

AJ-1543

M.A./M.Sc. (Previous) Term End Examination, 2021-22

MATHEMATICS (Paper-VI)

(Differential Geometry of Manifolds)

Time : 3 hours]

[Maximum Marks : 100

Note : Answer any five questions. All questions carry equal marks.

1. Define a differentiable manifold. Show that the torus T^2 is a 2-dimensional differentiable manifold. Also show that the General linear group $GL(n, R)$ is a differentiable manifold of dimension n^2 .
2. Define Lie bracket of vector fields. State and prove Jacobi identity of Lie bracket.
3. Define cotangent space at a point p of a manifold M . Prove that $\dim T_p^* M = n$, where n is the dimension of M .
4. Prove that the exterior product satisfies the following properties :
 - (a) $w \wedge (\eta \wedge \pi) = (w \wedge \eta) \wedge \pi$
for $w \in \wedge^k(\mathcal{V}^*)$, $\eta \in \wedge^l(\mathcal{V}^*)$, $\pi \in \wedge^m(\mathcal{V}^*)$
 - (b) $w \wedge (\eta + \pi) = w \wedge \eta + w \wedge \pi$
for $w \in \wedge^k(\mathcal{V}^*)$, $\eta, \pi \in \wedge^l(\mathcal{V}^*)$
 - (c) $(\alpha, w) \wedge \eta = \alpha(w \wedge \eta)$
for $\alpha \in F$, $w \in \wedge^k(\mathcal{V}^*)$, $\eta \in \wedge^l(\mathcal{V}^*)$
 - (d) $w \wedge \eta = (-1)^{kl} \eta \wedge w$
for $w \in \wedge^k(\mathcal{V}^*)$, $\eta \in \wedge^l(\mathcal{V}^*)$.
5. If w is a 1-form on a smooth manifold M of dimension n , then show that
$$dw(X, Y) = \frac{1}{2} (Xw(Y) - Yw(X) - w([X, Y]))$$
for any $X, Y \in \chi(M)$; dw is a 2-form on M .
6. If a Lie group G has dimension n , then prove that its Lie algebra is also of dimension n .
7. State and prove two Bianchi identities for a Levi-Civita connection on a Riemannian manifold.
8. Prove that the curvature tensor of a Riemannian manifold (M, g) at a point p is uniquely determined by the sectional curvatures of all 2-dimensional tangent subspaces of $T_p M$ at p .
9. State and prove Gauss-Codazzi equation. Also derive Ricci equation.
10. (a) Show that every almost complex manifold is of even dimension.
(b) Let D' be a general F -connection with T' as its torsion tensor. Then prove that :

$$N(X, Y) = -T'(\bar{X}, \bar{Y}) + T'(X, Y) + T'(\bar{X}, \bar{Y}) + T'(\bar{X}, \bar{Y})$$