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H-5341

M.Sc. (Semester-IV) Examination, July, 2022

MATHEMATICS

(Integration Theory and Functional Analysis - II)

Time Allowed : Three Hours

Maximum Marks : 70

Minimum Passing Marks : 25

Note : Attempt **all** Sections as directed.

SECTION-A

(Objective/Multiple Choice Questions)

Note : Attempt **all** questions. Each question carries **1** mark. [1x10=10]

1. Choose the correct option :

(i) Let T be a closed linear map of Banach space X into Banach space Y, then :

(a) T is continuous

(b) T is open

(c) T is closed

(d) All of the above

(ii) Let X and Y be normed linear space and $D \subset X$, then linear transformation $T : D \to Y$ is closed if and only if its graph G_T is :

(a) closed

(b) open

(c) bounded

(d) None of the above

(iii) Let X be a normed space over field K and S be a linear subspace of X. Suppose that $Z \in X$ and dist $(Z, S) = d > 0$, then there exist $g \in X^*$ such that :

(a) $g(s) = \{0\},\ g(z) = d,\ \| g \| = 1$

(b) $g(s) \neq \{0\},\ g(z) = d,\ \| g \| = 1$

(c) $g(s) = \{0\},\ g(z) = d,\ \| g \| \neq 1$

(d) $g(s) \neq \{0\},\ g(z) = d,\ \| g \| \neq 1$

(iv) Let X and Y be normed space over the field K and $T : X \to Y$ be a bounded linear operator, then :

(a) The adjoint T^* is bounded linear from Y^* to X^*.

(b) $\| T^* \| = \| T \|$

(c) The mapping T of T^* is an isometric isomorphism of $B(X, Y)$ into $B(Y^*, X^*)$

(d) All of the above

(v) If x and y are any two vectors in inner product space X, then

$$| \langle x, y \rangle | \leq \| x \| \cdot \| y \|$$

The above inequality is known as :

(a) Parallelogram law

(b) Cauchy-Schwarz's inequality

(c) Polarisation identity

(d) Bessel's inequality

(vi) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then

$$\| x \|^2 = \sum | \langle x_i e_i \rangle |^2$$

This equality is known as :

(a) Parsevals's identity

(b) Apollonius identity

(c) Holder's inequality

(d) None of the above

(vii) Let X be an inner product space. Which is incorrect statement?

(a) If $x \perp y \Leftrightarrow y \perp x, \ \forall \ x, y \in X$

(b) $x \perp 0, \ \forall \ x \in X$

(c) 0 is the only vector in X which is orthogonal to itself.

(d) $A \cap A^{\perp} \neq \{0\}$ or ϕ

(viii) Let H_1 and H_2 be Hilbert space and T and S are elements of $B(H_1, H_2)$ and $\alpha \in K$, then which statement is incorrect?

(a) $(T+S)^* = T^* + S^*$

(b) $(TS)^* = S^*T^*$

(c) $T^{**} = T$

(d) $(\alpha T)^* = \alpha T^*$

(ix) Let $P \in B(H)$ be a projection operator, then :

(a) $R(P)$ and $N(P)$ are closed subspace of H

(b) $I–P$ is a projection

(c) $R(P) = N(I–P)$

(d) All of the above

(x) Which statement is incorrect?

(a) Every positive operator is self-adjoint.

(b) Every self-adjoint operator is normal.

(c) Every normal operator is unitary.

(d) Every unitary operator is normal.

SECTION-B

(Very Short Answer Type Questions)

Note : Attempt **any five** questions. Each question carries **2** marks. (Maximum word limit : **25-30** words) [5x2=10]

2. (i) Give an example of closed operator which is not bounded.

   (ii) Define complex and normed linear space.

   (iii) Define weak sequential compactness.

   (iv) Define inner product space with example.

   (v) Write the statement of Projection theorem.

   (vi) Prove that every positive operator is self-adjoint.

   (vii) Define normal operator on Hilbert space.

SECTION-C

(Short Answer Type Questions)

Note : Attempt **any five** questions. Each question carries **4** marks. (Maximum word limit **: 250** words) [5x4=20]

3. (i) Let X be a Banach space over the field K. If $\{T_n\} \in B(X,Y)$ be a sequence such that $\lim_{n\to\infty} T_n x = Tx$ where $x \in X$ exists, then prove that $T \in B(X,Y)$.

(ii) Show that in an inner product space X, $x \perp y$ if and only if $\| x+\alpha y \| \geq \| x \| \quad \forall \text{scalar} \alpha$ $\forall x, y \in X$.

(iii) Let H_1 and H_2 be Hilbert space and $T \in B(H_1, H_2)$, then prove that $\| T^*T \| = \| T \|^2 = \| TT^* \|$

(iv) State and prove the closed range theorem.

(v) A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometry and surjective.

(vi) Show that the product of two bounded self-adjoint operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute.

(vii) State and prove uniform boundedness principle.

SECTION-D

(Essay Type Questions)

Note : Attempt **any three** questions. Each question carries **10** marks. (Maximum word limit : More than **500** words) [3x10=30]

4. (i) State and prove closed graph theorem.

(ii) Let M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exist a functional f_0 in N^* such that $f_0(M) \neq 0$ and $f_0(x_0) \neq 0$.

(iii) Show that a closed convex subset C of Hilbert space H contains a unique vector of smallest norm.

(iv) Let y be a fixed vector in Hilbert space H and let f_y be scalar defined by

$$f_y(x) = \langle x, y \rangle \forall x \in H$$

then show that $f_y \in B(H, K)$

i.e. $f_y \in H$

Further show that $\| y \| = \| f_y \|$