frac{t}{1+t^2} \, dt \quad \text{અને} \quad I_2 = \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$I_2 = \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{હરેલે છે, } t = \frac{1}{u} \quad \text{આથી, } dt = -\frac{1}{u^2} du$$

$$t = \frac{1}{e} \Rightarrow u = e \quad \text{તથા } t = \cot x \Rightarrow u = \tan x$$

$$\therefore I_2 = \int_e^{\tan x} \frac{1}{\frac{1}{u}\left(1+\frac{1}{u^2}\right)} \left(-\frac{1}{u^2}\right) du$$

$$= - \int_e^{\tan x} \frac{u \, du}{u^2 + 1}$$

$$= - \left\{ \int_e^{\tan x} \frac{u \, du}{1+u^2} + \int_{\frac{1}{e}}^{\tan x} \frac{u}{1+u^2} \, du \right\}$$

$$I_2 = -\int_e^{\frac{1}{e}} \frac{t}{1+t^2} dt - \int_e^{\tan x} \frac{t}{1+t^2} dt = -\int_e^{\frac{1}{e}} \frac{t}{1+t^2} dt - I_1$$

$$\therefore I_1 + I_2 = \frac{1}{2} \int_e^{\frac{1}{e}} \frac{2t}{1+t^2} dt$$

$$\begin{aligned} \therefore I &= \frac{-1}{2} \left[\log(t^2 + 1) \right]_e^{\frac{1}{e}} \\ &= \frac{-1}{2} \left[\log \left| \frac{1}{e^2} + 1 \right| - \log |e^2 + 1| \right] \\ &= \frac{-1}{2} \log \left(\frac{1}{e^2} \right) = \frac{-1}{2} \times -2 \times \log_e e = 1 \end{aligned}$$

જવાબ : (A)

$$(36) \text{ જો } \alpha f(x) + \beta f\left(\frac{1}{x}\right) = \frac{1}{x} - 5, x \neq 0 \text{ અને } \alpha \neq \beta, \text{ તો } \int_1^2 f(x) dx = \dots$$

$$(A) \frac{2\alpha \log 2 - 5\alpha + 3\beta}{2(\alpha^2 - \beta^2)} \quad (B) \frac{3\alpha \log 2 - 7\alpha + 5\beta}{2(\alpha^2 - \beta^2)} \quad (C) \frac{\alpha \log 2 - 5\alpha + 3\beta}{2(\alpha^2 - \beta^2)} \quad (D) \frac{2\alpha \log 2 - 10\alpha + 7\beta}{2(\alpha^2 - \beta^2)}$$

(IIT : 1995)

$$\text{ઉકેલ : } \alpha f(x) + \beta f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad (1)$$

$$x \text{ ની સ્થાને } \frac{1}{x} \text{ લેતાં, } \alpha f\left(\frac{1}{x}\right) + \beta f(x) = x - 5 \quad (2)$$

પરિણામ (1)ને α અને (2)ને β વડે ગુણી બાદબાકી કરતાં,

$$(\alpha^2 - \beta^2) f(x) = \frac{\alpha}{x} - 5\alpha - \beta x + 5\beta$$

$$f(x) = \frac{1}{(\alpha^2 - \beta^2)} \left\{ \frac{\alpha}{x} - \beta x - 5(\alpha - \beta) \right\}$$

$$\text{હવે, } \int_1^2 f(x) dx = \frac{1}{(\alpha^2 - \beta^2)} \int_1^2 \left[\frac{\alpha}{x} - \beta x - 5(\alpha - \beta) \right] dx$$

$$= \frac{1}{(\alpha^2 - \beta^2)} \left\{ \alpha [\log x]_1^2 - \beta \left[\frac{x^2}{2} \right]_1^2 - 5(\alpha - \beta) [x]_1^2 \right\}$$

$$= \frac{1}{(\alpha^2 - \beta^2)} \left\{ \alpha \log 2 - \frac{3}{2} \beta - 5(\alpha - \beta) \right\}$$

$$= \frac{2\alpha \log 2 - 10\alpha + 7\beta}{2(\alpha^2 - \beta^2)}$$

જવાબ : (D)

$$(37) \int_0^1 \frac{dx}{x^2 + (2\cos\alpha)x + 1} = \dots\dots$$

$$(A) \frac{1}{2\alpha \sin \alpha}$$

$$(B) \frac{\alpha}{2 \sin \alpha}$$

$$(C) \alpha \sin \alpha$$

$$(D) \frac{\alpha}{2} \sin \alpha$$

$$\text{જણાવો : } I = \int_0^1 \frac{dx}{x^2 + (2\cos\alpha)x + 1} \quad (0 < \alpha < \frac{\pi}{2})$$

$$I = \int_0^1 \frac{dx}{(x^2 + (2\cos\alpha)x + \cos^2\alpha) + (1 - \cos^2\alpha)}$$

$$= \int_0^1 \frac{dx}{(x + \cos\alpha)^2 + (\sin\alpha)^2}$$

$$= \left[\frac{1}{\sin\alpha} \tan^{-1} \left(\frac{x + \cos\alpha}{\sin\alpha} \right) \right]_0^1$$

$$= \frac{1}{\sin\alpha} \left[\tan^{-1} \left(\frac{1 + \cos\alpha}{\sin\alpha} \right) - \tan^{-1} \left(\frac{\cos\alpha}{\sin\alpha} \right) \right]$$

$$= \frac{1}{\sin\alpha} \left[\tan^{-1} \frac{2 \cos^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} - \tan^{-1} (\cot \alpha) \right]$$

$$= \frac{1}{\sin\alpha} \left[\tan^{-1} \left(\cot \frac{\alpha}{2} \right) - \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \alpha \right) \right) \right]$$

$$= \frac{1}{\sin\alpha} \left[\tan^{-1} \left(\tan \left(\frac{\pi}{2} - \frac{\alpha}{2} \right) \right) - \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \alpha \right) \right) \right]$$

$$= \frac{1}{\sin\alpha} \left[\left(\frac{\pi}{2} - \frac{\alpha}{2} \right) - \left(\frac{\pi}{2} - \alpha \right) \right]$$

$$= \frac{\alpha}{2 \sin \alpha}$$

જવાબ : (B)

$$(38) \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx = \dots\dots$$

$$(A) 2 \cos \frac{1}{2}$$

$$(B) \cos \frac{1}{3}$$

$$(C) 2 \log 2$$

$$(D) 0$$

$$\text{જણાવો : } I = \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

$$= \int_{-\log 2}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

ધારો કે, $f(x) = \sin \left(\frac{e^x - 1}{e^x + 1} \right)$

$$\therefore f(-x) = \sin \left(\frac{e^{-x} - 1}{e^{-x} + 1} \right)$$

$$\therefore f(-x) = \sin \left(\frac{1 - e^x}{1 + e^x} \right) = -\sin \left(\frac{e^x - 1}{e^x + 1} \right) = -f(x)$$

$\therefore f(x)$ અયુગ્મ વિધેય છે.

$$\therefore I = 0$$

જવાબ : (D)

બીજી રીત : $I = \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} \right) dx = \int_{-\log 2}^{\log 2} \sin \left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} \right) dx$

$$\therefore f(-x) = \sin \left(\frac{e^{-\frac{x}{2}} - e^{\frac{x}{2}}}{e^{-\frac{x}{2}} + e^{\frac{x}{2}}} \right)$$

$$= -\sin \left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} \right) = f(-x)$$

$$\therefore I = 0$$

જવાબ : (D)

(39) $\int_1^{16} \tan^{-1} \sqrt{(\sqrt{x} - 1)} dx = \dots\dots$

(A) $\frac{3\pi}{10} - 3\sqrt{2}$

(B) $\frac{4\pi}{15} - 3\sqrt{3}$

(C) $\frac{16\pi}{3} - 2\sqrt{3}$

(D) $\frac{9\pi}{16} - 2\sqrt{2}$

ઉકેલ : $I = \int_1^{16} \tan^{-1} \sqrt{(\sqrt{x} - 1)} dx$

ધારો કે, $x = \sec^4 \theta$. અર્થાત્, $dx = 4 \sec^3 \theta (\sec \theta \tan \theta) d\theta$

$$x = 1 \Rightarrow \theta = 0$$

$$x = 16 \Rightarrow \theta = \sec^{-1} \sqrt[4]{16} = \sec^{-1} 2 = \frac{\pi}{3}$$

$$\therefore I = 4 \int_0^{\frac{\pi}{3}} \theta \sec^3 \theta (\sec \theta \tan \theta) d\theta$$