

Measure of an Angle :

(i) $1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ 16' 22''$ (Approximately)

(ii) $1^\circ = \frac{\pi}{180} \text{ radian}$, $1^\circ = 60' \text{ (min)}$, $1' = 60'' \text{ (sec)}$

(iii) The length of an arc of a circle with radius r .

(i) $l = \frac{\pi r \alpha}{180}$, where r = radius of the circle

 α = The measure of the angle in degrees subtended by the arc at the centre of the circle.(iii) $l = r\theta$, where θ is the measure of the angle in radians subtended by the arc at the centre of circle.

(iv) Angle made by an hour hand (x) and minute hand (y) in degree is $\alpha = \frac{|60x - 11y|}{2}$

(v) The sum of the internal angles of n sided polygon is $(n - 2)\pi$.**(1) Domain, range, zeros and principal period of trigonometric functions :**

Functions	Domain	Range	Set of zeros	Principle period
$\sin$	$\mathbb{R}$	$[-1, 1]$	$\{k\pi k \in \mathbb{Z}\}$	2π
$\cos$	$\mathbb{R}$	$[-1, 1]$	$\{(2k + 1)\frac{\pi}{2} k \in \mathbb{Z}\}$	2π
$\tan$	$\mathbb{R} - \{(2k + 1)\frac{\pi}{2} k \in \mathbb{Z}\}$	$\mathbb{R}$	$\{k\pi k \in \mathbb{Z}\}$	π
$\cot$	$\mathbb{R} - \{k\pi k \in \mathbb{Z}\}$	$\mathbb{R}$	$\{(2k + 1)\frac{\pi}{2} k \in \mathbb{Z}\}$	π
$\sec$	$\mathbb{R} - \{(2k + 1)\frac{\pi}{2} k \in \mathbb{Z}\}$	$\mathbb{R} - (-1, 1)$	$\emptyset$	2π
$\csc$	$\mathbb{R} - \{k\pi k \in \mathbb{Z}\}$	$\mathbb{R} - (-1, 1)$	$\emptyset$	2π

(2) Increasing and decreasing functions :

Function/Quadrant	I	II	III	IV
$\cos$	↓	↓	↑	↑
$\sin$	↑	↓	↓	↑
$\tan$	↑	↑	↑	↑
$\cot$	↓	↓	↓	↓
$\sec$	↑	↑	↓	↓
$\csc$	↓	↑	↑	↓

Here,

↑ indicates increasing and

↓ indicates decreasing functions.

(3) Signs of trigonometric functions in different quadrants :

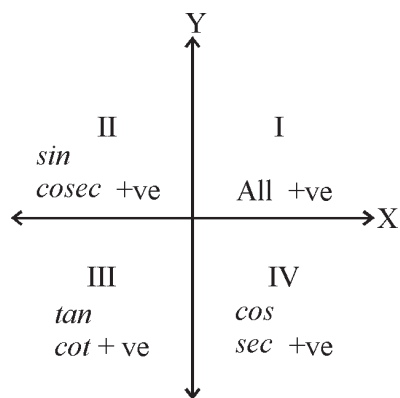


Figure 12.1

(4) Values of trigonometric functions :

θ Function	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	2π
$\sin$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	-1	0
$\cos$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	0	1
$\tan$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined	$-\sqrt{3}$	0	Undefined	0

(5) Values of trigonometric functions for special measures of angles :

Trigo. functions $\rightarrow$ measures of Angles $\downarrow$	$\sin$	$\cos$	$\tan$	$\cot$
$15^\circ = \frac{\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2 - \sqrt{3}$	$2 + \sqrt{3}$
$75^\circ = \frac{5\pi}{12}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$2 + \sqrt{3}$	$2 - \sqrt{3}$
$18^\circ = \frac{\pi}{10}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{5+2\sqrt{5}}$
$72^\circ = \frac{2\pi}{5}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\sqrt{5+2\sqrt{5}}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$
$36^\circ = \frac{\pi}{5}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\sqrt{5-2\sqrt{5}}$	$\frac{\sqrt{25+10\sqrt{5}}}{5}$
$54^\circ = \frac{3\pi}{10}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{25+10\sqrt{5}}}{5}$	$\sqrt{5-2\sqrt{5}}$
$22\frac{1}{2}^\circ = \frac{\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\sqrt{2} - 1$	$\sqrt{2} + 1$
$67\frac{1}{2}^\circ = \frac{3\pi}{8}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\sqrt{2} + 1$	$\sqrt{2} - 1$

Addition formulae :

$$(1) \cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$(2) \cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$(3) \sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$(4) \sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

$$(5) \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$(6) \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

$$(7) \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$$

$$(8) \cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$$

$$(9) \cot(\alpha + \beta) = \frac{\cot\alpha \cot\beta - 1}{\cot\beta + \cot\alpha}$$

$$(10) \cot(\alpha - \beta) = \frac{\cot\alpha \cot\beta + 1}{\cot\beta - \cot\alpha}$$

$$(11) \sin(-\theta) = -\sin\theta$$

(Odd function)

$$\cos(-\theta) = \cos\theta$$

(Even function)**Note :** *cos* and *sec* are even functions and *sin*, *cosec*, *tan* and *cot* are odd functions.

$$(12) \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \text{ and } \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Factor formulae :

$$(1) 2\sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

$$(2) 2\cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

$$(3) 2\cos\alpha \cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

$$(4) 2\sin\alpha \sin\beta = -\cos(\alpha + \beta) + \cos(\alpha - \beta)$$

$$(5) \sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$(6) \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

$$(7) \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$(8) \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

Values of trigonometric functions for multiples and submultiples :

$$(1) \sin 2\theta = 2\sin\theta \cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$(2) \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$(3) \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$(4) \cos^2\theta = \frac{1 + \cos 2\theta}{2}, \sin^2\theta = \frac{1 - \cos 2\theta}{2}, \tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

$$(5) \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$(6) \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$(7) \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$(8) \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$(9) \cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$

Properties of a triangle :

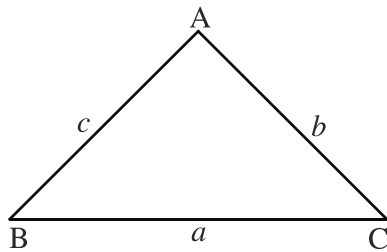


Figure 12.2

(1) **sine rule :**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

(2) **cosine rule :**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

(3) **Projection rule :**

$$a = b \cos C + c \cos B, b = a \cos C + c \cos A, c = a \cos B + b \cos A$$

(4) **Half angle formulae :**

$$(i) \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$(ii) \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$(iii) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

(5) **Area of a triangle :**

$$\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}$$

(6) Formulae for inradius :

(i) $r = \frac{\Delta}{s}$

$$\begin{array}{l} \text{(ii) } r = (s - a)\tan\frac{A}{2} \\ r = (s - b)\tan\frac{B}{2} \\ r = (s - c)\tan\frac{C}{2} \end{array} \quad \left| \begin{array}{l} \cot A = \frac{b^2 + c^2 - a^2}{4\Delta} \\ \cot B = \frac{a^2 + c^2 - b^2}{4\Delta} \\ \cot C = \frac{a^2 + b^2 - c^2}{4\Delta} \end{array} \right.$$

(iii) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Trigonometric equations :

(i) Solution of $\cos\theta = a$, $|a| \leq 1$

$$\theta = 2k\pi \pm \cos^{-1}a, \quad k \in \mathbb{Z}$$

(ii) Solution of $\sin\theta = a$, $|a| \leq 1$

$$\theta = k\pi + (-1)^k \sin^{-1}a, \quad k \in \mathbb{Z}$$

(iii) Solution of $\tan\theta = a$, $a \in \mathbb{R}$

$$\theta = k\pi + \tan^{-1}a, \quad k \in \mathbb{Z}$$

(iv) Condition for existence of solution of equation $a \cos\theta + b \sin\theta = c$, $\left| \frac{c}{r} \right| \leq 1$, where $r = \sqrt{a^2 + b^2}$

$$\theta = 2k\pi \pm \cos^{-1} \frac{c}{r} - \beta, \quad k \in \mathbb{Z}, \quad \text{where } \cos\beta = \frac{a}{r}, \sin\beta = \frac{b}{r}.$$

Note : If $c^2 > a^2 + b^2$ or $c^2 > r^2$, then the solution set is $\emptyset$.

Formulae for inverse trigonometric functions :

1. (i) $\sin^{-1}(-x) = -\sin^{-1}x$, $|x| \leq 1$
(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \leq 1$
(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in \mathbb{R}$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in \mathbb{R}$
(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$
(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, $|x| \geq 1$

2. (i) $\sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1}x$; $|x| \geq 1$

(ii) $\cos^{-1} \frac{1}{x} = \sec^{-1}x$, $|x| \geq 1$

(iii) $\tan^{-1} \frac{1}{x} = \begin{cases} \cot^{-1}x, & x > 0 \\ \cot^{-1}x - \pi, & x < 0 \end{cases}$

3. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $|x| \leq 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $x \in \mathbb{R}$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ $|x| \geq 1$

4. If $x > 0, y > 0$, then

$$(i) \quad \tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & xy > 1 \\ \frac{\pi}{2} & xy = 1 \end{cases}$$

$$(ii) \quad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$(iii) \quad \tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Note : (i) If $xy + yz + zx = 1$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$

(ii) If $x + y + z = xyz$, then $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

5. **Inter-relations :** $0 < x < 1$

$$(a) \quad \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$(b) \quad \cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$(c) \quad \tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

Special formulae :

1. **Formulae for summation of three angles :**

$$(1) \quad \sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

$$(2) \quad \cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C \\ = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan A \tan C)$$

$$(3) \quad \tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan A \tan C}$$

$$2. (i) \quad \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \sec\theta + \tan\theta$$

$$(ii) \quad \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sec\theta - \tan\theta$$

$$(iii) \quad \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \cot \frac{\theta}{2} = \operatorname{cosec}\theta + \cot\theta$$

$$(iv) \quad \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan \frac{\theta}{2} = \operatorname{cosec}\theta - \cot\theta$$

$$(v) \quad \cos\theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}, \quad n \in \mathbb{Z} \quad (\theta \neq n\pi)$$

$$(vi) \cos A + \cos(A + B) + \cos(A + 2B) + \dots + \cos(A + (n - 1)B)$$

$$= \frac{\sin \frac{nB}{2}}{\sin \frac{B}{2}} \cos \left\{ A + (n - 1) \frac{B}{2} \right\}$$

$$(vii) \tan \theta \tan \left(\frac{\pi}{3} - \theta \right) \tan \left(\frac{\pi}{3} + \theta \right) = \tan 3\theta$$

$$(viii) A + B + C = \pi \Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

: Multiple Choice Questions :

- (1) If for two regular polygons, the ratio of the number of sides is 5:4 and difference of the measures of internal angles in radians is $\frac{\pi}{30}$, then the number of sides of both the polygons are

(A) 15, 12 (B) 20, 16 (C) 10, 8 (D) 25, 20

Sol. : The sum of the internal angles of N sided polygon is $(N - 2)\pi$ and that of n sided polygon is $(n - 2)\pi$ ($N > n$).

$$\therefore \text{The measure of each internal angle of N sided polygon} = \frac{(N - 2)\pi}{N}$$

$$\text{and that of } n \text{ sided polygon} = \frac{(n - 2)\pi}{n}$$

$$\text{Now, difference between them} = \frac{\pi}{30}$$

$$\therefore \frac{(N - 2)\pi}{N} - \frac{(n - 2)\pi}{n} = \frac{\pi}{30}$$

$$\therefore \frac{2}{n} - \frac{2}{N} = \frac{1}{30}$$

$$\text{and } \frac{N}{n} = \frac{5}{4} \quad \text{So., } N = \frac{5}{4}n$$

$$\therefore \frac{2}{n} - \frac{2}{\left(\frac{5}{4}\right)n} = \frac{1}{30}$$

$$\therefore \frac{2}{n} - \frac{8}{5n} = \frac{1}{30}$$

$$\therefore \frac{2}{5n} = \frac{1}{30}$$

$$\therefore n = 12 \text{ and } N = \frac{5}{4}n = 15 \quad \text{Ans. : (A)}$$

- (2) The length of the side of an equilateral triangle is 4 cm. Its one of the vertex is centre of a circle. The arc of this circle divides the triangle in two congruent parts. Then the radius of circle is

(A) $\sqrt{12\sqrt{3}}$ cm (B) $\frac{\sqrt{12\sqrt{3}}}{\pi}$ cm (C) $\sqrt{\frac{\pi}{12\sqrt{3}}}$ cm (D) $\sqrt{\frac{12\sqrt{3}}{\pi}}$ cm

$$\text{Sol. : The area of the equilateral triangle} = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}$$

$$\therefore \frac{1}{2} \text{ (The area of the equilateral triangle)} = 2\sqrt{3}$$

Let the arc $\widehat{DE}$ of the circle drawn by taking A as centre and radius r divide equilateral $\triangle ABC$ in two congruent parts.

$$\text{Now, the area of sector ADE} = \frac{1}{2}r^2\theta = 2\sqrt{3}, \text{ where } \theta = \frac{\pi}{3}$$

$$\therefore r^2 = \frac{2\sqrt{3} \times 6}{\pi}$$

$$\therefore r = \sqrt{\frac{12\sqrt{3}}{\pi}} \text{ cm}$$

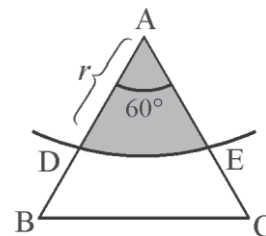


Figure 12.3

Ans. : (D)

- (3) In a quadrilateral, the measure of the angles are in A.P. If the ratio of the measures of the greatest and the smallest angles is 2:1, then the measure of the greatest angle in radians is

(A) $\frac{5\pi}{6}$ (B) $\frac{2\pi}{3}$ (C) $\frac{5\pi}{9}$ (D) None of these

Sol. : Let the measures of four angles of the quadrilateral be $a - 3d$, $a - d$, $a + d$, $a + 3d$.

$\therefore$ The sum of the measures of all angles is 2π .

$$a - 3d + a - d + a + d + a + 3d = 2\pi$$

$$\therefore 4a = 2\pi$$

$$\therefore a = \frac{\pi}{2}$$

$$\text{and } \frac{\text{the measure of the greatest angle}}{\text{the measure of the smallest angle}} = \frac{2}{1}$$

$$\therefore \frac{a + 3d}{a - 3d} = 2 \quad \left| \quad \therefore 9d = \frac{\pi}{2} \right.$$

$$\therefore \frac{\pi}{2} + 3d = 2\left(\frac{\pi}{2} - 3d\right) \quad \left| \quad \therefore d = \frac{\pi}{18} \right.$$

$$\therefore \text{The measure of the greatest angle} = a + 3d = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3} \quad \text{Ans. : (B)}$$

- (4) In a circle of radius r , the length of an arc equals closed perimeter of $\frac{1}{8}$ of the circle. The measure of the angle made by the arc at the centre of the circle is

(A) π (B) $\pi - 2$ (C) $\frac{\pi}{4} + 2$ (D) $\pi + 3$

Sol. : The length of the arc l of the circle with radius, r = closed perimeter of $\frac{1}{8}$ of the circle

$$= \frac{\pi r}{4} + 2r$$

$$= \left(\frac{\pi}{4} + 2\right)r$$

But, if the measure of the angle made by the arc at the centre is θ , in radians, then the length of the arc $l = r\theta$

$$\therefore r\theta = \left(\frac{\pi}{4} + 2\right)r$$

$$\therefore \theta = \left(\frac{\pi}{4} + 2\right) \text{ radians}$$

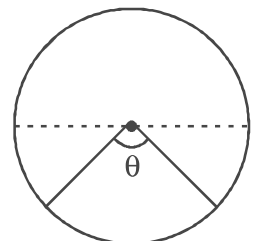


Figure 12.4

Ans. : (C)

- (5) If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

(A) $\tan^2 x = \frac{1}{5}$

(B) $\tan^2 x = \frac{1}{3}$

(C) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

Sol. : $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$

$$\therefore \frac{\sin^4 x}{2} + \frac{(1 - \sin^2 x)^2}{3} = \frac{1}{5}$$

$$\text{Let } \sin^2 x = m$$

$$\therefore \frac{m^2}{2} + \frac{(1-m)^2}{3} = \frac{1}{5}$$

$$\therefore \frac{3m^2 + 2(1-m)^2}{6} = \frac{1}{5}$$

$$\therefore 5[3m^2 + 2(1 - 2m + m^2)] = 6$$

$$\therefore 25m^2 - 20m + 4 = 0$$

$$\therefore (5m - 2)^2 = 0$$

$$\therefore m = \frac{2}{5} = \sin^2 x$$

$$\therefore \sin^2 x = \frac{2}{5} \Rightarrow \cos^2 x = \frac{3}{5}$$

$$\therefore \tan^2 x = \frac{2}{3}$$

$\therefore$ (A), (B) are not possible.

$$\text{Now, } \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{(\sin^2 x)^4}{8} + \frac{(\cos^2 x)^4}{27} = \frac{\left(\frac{2}{5}\right)^4}{8} + \frac{\left(\frac{3}{5}\right)^4}{27} = \frac{2+3}{5^4} = \frac{1}{125} \quad \text{Ans. : (C)}$$

(6) If $x = \sec\theta - \tan\theta$, $y = \operatorname{cosec}\theta + \cot\theta$, then $xy + 1 = \dots$

(A) $x + y$

(B) $x - y$

(C) $y - x$

(D) $-x - y$

Sol. : $xy + 1 = (\sec\theta - \tan\theta)(\operatorname{cosec}\theta + \cot\theta) + 1$

$$= \left(\frac{1 - \sin\theta}{\cos\theta}\right) \left(\frac{1 + \cos\theta}{\sin\theta}\right) + 1$$

$$= \frac{1 + \cos\theta - \sin\theta - \sin\theta\cos\theta + \sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{1 + \cos\theta - \sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + \cos\theta - \sin\theta}{\sin\theta\cos\theta}$$

$$= \tan\theta + \cot\theta + \operatorname{cosec}\theta - \sec\theta$$

$$= (\cot\theta + \operatorname{cosec}\theta) - (\sec\theta - \tan\theta)$$

$$= y - x$$

Ans. : (C)

(7) θ and ϕ are acute angles. If $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$, then $\theta + \phi \in \dots$

[IIT : 2004]

(A) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

(B) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(C) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right)$

(D) $\left(\frac{5\pi}{6}, \pi\right)$

Sol. : Both θ and ϕ are acute angles and

$$\sin\theta = \frac{1}{2}. \quad \text{So, } \theta = \frac{\pi}{6} \quad (1)$$

$\cos$ function is decreasing in the first quadrant,

$$\cos\phi = \frac{1}{3} < \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\text{Now, } \cos\phi = \frac{1}{3}$$

$$\text{So clearly, } \frac{\pi}{3} < \phi < \frac{\pi}{2}$$

$$\therefore \frac{\pi}{3} + \theta < \theta + \phi < \frac{\pi}{2} + \theta$$

$$\therefore \frac{\pi}{3} + \frac{\pi}{6} < \theta + \phi < \frac{\pi}{2} + \frac{\pi}{6} \quad (\text{From (1)})$$

$$\therefore \frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$$

$$\therefore \theta + \phi \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$$

Ans. : (B)

(8) If $\sin^2\theta + 3\cos\theta - 2 = 0$, then $\cos^3\theta + \sec^3\theta = \dots$, $\theta \neq (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

(A) 18

(B) 9

(C) 4

(D) $\frac{1}{4}$

Sol. : $\sin^2\theta + 3\cos\theta - 2 = 0$

$$\therefore 1 - \cos^2\theta + 3\cos\theta - 2 = 0$$

$$\therefore \cos^2\theta - 3\cos\theta + 1 = 0$$

$$\therefore \cos\theta - 3 + \sec\theta = 0 \quad (\cos\theta \neq 0)$$

$$\therefore \cos\theta + \sec\theta = 3$$

$$\therefore (\cos\theta + \sec\theta)^3 = \cos^3\theta + \sec^3\theta + 3\cos\theta \sec\theta(\cos\theta + \sec\theta)$$

$$\therefore 27 = \cos^3\theta + \sec^3\theta + 3(3)$$

$$\therefore \cos^3\theta + \sec^3\theta = 18$$

Ans. : (A)

(9) If $x = \sin^2\theta \cos\theta$ and $y = \cos^2\theta \sin\theta$, then which of the following is true ? $0 < \theta < \frac{\pi}{2}$, $x > y > 0$

(A) $(x^2y)^{\frac{2}{3}} + (xy^2)^{\frac{2}{3}} = 1$

(B) $\left(\frac{x^2}{y}\right)^{\frac{2}{3}} + \left(\frac{y^2}{x}\right)^{\frac{2}{3}} = 1$

(C) $x^2 + y^2 = x^2y^2$

(D) None of these

Sol. : Here, $\frac{x}{y} = \frac{\sin^2\theta \cos\theta}{\sin\theta \cos^2\theta} = \tan\theta$.

$$\therefore \frac{x^2}{y^2} + 1 = \sec^2\theta$$

$$\therefore \cos^2\theta = \frac{y^2}{x^2 + y^2}, \sin^2\theta = \frac{x^2}{x^2 + y^2}$$

$$\text{So, from } x = \sin^2\theta \cos\theta, x = \frac{x^2}{x^2 + y^2} \cdot \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore (x^2 + y^2)^{\frac{3}{2}} = xy. \quad \text{So, } x^2 + y^2 = (xy)^{\frac{2}{3}}$$

$$\therefore \frac{x^2}{(xy)^{\frac{2}{3}}} + \frac{y^2}{(xy)^{\frac{2}{3}}} = 1. \quad \text{So, } \frac{x^{\frac{4}{3}}}{y^{\frac{2}{3}}} + \frac{y^{\frac{4}{3}}}{x^{\frac{2}{3}}} = 1$$

$$\therefore \left(\frac{x^2}{y}\right)^{\frac{2}{3}} + \left(\frac{y^2}{x}\right)^{\frac{2}{3}} = 1$$

Ans. : (B)

(10) The minimum value of $\sin\theta \cos\theta$ is

(A) 1

(B) 0

(C) $-\frac{1}{2}$

(D) $\frac{1}{2}$

Sol. : $(\sin\theta + \cos\theta)^2 \geq 0$

$$\therefore \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \geq 0$$

$$\therefore 1 + 2\sin\theta\cos\theta \geq 0$$

$$\therefore \sin\theta\cos\theta \geq -\frac{1}{2}$$

The minimum value is $-\frac{1}{2}$.

Second method :

$$\sin\theta\cos\theta = \frac{1}{2} \cdot 2\sin\theta\cos\theta = \frac{1}{2} \cdot \sin 2\theta$$

Now, $\sin 2\theta \geq -1$. Hence, $\frac{1}{2}\sin 2\theta \geq -\frac{1}{2}$

The minimum value is $-\frac{1}{2}$.

Ans. : (C)

Note : For $\theta = \frac{3\pi}{4}$, $\sin\theta \cos\theta = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = -\frac{1}{2}$ does occur.

- (11) If $\sin x + \sin^2 x = 1$, then $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x + 2\cos^4 x + \cos^2 x - 2 = \dots$
 (A) 0 (B) 1 (C) 2 (D) $\sin^2 x$

Sol. : $\sin x + \sin^2 x = 1 \Rightarrow \sin x = 1 - \sin^2 x = \cos^2 x$

$$\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x + 2\cos^4 x + \cos^2 x - 2$$

$$= \sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x + 2\sin^2 x + \sin x - 2$$

$$= \left[(\sin^2 x)^3 + \binom{3}{1} (\sin^2 x)^2 (\sin x) + \binom{3}{2} (\sin^2 x) (\sin x)^2 + \binom{3}{3} (\sin x)^3 \right] + (\sin^2 x + \sin x) - 2 + \sin^2 x$$

$$= [\sin^2 x + \sin x]^3 + [\sin^2 x + \sin x] - 2 + \sin^2 x$$

$$= 1 + 1 - 2 + \sin^2 x$$

$$= \sin^2 x$$

Ans. : (D)

- (12) In which intervals out of the given intervals (p) and (q) values of $\frac{\sin 3\alpha}{\cos 2\alpha}$ is positive or negative ?

$$(p) \left(\frac{13\pi}{48}, \frac{14\pi}{48} \right) \quad (q) \left(\frac{18\pi}{48}, \frac{23\pi}{48} \right)$$

(A) (p) positive and (q) negative

(B) (p) negative (q) positive

(C) both positive

(D) both negative

Sol. : (p) Let $\frac{13\pi}{48} < \alpha < \frac{14\pi}{48}$

$$\therefore \frac{13\pi}{16} < 3\alpha < \frac{14\pi}{16}$$

$\therefore P(3\alpha)$ lies in the second quadrant.

$$\therefore \sin 3\alpha > 0$$

$$\text{Now, } \frac{13\pi}{48} < \alpha < \frac{14\pi}{48}$$

$$\frac{13\pi}{24} < 2\alpha < \frac{14\pi}{24}$$

$\therefore P(2\alpha)$ lies in the second quadrant.

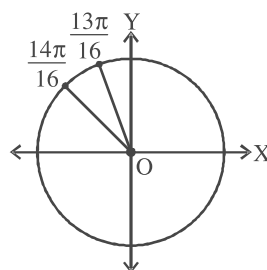


Figure 12.5

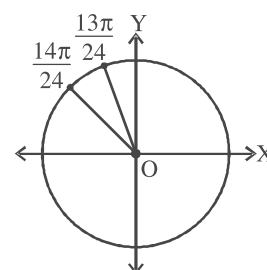


Figure 12.6

$$\cos 2\alpha < 0$$

$$\therefore \frac{\sin 3\alpha}{\cos 2\alpha} < 0$$

$\therefore$ Negative value for interval (p).

$$(q) \quad \frac{18\pi}{48} < \alpha < \frac{23\pi}{48}$$

$$\therefore \frac{18\pi}{16} < 3\alpha < \frac{23\pi}{16}$$

$\therefore P(3\alpha)$ lies in the third quadrant.

$$\therefore \sin 3\alpha < 0$$

$$\frac{18\pi}{48} < \alpha < \frac{23\pi}{48}$$

$$\therefore \frac{18\pi}{24} < 2\alpha < \frac{23\pi}{24}$$

$\therefore P(2\alpha)$ lies in the second quadrant.

$$\therefore \cos 2\alpha < 0$$

$$\therefore \frac{\sin 3\alpha}{\cos 2\alpha} > 0$$

$\therefore$ Positive value for interval (q).

(13) Which of the following value is the greatest of all four ?

(A) $\tan 1$

(B) $\tan 2$

(C) $\tan 3$

(D) $\tan 4$

Sol. : $\tan 2 < 0$, because $\frac{\pi}{2} < 2 < \pi$

$\tan 3 < 0$, because $\frac{\pi}{2} < 3 < \pi$

$\tan 4 = \tan(4 - \pi) > 0$ because $0 < 4 - \pi < \frac{\pi}{2}$. Also, $0 < 4 - \pi < 1 < \frac{\pi}{2}$

$\therefore \tan(4 - \pi) = \tan 4 < \tan 1$ ($\tan$ function is increasing in the first quadrant.)

$\therefore$ The value of $\tan 1$ is the greatest.

Ans. : (A)

(14) If $y = \frac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta}$, then $y \in$

(A) $\left(\frac{1}{3}, 3\right)$

(B) $\left(-3, -\frac{1}{3}\right)$

(C) $\left[\frac{1}{3}, 3\right]$

(D) $\left(\frac{1}{3}, 4\right)$

Sol. : Let $y = \frac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta}$

$$\therefore y = \frac{1 + \tan^2 \theta - \tan \theta}{1 + \tan^2 \theta + \tan \theta}$$

$$\therefore y + y \tan^2 \theta + y \tan \theta = 1 + \tan^2 \theta - \tan \theta$$

$$\therefore (y - 1) \tan^2 \theta + (y + 1) \tan \theta + (y - 1) = 0$$

For real roots of the quadratic equation in $\tan \theta$, $\Delta \geq 0$ is necessary.

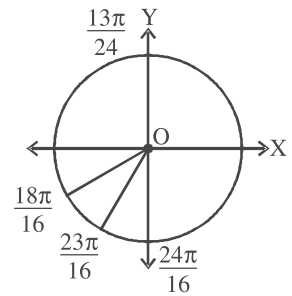


Figure 12.7

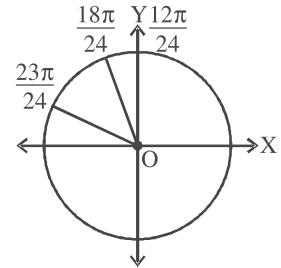


Figure 12.8

Ans. : (B)

$$\text{i.e. } (y + 1)^2 - 4(y - 1)^2 \geq 0$$

$$\text{i.e. } y^2 + 2y + 1 - 4(y^2 - 2y + 1) \geq 0$$

$$\text{i.e. } -3y^2 + 10y - 3 \geq 0$$

$$\text{i.e. } 3y^2 - 10y + 3 \leq 0$$

$$\text{i.e. } (3y - 1)(y - 3) \leq 0 \Rightarrow y \in \left[\frac{1}{3}, 3\right]$$

$$\therefore y \in \left[\frac{1}{3}, 3\right]$$

Ans. : (C)

(15) If $\cos\theta = x + \frac{1}{x}$ for some real $\theta \in \mathbb{R}$, $x \neq 0$, then which of following is true ?

(A) θ is acute. (B) θ is obtuse (C) $\theta = \frac{\pi}{2}$ (D) does not exist.

Sol. : $\cos\theta = x + \frac{1}{x}$

$$\therefore \cos^2\theta = x^2 + 2 + \frac{1}{x^2} > 2 \text{ is not possible. } \theta \text{ does not exist.}$$

Ans. : (D)

(16) Range of $\sqrt{\cos^2 x - 10\cos x + 25} + 2$ is

(A) [4, 6] (B) (6, 8) (C) [6, 8] (D) [8, 10]

Sol. : $\sqrt{\cos^2 x - 10\cos x + 25} + 2$

$$= \sqrt{(\cos x - 5)^2} + 2$$

$$= |\cos x - 5| + 2$$

$$-1 \leq \cos x \leq 1 \Leftrightarrow -6 \leq \cos x - 5 \leq -4$$

$$\Leftrightarrow 4 \leq |\cos x - 5| \leq 6$$

$$\Leftrightarrow 6 \leq |\cos x - 5| + 2 \leq 8$$

$$\Leftrightarrow 6 \leq f(x) \leq 8, \text{ where } f(x) = \sqrt{\cos^2 x - 10\cos x + 25} + 2$$

$\therefore$ Range is [6, 8]

Ans. : (C)

(17) If $\sum_{i=1}^3 \sin^2\theta_i = 0$, then which of the following cannot be the value of $\sum_{i=1}^3 \cos\theta_i$?

(A) 3 (B) -3 (C) -1 (D) -2

Sol. : $\sum_{i=1}^3 \sin^2\theta_i = 0$

$$\therefore \sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$$

$$\therefore \sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 0$$

$$\therefore \cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \pm 1 \quad (1)$$

$\cos\theta_1$	$\cos\theta_2$	$\cos\theta_3$	The sum
1	1	1	3
-1	-1	-1	-3
1	1	-1	1
-1	-1	1	-1

From the table, it is clear that the sum cannot be -2.

Ans. : (D)

(18) If $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true, then which of following is true ? ($x \neq -y$) [IIT : 1996]

- (A) $x + y = 0$ (B) $x = y, x \neq 0$ (C) $x = y$ (D) $x \neq y$

Sol. : $\sec^2\theta \geq 1$

$$\frac{4xy}{(x+y)^2} \geq 1 \Leftrightarrow 4xy \geq (x+y)^2 \quad ((x+y)^2 > 0)$$

$$\Leftrightarrow x^2 - 2xy + y^2 \leq 0$$

$$\Leftrightarrow (x-y)^2 \leq 0$$

Now, $(x-y)^2 < 0$ is not possible.

$$\therefore x - y = 0$$

$$\therefore x = y \text{ and } x \neq 0$$

Ans. : (B)

(19) $\sqrt{\frac{1-\sin A}{1+\sin A}} = \dots\dots$ $-\frac{\pi}{2} < A < \frac{\pi}{2}$

- (A) $\sec A + \tan A$ (B) $\sec A - \tan A$ (C) $-\sec A + \tan A$ (D) $-(\sec A + \tan A)$

Sol. : $\sqrt{\frac{1-\sin A}{1+\sin A}} \times \sqrt{\frac{1-\sin A}{1-\sin A}}$

$$= \frac{1-\sin A}{\sqrt{1-\sin^2 A}}$$

$$= \frac{1-\sin A}{|\cos A|}$$

$$= \frac{1-\sin A}{\cos A} \quad \left[\left(-\frac{\pi}{2} < A < \frac{\pi}{2} \right) \Rightarrow \cos A > 0 \right]$$

$$= \sec A - \tan A$$

Ans. : (B)

(20) If $A = \sin^8\theta + \cos^{14}\theta, \forall \theta \in \mathbb{R}$, then which of the following is true ?

- (A) $A \geq 1$ (B) $0 < A \leq 1$ (C) $\frac{1}{2} \leq A \leq \frac{3}{2}$ (D) None of these

Sol. : We know that,

$$0 \leq \sin^2 A \leq 1 \text{ and } 0 \leq \cos^2 A \leq 1$$

$$\therefore \sin^4 A \leq \sin^2 A$$

$$\therefore \sin^6 A \leq \sin^4 A \leq \sin^2 A$$

$$\therefore \sin^8 A \leq \sin^2 A$$

$$\cos^4 A \leq \cos^2 A$$

$$\therefore \cos^6 A \leq \cos^4 A \leq \cos^2 A$$

$$\therefore \cos^{14} A \leq \cos^2 A$$

$$0 < \sin^8 A + \cos^{14} A \leq \sin^2 A + \cos^2 A$$

($\sin^8 A + \cos^{14} A$ cannot be zero.)

$$\Rightarrow 0 < \sin^8 A + \cos^{14} A \leq 1$$

Ans. : (B)

Note : Value of $\sin^8 A + \cos^{14} A$ can be zero, only when both $\sin A$ and $\cos A$ are zero, which is not possible as $\sin^2 A + \cos^2 A = 1$.

$$(21) \quad \cos\theta - \sin\theta = \frac{1}{5}, \quad 0 < \theta < \frac{\pi}{2}$$

I		II	
(i)	$\frac{(\cos\theta + \sin\theta)}{2}$	(a)	$\frac{4}{5}$
(ii)	$\cos\theta$	(b)	$\frac{7}{10}$
(c)		(c)	$\frac{24}{25}$
(d)		(d)	$\frac{7}{25}$
(A)	(i) (a) (ii) (c)	(B)	(i) (b) (ii) (d)
(C)	(i) (a) (ii) (d)	(D)	(i) (b) (ii) (a)

Sol. : $\cos\theta = \frac{1}{5} + \sin\theta$

$$\therefore 5\cos\theta = 1 + 5\sin\theta$$

$$\therefore 25\cos^2\theta = 1 + 10\sin\theta + 25\sin^2\theta$$

$$\therefore 25(1 - \sin^2\theta) = 1 + 10\sin\theta + 25\sin^2\theta$$

$$\therefore 50\sin^2\theta + 10\sin\theta - 24 = 0$$

$$\therefore 25\sin^2\theta + 5\sin\theta - 12 = 0$$

$$\therefore (5\sin\theta - 3)(5\sin\theta + 4) = 0$$

$$\therefore \sin\theta = \frac{3}{5} \text{ because } 0 < \theta < \frac{\pi}{2}$$

$$\cos\theta = \frac{4}{5}$$

$$\therefore \frac{\sin\theta + \cos\theta}{2} = \frac{7}{10}$$

$$\therefore (i) \rightarrow (b) \quad (ii) \rightarrow (a)$$

Ans. : (D)

(22) The ratio of the maximum and the minimum values of $2 - \cos x + \sin^2 x$ is

- (A) $\frac{7}{4}$ (B) $\frac{11}{4}$ (C) $\frac{13}{4}$ (D) None of these

Sol. : Let $f(x) = 2 - \cos x + \sin^2 x$
 $= 2 - \cos x + 1 - \cos^2 x$
 $= 3 - \cos x - \cos^2 x$
 $= 3 + \frac{1}{4} - \frac{1}{4} - \cos x - \cos^2 x$
 $= \frac{13}{4} - \left(\cos x + \frac{1}{2}\right)^2$

To find the maximum value of $f(x)$, value of $\left(\cos x + \frac{1}{2}\right)^2$ should be minimum.

$$\therefore \left(\cos x + \frac{1}{2}\right) \text{ should be } 0. \text{ So, maximum value of } f(x) \text{ is } \frac{13}{4} \text{ for } \cos x = -\frac{1}{2}.$$

$$f(x)_{\max} = \frac{13}{4}$$

To find minimum value of $f(x)$, value of $\left(\cos x + \frac{1}{2}\right)^2$ should be maximum, which is possible for $\cos x = 1$.

$$\therefore f(x) = \frac{13}{4} - \left(1 + \frac{1}{2}\right)^2 = \frac{13}{4} - \frac{9}{4} = 1$$

$$\therefore f(x)_{\min} = 1$$

$$\therefore \frac{f(x)_{\max}}{f(x)_{\min}} = \frac{13}{4}$$

Ans. : (C)

- (23) The number of ordered pairs (a, x) satisfying the equation $\sec^2(a + 2)x + a^2 - 1 = 0$, $-\pi < x < \pi$ is

(A) 2 (B) 1 (C) 3 (D) infinite

Sol. : $\sec^2(a + 2)x + a^2 - 1 = 0$

$$\therefore \sec^2(a + 2)x = 1 - a^2$$

$$\therefore \text{This is possible for } a = 0 \text{ only. } (|\sec x| \geq 1) \quad \text{So, } a = 0$$

$$a = 0 \Rightarrow \sec^2 2x = 1 \Rightarrow \sec 2x = \pm 1$$

$$-\pi < x < \pi. \Rightarrow -2\pi < 2x < 2\pi$$

$$\sec 2x = 1 \Rightarrow x = 0 \text{ and } \sec 2x = -1 \Rightarrow 2x = \pi, -\pi \Rightarrow x = \frac{\pi}{2}, \frac{-\pi}{2}$$

$$\therefore \text{The number of solutions} = 3 \text{ namely } (0, 0) \left(0, \pm \frac{\pi}{2}\right). \quad \text{Ans. : (C)}$$

- (24) If the value of $\tan x - \tan^2 x = 1$, then $\tan^4 x - 2\tan^3 x - \tan^2 x + 2\tan x + 1$ is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\tan^4 x - 2\tan^3 x + \tan^2 x - 2\tan^2 x + 2\tan x + 1$

$$= (\tan^2 x - \tan x)^2 - 2(\tan^2 x - \tan x) + 1 = (-1)^2 - 2(-1) + 1 = 1 + 2 + 1 = 4 \quad \text{Ans. : (D)}$$

- (25) $2\cos^2 \frac{x}{2} \sin^2 x = x^2 + x^{-2}$, $0 < x \leq \frac{\pi}{2}$ has [IIT : 1984]

(A) not a single real solution (B) only one real solution

(C) more than one real solution (D) none of these

Sol. : Since $\left(x - \frac{1}{x}\right)^2 \geq 0$,

$$\therefore x^2 - 2 + \frac{1}{x^2} \geq 0$$

$$\therefore x^2 + \frac{1}{x^2} \geq 2$$

$$\therefore 2\cos^2 \frac{x}{2} \sin^2 x \geq 2$$

$$\therefore \cos^2 \frac{x}{2} \sin^2 x \geq 1$$

$$\cos^2 \frac{x}{2} \sin^2 x > 1 \text{ is not possible.}$$

$$\text{Also, } \cos^2 \frac{x}{2} \sin^2 x = 1 \text{ is not possible, as } \cos^2 \frac{x}{2} \leq 1, \sin^2 x \leq 1$$

$$\therefore \sin^2 x = \cos^2 \frac{x}{2} = 1 \text{ may be possible.}$$

$$\text{But if } \cos^2 \frac{x}{2} = 1, \text{ then } \sin^2 x = 4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2} = 4(0)(1) = 0.$$

$$\therefore \text{The number of real solutions} = 0.$$

Ans. : (A)

- (26) If $\tan \alpha = \frac{m}{m+1}$ and $\tan \beta = \frac{1}{2m+1}$, then a possible value of $(\alpha + \beta)$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Sol. : $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

$$= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \left(\frac{m}{m+1}\right)\left(\frac{1}{2m+1}\right)}$$

$$= \frac{m(2m+1) + (m+1)}{(m+1)(2m+1) - m} = \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1$$

$\therefore \alpha + \beta = \frac{\pi}{4}$ is a possible value.

Ans. : (B)

(27) $\frac{2(\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \dots + \sin 89^\circ)}{2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1} = \dots$

(A) $\sqrt{2}$

(B) $\frac{1}{\sqrt{2}}$

(C) $\frac{1}{2}$

(D) 1

Sol. : $2(\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \dots + \sin 89^\circ)$

$$= 2[(\sin 1^\circ + \sin 89^\circ) + (\sin 2^\circ + \sin 88^\circ) + \dots + (\sin 44^\circ + \sin 46^\circ) + \sin 45^\circ]$$

$$= 2\left[2\sin 45^\circ \cos 44^\circ + 2\sin 45^\circ \cos 43^\circ + \dots + 2\sin 45^\circ \cos 1^\circ + \frac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{2}(\cos 44^\circ + \cos 43^\circ + \dots + \cos 1^\circ) + \sqrt{2} \quad (1)$$

Now, $\frac{2(\sin 1^\circ + \sin 2^\circ + \dots + \sin 89^\circ)}{2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1}$

$$= \frac{[2\sqrt{2}(\cos 44^\circ + \cos 43^\circ + \dots + \cos 1^\circ) + \sqrt{2}]}{2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1} \quad (\text{From (1)})$$

$$= \frac{\sqrt{2}[2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1]}{2(\cos 1^\circ + \cos 2^\circ + \dots + \cos 44^\circ) + 1}$$

$$= \sqrt{2}$$

Ans. : (A)

(28) If $\sin(y + z - x)$, $\sin(z + x - y)$, $\sin(x + y - z)$ are in A.P., then $\tan x$, $\tan y$, $\tan z$ are in

(A) A.P.

(B) G.P.

(C) H.P.

(D) None of these

Sol. : $\sin(y + z - x)$, $\sin(z + x - y)$, $\sin(x + y - z)$ are in A.P.

$$\therefore \sin(y + z - x) - \sin(z + x - y) = \sin(z + x - y) - \sin(x + y - z)$$

$$\therefore 2\cos z \sin(y - x) = 2\cos x \sin(z - y)$$

$$\therefore \cos z[\sin y \cos x - \cos y \sin x] = \cos x[\sin z \cos y - \cos z \sin y]$$

$$\therefore \cos z \sin y \cos x - \cos z \cos y \sin x = \cos x \sin z \cos y - \cos x \cos z \sin y$$

$$\therefore \tan y - \tan x = \tan z - \tan y \quad (\cos x \cos y \cos z \neq 0)$$

$$\therefore 2\tan y = \tan x + \tan z$$

$$\therefore \tan x, \tan y, \tan z \text{ are in A.P.}$$

Ans. : (A)

(29) If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$; $0 < \alpha, \beta < \frac{\pi}{4}$, then $\tan 2\alpha = \dots$

(A) $\frac{33}{56}$

(B) $\frac{56}{33}$

(C) $\frac{55}{32}$

(D) $\frac{32}{55}$

Sol. : $0 < \alpha < \frac{\pi}{4}$

$0 < \beta < \frac{\pi}{4}$

$0 < \alpha + \beta < \frac{\pi}{2}$

$0 < \alpha < \frac{\pi}{4}$

$-\frac{\pi}{4} < -\beta < 0$

$-\frac{\pi}{4} < \alpha - \beta < \frac{\pi}{4}$ and $\sin(\alpha - \beta) = \frac{5}{13} > 0$

$\therefore P(\alpha - \beta)$ lies in the first quadrant.

$\therefore \tan(\alpha + \beta) = \frac{3}{4}$ and $\tan(\alpha - \beta) = \frac{5}{12}$

$2\alpha = (\alpha + \beta) + (\alpha - \beta)$

$\therefore \tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \tan(\alpha - \beta)}$

$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4}\right)\left(\frac{5}{12}\right)} = \frac{56}{33}$

Ans. : (B)

(30) If $\sin^{-1}x \leq \cos^{-1}x$, then $x \in \dots$

(A) $\left(-1, \frac{1}{\sqrt{2}}\right)$

(B) $\left[-1, \frac{1}{\sqrt{2}}\right]$

(C) $\left[\frac{1}{\sqrt{2}}, 1\right]$

(D) None of these

Sol. : $\sin^{-1}x \leq \cos^{-1}x$

$\sin^{-1}x \leq \frac{\pi}{2} - \sin^{-1}x$

$\therefore 2\sin^{-1}x \leq \frac{\pi}{2}$

$\therefore \sin^{-1}x \leq \frac{\pi}{4}$

Also, $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{4}$

$\therefore \sin\left(\frac{-\pi}{2}\right) \leq x \leq \sin\frac{\pi}{4}$

$\therefore -1 \leq x \leq \frac{1}{\sqrt{2}}$

Ans. : (B)

(31) The solution of $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ is

(A) $\left[\frac{1}{\sqrt{2}}, 1\right]$

(B) $\left[\frac{-1}{\sqrt{2}}, 1\right]$

(C) $\left[-1, \frac{-1}{\sqrt{2}}\right]$

(D) $\left(\frac{1}{\sqrt{2}}, 1\right)$

Sol. : $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

Let, $\cos^{-1}x = \theta$, $\theta \in [0, \pi]$

$\therefore x = \cos\theta$

$\therefore 2\theta = \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$$\therefore 2\theta = \sin^{-1}(2\cos\theta\sin\theta) \quad (\sin\theta \geq 0 \text{ because } 0 \leq \theta \leq \pi)$$

$$\therefore 2\theta = \sin^{-1}(\sin 2\theta)$$

$$\text{Now, } -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$$

$$\therefore -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \text{ and } \theta \in [0, \pi] \text{ and so, } \theta \in \left[0, \frac{\pi}{4}\right].$$

Now, $\cos$ function is decreasing in the first quadrant.

$$0 \leq \theta \leq \frac{\pi}{4} \Rightarrow \cos 0 \geq \cos \theta \geq \cos \frac{\pi}{4}$$

$$\therefore \frac{1}{\sqrt{2}} \leq \cos \theta \leq 1$$

$$\therefore x \in \left[\frac{1}{\sqrt{2}}, 1\right]$$

Ans. : (A)

$$(32) \quad f(x) = \sin^{-1}\left(\frac{1+x^2}{2|x|}\right), \text{ is possible for } x \in \dots\dots \text{ (largest domain).}$$

$$(A) \{-1, 1\}$$

$$(B) [-1, 1]$$

$$(C) [0, 1]$$

$$(D) [-1, 0]$$

Sol. : We know that, $(1 \pm x)^2 \geq 0$

$$\therefore 1 \pm 2x + x^2 \geq 0$$

$$\therefore 1 + x^2 \geq 2|x|$$

$$\therefore \frac{1+x^2}{2|x|} \geq 1$$

$$f(x) = \sin^{-1}\left(\frac{1+x^2}{2|x|}\right) \text{ is not possible, if } \frac{1+x^2}{2|x|} > 1.$$

$$\therefore \frac{1+x^2}{2|x|} = 1$$

$$\therefore x = \pm 1$$

$$\therefore x \in \{-1, 1\}$$

Ans. : (A)

$$(33) \quad \sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right) = \dots\dots$$

$$(A) 1$$

$$(B) 0$$

$$(C) -1$$

$$(D) 2$$

$$\text{Sol. : } \sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{3}}{2} + \sec^{-1}\sqrt{2}\right)\right)$$

$$= \sin^{-1}\left(\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos^{-1}\frac{\sqrt{3}}{2} + \sec^{-1}\sqrt{2}\right)\right) \quad \left(\sqrt{\frac{2-\sqrt{3}}{4}} = \sqrt{\frac{4-2\sqrt{3}}{8}} = \frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\cot\left(\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right)\right) = \sin^{-1}\left(\cot\frac{\pi}{2}\right) = \sin^{-1}(0) = 0$$

Ans. : (B)

- (34) If $f(x) = x^{11} + x^9 + x^7 + x^3 + 1$ and $f(\sin^{-1}(\sin 8)) = \alpha$, α is a constant, then $f(\tan^{-1}(\tan 8)) = \dots$
- (A) α (B) $\alpha - 2$ (C) $\alpha + 2$ (D) $2 - \alpha$

Sol. : $\frac{5\pi}{2} < 8 < 3\pi$

$$\therefore -\frac{\pi}{2} < 8 - 3\pi < 0$$

$$\sin(8 - 3\pi) = -\sin 8$$

$$\begin{aligned}\therefore \alpha &= f(\sin^{-1}(\sin 8)) \\ &= f(\sin^{-1}(-\sin(8 - 3\pi))) \\ &= f(-\sin^{-1}(\sin(8 - 3\pi))) \\ &= f(-8 + 3\pi)\end{aligned}\quad (1)$$

Now, $\tan(8 - 3\pi) = \tan 8$

$$\begin{aligned}f(\tan^{-1}(\tan 8)) &= f(\tan^{-1}(\tan(8 - 3\pi))) \\ &= f(8 - 3\pi)\end{aligned}\quad (2)$$

$$\begin{array}{l}f(x) = x^{11} + x^9 + x^7 + x^3 + 1 \\ f(-x) = -x^{11} - x^9 - x^7 - x^3 + 1 \\ \hline f(x) + f(-x) = 2\end{array}$$

$$\therefore f(-8 + 3\pi) + f(8 - 3\pi) = 2$$

$$\therefore \alpha + f(8 - 3\pi) = 2 \quad (\text{From (1)})$$

$$\therefore f(8 - 3\pi) = 2 - \alpha$$

$$\therefore f(\tan^{-1}(\tan 8)) = 2 - \alpha \quad (\text{From (2)})$$

Ans. : (D)

- (35) The solution of $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is [IIT : 1989]

(A) $n\pi + \frac{\pi}{8}, n \in \mathbb{Z}$

(B) $\frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbb{Z}$

(C) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbb{Z}$

(D) $2n\pi + \cos^{-1} \frac{2}{3}, n \in \mathbb{Z}$

Sol. : $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$$\therefore (\sin 3x + \sin x) - 3\sin 2x = (\cos 3x + \cos x) - 3\cos 2x$$

$$\therefore 2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \cos x - 3\cos 2x$$

$$\therefore \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\therefore (\sin 2x - \cos 2x)(2\cos x - 3) = 0. \quad \text{So, } \sin 2x = \cos 2x \text{ or } \cos x = \frac{3}{2}$$

But, $\cos x = \frac{3}{2}$ is not possible.

$$\therefore \sin 2x - \cos 2x = 0$$

$$\therefore \tan 2x = 1 \quad (\text{If } \cos 2x = 0, \text{ then } \sin 2x = 0 \text{ which is not possible. So, } \cos 2x \neq 0)$$

$$\therefore \tan 2x = \tan \frac{\pi}{4}$$

$$\therefore 2x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\therefore x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbb{Z}$$

Ans. : (B)

- (36) The number of real roots of $\tan x + \sec x = 2\cos x$, $x \in [0, 2\pi]$ is/are

(A) 0

(B) 1

(C) 2

(D) 3

Sol. : $\tan x + \sec x = 2\cos x$

$$\therefore \frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x$$

$$\therefore \sin x + 1 = 2\cos^2 x$$

$$\therefore \sin x + 1 = 2(1 - \sin^2 x)$$

$$\begin{aligned}\therefore 2\sin^2x + \sin x - 1 &= 0 \\ \therefore (2\sin x - 1)(\sin x + 1) &= 0 \\ \therefore \sin x &= \frac{1}{2} \text{ or } \sin x = -1\end{aligned}$$

As $x \in [0, 2\pi]$,

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

But $\sin x = -1 \Rightarrow \cos x = 0$ is not possible.

($\tan x$ is well defined.)

$\therefore \sin x = -1$ is not possible.

$\therefore$ The number of solutions is 2.

Ans. : (C)

(37) The general solution of $2\sin^2\theta - 3\sin\theta - 2 = 0$ is

(A) $n\pi + (-1)^n \frac{\pi}{6}; n \in \mathbb{Z}$

(B) $n\pi + (-1)^n \frac{\pi}{2}; n \in \mathbb{Z}$

(C) $n\pi + (-1)^n \frac{5\pi}{6}; n \in \mathbb{Z}$

(D) $n\pi + (-1)^n \frac{7\pi}{6}; n \in \mathbb{Z}$

Sol. : $2\sin^2\theta - 3\sin\theta - 2 = 0$

$$\therefore (2\sin\theta + 1)(\sin\theta - 2) = 0$$

$$\therefore \sin\theta = -\frac{1}{2} \text{ or } \sin\theta = 2$$

The solution set of $\sin\theta = 2 > 1$ is $\emptyset$.

$$\left. \begin{aligned} \sin\theta &= -\frac{1}{2} \Rightarrow \sin\theta = \sin\left(\frac{7\pi}{6}\right) \\ \Rightarrow \theta &= n\pi + (-1)^n \left(\frac{7\pi}{6}\right), n \in \mathbb{Z} \end{aligned} \right\}$$

Ans. : (D)

(38) The number of solutions in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for the equation $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ is [IIT : 2001]

(A) 0

(B) 2

(C) 1

(D) 3

Sol. : $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} \sin x - \cos x & 0 & \cos x \\ \cos x - \sin x & \sin x - \cos x & \cos x \\ 0 & \cos x - \sin x & \sin x \end{vmatrix} = 0$$

$$C_1 - C_2, C_2 - C_3$$

$$\therefore (\sin x - \cos x)^2 \begin{vmatrix} 1 & 0 & \cos x \\ -1 & 1 & \cos x \\ 0 & -1 & \sin x \end{vmatrix} = 0$$

$$\therefore (\sin x - \cos x)^2 [1(\sin x + \cos x) + \cos x(1 - 0)] = 0$$

$$\therefore (\sin x - \cos x)^2 (\sin x + 2\cos x) = 0$$

$$\therefore \sin x = \cos x \text{ or } \sin x = -2\cos x$$

$$\therefore \tan x = 1 \text{ or } \tan x = -2$$

Here, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Hence, $\tan x \in [-1, 1]$

$\therefore \tan x = -2$ is not possible.

$$\therefore \tan x = 1 \Rightarrow x = \frac{\pi}{4}.$$

So, number of solution is 1.

Ans. : (C)

- (39) The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is [IIT : 2002]

(A) 4 (B) 8 (C) 10 (D) 12

Sol. : $7\cos x + 5\sin x = 2k + 1$

$$a = 7, b = 5$$

$$\therefore r^2 = a^2 + b^2 = 49 + 25 = 74. \text{ So, } r = \sqrt{74}$$

$$\therefore -\sqrt{74} \leq 2k + 1 \leq \sqrt{74} \text{ is necessary.}$$

$$\therefore |2k + 1| \leq \sqrt{74}$$

$$\therefore \text{Possible integers } k = 0, 1, 2, 3, -1, -2, -3, -4$$

$$\therefore 8 \text{ integral values are possible.}$$

Ans. : (B)

- (40) $\cos(\alpha - \beta) = 1, \cos(\alpha + \beta) = \frac{1}{e}, \alpha, \beta \in [-\pi, \pi]$. The number of pairs (α, β) satisfying both the equations is [IIT : 2005]

(A) 0 (B) 1 (C) 2 (D) 4

Sol. : $-\pi \leq \alpha \leq \pi$

$$-\pi \leq \beta \leq \pi \Rightarrow \pi \geq -\beta \geq -\pi$$

$$\therefore -\pi \leq \alpha \leq \pi, -\pi \leq -\beta \leq \pi$$

$$\underline{-2\pi \leq \alpha - \beta \leq 2\pi}$$

$$\cos(\alpha - \beta) = 1 \text{ and } \alpha - \beta \in [-2\pi, 2\pi] \Rightarrow \alpha - \beta = 0, 2\pi, -2\pi$$

Three cases : (i) $\alpha - \beta = 0$ (ii) $\alpha = \beta + 2\pi$ (iii) $\alpha = \beta - 2\pi$.

If $\alpha = \beta + 2\pi$, then

$$-\pi \leq \beta \leq \pi \Rightarrow \pi \leq 2\pi + \beta = \alpha \leq 3\pi \text{ which is not possible as, } -\pi \leq \alpha \leq \pi$$

$$[\alpha = \pi, \beta = -\pi \text{ is possible, but } \cos(\alpha + \beta) = \frac{1}{e} \neq 1]$$

Similarly, $\alpha = \beta - 2\pi$ is also not possible.

$$\therefore \text{Let } \alpha = \beta,$$

$$\cos(\alpha + \beta) = \frac{1}{e} \Rightarrow \cos 2\alpha = \frac{1}{e} \text{ where } -\pi \leq \alpha \leq \pi \Rightarrow -2\pi \leq 2\alpha \leq 2\pi$$

$$\therefore -2\pi \leq 2\alpha \leq 2\pi \text{ and } \cos 2\alpha = \frac{1}{e}$$

Two solutions are possible in $[-2\pi, 0]$ and also in $[0, 2\pi]$.

$$\therefore \text{The number of solutions} = 4$$

Ans. : (D)

- (41) The value of $\theta \in (0, 2\pi)$ for which the value of $2\sin^2\theta - 5\sin\theta + 2 > 0$ is [IIT : 2006]

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ (C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{8}, \pi\right)$

Sol. : $2\sin^2\theta - 5\sin\theta + 2 > 0$

$$\therefore 2\left(\sin\theta - \frac{1}{2}\right)(\sin\theta - 2) > 0$$

$$\therefore \sin\theta < \frac{1}{2} \text{ or } \sin\theta > 2$$

$\sin\theta > 2$ is not possible.

$$\therefore \sin\theta < \frac{1}{2}$$

$\therefore$ The solution set is $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$. **Ans. : (A)**

- (42) The number of solutions in $[0, 2\pi]$ for the equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ is

(A) 0 (B) 1 (C) 2 (D) 4

Sol. : $2\sin^2\theta - \cos 2\theta = 0$

$$\therefore 2\sin^2\theta - (1 - 2\sin^2\theta) = 0$$

$$\therefore 4\sin^2\theta = 1$$

$$\therefore \sin\theta = \pm \frac{1}{2}$$

Now, $2\cos^2\theta - 3\sin\theta = 0$

Thus, the general solution is $\sin\theta = \frac{1}{2}$. $\theta \in [0, 2\pi] \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\therefore \text{The number of solutions} = 2$$

Ans. : (C)

- (43) The number of solutions for the equation $3\sin^2x - 7\sin x + 2 = 0$ in $[0, 6\pi]$ is

(A) 0 (B) 5 (C) 6 (D) 10

Sol. : $3\sin^2x - 7\sin x + 2 = 0$

$$\therefore (3\sin x - 1)(\sin x - 2) = 0$$

$$\therefore \sin x = \frac{1}{3} \text{ or } \sin x = 2. \text{ But } \sin x = 2 \text{ is not possible.}$$

Now, $\sin x = \frac{1}{3} > 0$, so, two solutions are possible in $[0, \pi]$, two in $[2\pi, 3\pi]$ and two in $[4\pi, 5\pi]$. So, The total number of solutions is 6.

Ans. : (C)

- (44) The smallest value of x for which $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$ is $x = \dots$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Sol. : Let $\log_{\cos x} \sin x = a$,

$$a + \frac{1}{a} = 2$$

$$\therefore a^2 - 2a + 1 = 0$$

$$\therefore a = 1$$

$$\therefore \log_{\cos x} \sin x = 1$$

$$\therefore \sin x = \cos x$$

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4} \text{ is the smallest positive value.}$$

Ans. : (C)

- (45) The number of ordered pairs (x, y) in which solution of $\sin x \cos y = 1$ is possible is
 $x, y \in [0, 2\pi]$

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\sin x \cos y = 1$ is true for following two options.

(i) $\sin x = \cos y = 1$ or (ii) $\sin x = \cos y = -1$

(i) If $\sin x = \cos y = 1$ then,

$$(x, y) = \left(\frac{\pi}{2}, 0\right) \text{ or } \left(\frac{\pi}{2}, 2\pi\right) \quad (x, y \in [0, 2\pi])$$

(ii) If $\sin x = \cos y = -1$, then $(x, y) = \left(\frac{3\pi}{2}, \pi\right)$

$$\therefore \text{The number of solutions} = 3.$$

Ans. : (C)

- (46) If $\sin^3 x \sin 3x = \sum_{m=0}^n c_m \cos mx$, then $n = \dots$ where $c_0, c_1, \dots, c_n$ are constants, $c_n \neq 0$ [IIT : 1981]
 (A) 4 (B) 5 (C) 6 (D) 7

Sol. : $\sin^3 x \sin 3x = \sum_{m=0}^n c_m \cos mx$

$$\begin{aligned}\therefore \sum_{m=0}^n c_m \cos mx &= \frac{1}{4}(3\sin x - \sin 3x)\sin 3x \\ &= \frac{1}{4}[3\sin 3x \sin x - \sin^2 3x] \\ &= \frac{1}{8}[3(2\sin 3x \sin x) - 2\sin^2 3x] \\ &= \frac{1}{8}[3(-\cos 4x + \cos 2x) - (1 - \cos 6x)]\end{aligned}$$

$$c_0 + c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x + \dots + c_n \cos nx = \frac{1}{8}[-3\cos 4x + 3\cos 2x - 1 + \cos 6x]$$

$$\therefore n = 6$$

Ans. : (C)

- (47) $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14} = \dots$
 (A) $\frac{1}{32}$ (B) $\frac{1}{128}$ (C) $\frac{1}{16}$ (D) $\frac{1}{64}$

Sol. : $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$

$$= \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{\pi}{2} \sin \left(\pi - \frac{5\pi}{14}\right) \cdot \sin \left(\pi - \frac{3\pi}{14}\right) \cdot \sin \left(\pi - \frac{\pi}{14}\right)$$

$$= \left(\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}\right)^2$$

$$= \left(\cos \left(\frac{\pi}{2} - \frac{\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{3\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right)^2$$

$$= \left[\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7}\right]^2$$

$$= \left[\frac{1}{2\sin \frac{\pi}{7}} \left(2\sin \frac{\pi}{7} \cos \frac{\pi}{7}\right) \cos \frac{2\pi}{7} \cos \frac{3\pi}{7}\right]^2$$

$$= \left[\frac{1}{2\sin \frac{\pi}{7}} \left(\sin \frac{2\pi}{7} \cos \frac{2\pi}{7}\right) \cos \left(\pi - \frac{4\pi}{7}\right)\right]^2$$

$$= \left[\frac{1}{2^2 \sin \frac{\pi}{7}} \left(2\sin \frac{2\pi}{7} \cos \frac{2\pi}{7}\right) \left(-\cos \frac{4\pi}{7}\right)\right]^2$$

$$= \left[\frac{1}{2 \times 4 \sin \frac{\pi}{7}} \left(2\sin \frac{4\pi}{7} \cos \frac{4\pi}{7}\right)\right]^2$$

$$= \left[\frac{1}{8 \sin \frac{\pi}{7}} \left(\sin \frac{8\pi}{7}\right)\right]^2$$

$$= \frac{1}{64 \sin^2 \frac{\pi}{7}} \sin^2 \left(\pi + \frac{\pi}{7}\right)$$

$$= \frac{1}{64 \sin^2 \frac{\pi}{7}} \cdot \sin^2 \frac{\pi}{7}$$

$$= \frac{1}{64}$$

Ans. : (D)

- (48) The number of solutions of the equations $x + y = \frac{2\pi}{3}$ and $\cos x + \cos y = \frac{3}{2}$ is [IIT : 1987]
 (A) R (B) $\emptyset$ (C) $[-1, 1]$ (D) $\{-1, 1\}$

Sol. : $\cos x + \cos y = \frac{3}{2}$

$$\begin{aligned} \therefore 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \frac{3}{2} & \therefore 2\left(\frac{1}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \frac{3}{2} \\ \therefore 2\cos\left(\frac{2\pi}{3(2)}\right)\cos\left(\frac{x-y}{2}\right) &= \frac{3}{2} & \therefore \cos\left(\frac{x-y}{2}\right) &= \frac{3}{2} > 1 \text{ which is not possible.} \end{aligned}$$

- (49) $\sin\frac{\pi}{18} \sin\frac{5\pi}{18} \sin\frac{7\pi}{18} = \dots$ **Ans.** : (B)
 (A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) None of these

Sol. : $\sin\frac{\pi}{18} \sin\frac{5\pi}{18} \sin\frac{7\pi}{18} = \sin 10^\circ \sin 50^\circ \sin 70^\circ$

$$\begin{aligned} &= \frac{1}{2}(2\sin 70^\circ \sin 10^\circ)\sin 50^\circ & &= \frac{1}{4}[\cos 60^\circ - \cos 40^\circ] + \frac{1}{4}\sin 50^\circ \\ &= \frac{1}{2}(-\cos 80^\circ + \cos 60^\circ)\sin 50^\circ & &= \frac{1}{4}\left[\frac{1}{2} - \cos 40^\circ + \sin 50^\circ\right] \\ &= \frac{1}{2}\left(-\sin 10^\circ + \frac{1}{2}\right)\sin 50^\circ & &= \frac{1}{4}\left[\frac{1}{2} - \cos 40^\circ + \cos 40^\circ\right] \\ &= -\frac{1}{4}[2\sin 50^\circ \sin 10^\circ] + \frac{1}{4}\sin 50^\circ & &= \frac{1}{8} \end{aligned} \quad \text{Ans. : (C)}$$

Note : $= \sin 10^\circ \sin(60^\circ - 10^\circ) \sin(60^\circ + 10^\circ) = \frac{1}{4} \sin 3(10^\circ) = \frac{1}{8}$

- (50) If the $\cos(x - y)$, $\cos x$, $\cos(x + y)$ are in H.P., then $\cos x \cdot \sec \frac{y}{2} = \dots$ $\left(\sin \frac{y}{2} \neq 0\right)$
 (A) $\pm\sqrt{2}$ (B) ± 2 (C) $\pm\sqrt{3}$ (D) ± 4

Sol. : $\cos(x - y)$, $\cos x$, $\cos(x + y)$ are in H.P.

$\therefore \frac{1}{\cos(x - y)}, \frac{1}{\cos x}, \frac{1}{\cos(x + y)}$ are in A.P.

$\therefore \frac{1}{\cos(x - y)} + \frac{1}{\cos(x + y)} = \frac{2}{\cos x}$

$\therefore \frac{\cos(x + y) + \cos(x - y)}{\cos(x + y)\cos(x - y)} = \frac{2}{\cos x}$

$\therefore \frac{2\cos x \cos y}{\cos^2 x - \sin^2 y} = \frac{2}{\cos x}$

$\therefore \cos^2 x \cos y = \cos^2 x - \sin^2 y$

$\therefore \cos^2 x - \cos^2 x \cos y = \sin^2 y$

$\therefore \cos^2 x (1 - \cos y) = \sin^2 y$

$\therefore \cos^2 x \left(2\sin^2 \frac{y}{2}\right) = \left(2\sin \frac{y}{2} \cos \frac{y}{2}\right)^2$

$\therefore \cos^2 x \left(2\sin^2 \frac{y}{2}\right) = 4\sin^2 \frac{y}{2} \cos^2 \frac{y}{2}$

$\therefore \cos^2 x = 2\cos^2 \frac{y}{2}$

$\therefore \frac{\cos^2 x}{\cos^2 \frac{y}{2}} = 2$

$\therefore \cos^2 x \sec^2 \frac{y}{2} = 2$

$\therefore \cos x \sec \frac{y}{2} = \pm\sqrt{2}$

Ans. : (A)

(51) If $A > 0$, $B > 0$, $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{1}{\sqrt{3}}$

Sol. : Let $\tan A \tan B = y$

Now, $A + B = \frac{\pi}{3}$

$\therefore \tan(A + B) = \tan \frac{\pi}{3}$

$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$

$\therefore \tan A + \tan B = \sqrt{3}(1 - \tan A \tan B)$

$\therefore \frac{y}{\tan B} + \tan B = \sqrt{3}(1 - y)$

$\therefore y + \tan^2 B = \sqrt{3} \tan B(1 - y)$

$\therefore \tan^2 B - \sqrt{3} \tan B(1 - y) + y = 0$

This is a quadratic equation in $\tan B$.

For real solution, $\Delta \geq 0$.

$\therefore 3(1 - y)^2 - 4y \geq 0$

$\therefore 3(1 - 2y + y^2) - 4y \geq 0$

$\therefore 3y^2 - 10y + 3 \geq 0$

$\therefore (y - 3)(3y - 1) \geq 0$

$A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$

$\therefore 0 < A < \frac{\pi}{3}$ and $0 < B < \frac{\pi}{3}$

$\therefore 0 < \tan A < \sqrt{3}$ and $0 < \tan B < \sqrt{3}$

$\therefore 0 < \tan A \tan B < 3$

$\therefore \tan A \tan B \geq 3$ is not possible.

$\therefore \tan A \tan B \leq \frac{1}{3}$

$\therefore$ The maximum value of $\tan A \tan B$ is $\frac{1}{3}$.

We can observe that $\tan A = \tan B = \frac{1}{\sqrt{3}}$ is possible for $A = B = \frac{\pi}{6}$

$\therefore$ The maximum value for $A = B = \frac{\pi}{6}$ is $\frac{1}{3}$.

Ans. : (A)

(52) If $\tan A = \frac{(1 - \cos B)}{\sin B}$, then $\tan 2A = \dots\dots$

(A) $\tan B$

(B) $\sin B$

(C) $\tan 2B$

(D) $\tan 4B$

Sol. : $\tan A = \frac{(1 - \cos B)}{\sin B} = \frac{2 \sin^2 \frac{B}{2}}{2 \sin \frac{B}{2} \cos \frac{B}{2}} = \tan \frac{B}{2}$

$\therefore \tan A = \tan \frac{B}{2}$

$\therefore A = \frac{B}{2} + n\pi$

$\therefore 2A = B + 2n\pi$

$\therefore \tan 2A = \tan B$

Ans. : (A)

(53) The number of solutions of the equation $\sin^4 \theta - 2 \sin^2 \theta - 1 = 0$ in $[0, 2\pi]$ is

(A) 0

(B) 1

(C) 2

(D) 3

Sol. : $\sin^4 \theta - 2 \sin^2 \theta - 1 = 0$

$\therefore \sin^4 \theta - 2 \sin^2 \theta + 1 - 2 = 0$

$\therefore (1 - \sin^2 \theta)^2 = 2$

$\therefore \cos^4 \theta = 2$ which is not possible.

$\therefore$ The number of solutions is 0.

Ans. : (A)

- (54) If for some positive integer n , $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$, then possible values of n are

[IIT : 1994]

- (A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$

$$\therefore \left(\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} \right)^2 = \frac{n}{4}$$

$$\therefore 1 + \sin \frac{2\pi}{2n} = \frac{n}{4}$$

$$\therefore \sin \frac{\pi}{n} = \frac{n}{4} - 1$$

$$\therefore \sin \frac{\pi}{n} = \frac{n-4}{4} \quad (1)$$

Now, n is positive integer.

$$0 \leq \sin \frac{\pi}{n} \leq 1$$

If $\sin \frac{\pi}{n} = 0$, then $n = 4$. But $\sin \frac{\pi}{4} \neq 0$ (From (1))

If $\sin \frac{\pi}{n} = 1$ then $n = 8$. But $\sin \frac{\pi}{8} \neq 1$

$$\therefore 0 < \sin \frac{\pi}{n} < 1 \Rightarrow 0 < \frac{n-4}{4} < 1$$

$$\therefore 0 < n - 4 < 4$$

$$\therefore 4 < n < 8$$

$\therefore n = 5, 6, 7$ will be possible.

$\therefore$ 3 values of n are possible. **Ans. : (C)**

- (55) If the complex number ω is a cube root of unity then, $\sin \left((\omega^{10} + \omega^{23})\pi - \frac{\pi}{4} \right) = \dots\dots$ ($\omega \neq 1$)

- (A) $-\frac{\sqrt{3}}{2}$ (B) $-\frac{1}{\sqrt{2}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{\sqrt{3}}{2}$

Sol. : $\sin \left((\omega^{10} + \omega^{23})\pi - \frac{\pi}{4} \right)$

$$\omega^{10} = \omega^9 \cdot \omega = \omega$$

$$\omega^{23} = \omega^{21} \cdot \omega^2 = \omega^2$$

$$\sin \left((\omega + \omega^2)\pi - \frac{\pi}{4} \right)$$

$$= \sin \left(-\pi - \frac{\pi}{4} \right) \quad (1 + \omega + \omega^2 = 0. \Rightarrow \omega + \omega^2 = -1)$$

$$= -\sin \left(\pi + \frac{\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{Ans. : (C)}$$

- (56) In ΔPQR , if $R = \frac{\pi}{2}$ and $\tan \frac{P}{2}$ and $\tan \frac{Q}{2}$ are the roots of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$), then which of the following is true ?

- (A) $a + b = c$ (B) $a + c = b$ (C) $b + c = a$ (D) $b = c$

Sol. : The roots of the equation $ax^2 + bx + c = 0$ are $\tan \frac{P}{2}$ and $\tan \frac{Q}{2}$.

$$\therefore \tan \frac{P}{2} + \tan \frac{Q}{2} = -\frac{b}{a} \quad \text{and} \quad \tan \frac{P}{2} \tan \frac{Q}{2} = \frac{c}{a}$$

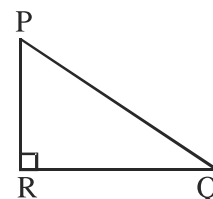


Figure 12.9

Also, in $\triangle PQR$, $R = \frac{\pi}{2}$.

So, $P + Q = \frac{\pi}{2}$

$$\therefore \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$$

$$\therefore \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = 1$$

$$\therefore \frac{\tan\frac{P}{2} + \tan\frac{Q}{2}}{1 - \tan\frac{P}{2} \tan\frac{Q}{2}} = 1$$

$$\therefore \frac{\frac{-b}{a}}{1 - \frac{c}{a}} = 1$$

$$\therefore \frac{-b}{a - c} = 1$$

$$\therefore -b = a - c$$

$$\therefore c = a + b$$

Ans. : (A)

(57) For $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$ which of the following is true ?

(A) $f(\theta) < 0, \forall \theta \geq 0$

(B) $f(\theta) < 0, \forall \theta < 0$

(C) $f(\theta) \geq 0, \forall \theta \in \mathbb{R}$

(D) $f(\theta) < 0, \forall \theta \in \mathbb{R}$

Sol. : $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$$= \sin\theta(2\sin 2\theta \cos\theta)$$

$$= (2\sin\theta \cos\theta)\sin 2\theta$$

$$= \sin^2 2\theta \geq 0, \forall \theta \in \mathbb{R}$$

Ans. : (C)

(58) If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha = \dots$

(A) $2(\tan\beta + \tan\gamma)$

(B) $\tan\beta + 2\tan\gamma$

(C) $\tan\beta + \tan\gamma$

(D) $2\tan\beta + \tan\gamma$

Sol. : Here, $\alpha + \beta = \frac{\pi}{2}$.

$$\therefore \alpha = \frac{\pi}{2} - \beta$$

$$\tan\alpha = \tan\left(\frac{\pi}{2} - \beta\right) = \cot\beta = \frac{1}{\tan\beta}$$

$$\therefore \tan\alpha \tan\beta = 1$$

(1)

$$\text{Now, } \beta + \gamma = \alpha$$

$$\therefore \gamma = \alpha - \beta$$

$$\therefore \tan\gamma = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

(From (1))

$$\therefore \tan\gamma = \frac{\tan\alpha - \tan\beta}{2}$$

$$\therefore \tan\alpha = \tan\beta + 2\tan\gamma$$

Ans. : (B)

(59) The maximum value of $(\cos\alpha_1)(\cos\alpha_2) \dots (\cos\alpha_n)$ is

where $0 \leq \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \leq \frac{\pi}{2}$ and $\cot\alpha_1 \cdot \cot\alpha_2 \dots \cot\alpha_n = 1$

(A) $\frac{1}{2^n}$

(B) $\frac{1}{2^n}$

(C) $\frac{1}{2^n}$

(D) 1

Sol. : Let $y = (\cos\alpha_1)(\cos\alpha_2) \dots (\cos\alpha_n)$

Now, $\cot\alpha_1 \cot\alpha_2 \dots \cot\alpha_n = 1$

$$\therefore \frac{\cos\alpha_1}{\sin\alpha_1} \cdot \frac{\cos\alpha_2}{\sin\alpha_2} \dots \frac{\cos\alpha_n}{\sin\alpha_n} = 1$$

$$\therefore \cos\alpha_1 \cdot \cos\alpha_2 \dots \cos\alpha_n = \sin\alpha_1 \cdot \sin\alpha_2 \dots \sin\alpha_n = y$$

$$y^2 = (\cos\alpha_1 \cos\alpha_2 \cos\alpha_3 \dots \cos\alpha_n)(\sin\alpha_1 \sin\alpha_2 \dots \sin\alpha_n)$$

$$\therefore y^2 = \frac{(2\sin\alpha_1 \cos\alpha_1) (2\sin\alpha_2 \cos\alpha_2) \dots (2\sin\alpha_n \cos\alpha_n)}{2^n}$$

$$\therefore y^2 = \frac{\sin 2\alpha_1 \sin 2\alpha_2 \dots \sin 2\alpha_n}{2^n}$$

$$0 \leq \alpha_1, \alpha_2, \dots, \alpha_n \leq \frac{\pi}{2}$$

$$\therefore 0 \leq 2\alpha_1, 2\alpha_2, \dots, 2\alpha_n \leq \pi$$

$$\therefore 0 \leq \sin 2\alpha_1, \sin 2\alpha_2, \dots, \sin 2\alpha_n \leq 1$$

$$\therefore 0 \leq \frac{\sin 2\alpha_1 \sin 2\alpha_2 \dots \sin 2\alpha_n}{2^n} \leq \frac{1}{2^n}$$

$$\therefore 0 \leq y^2 \leq \frac{1}{2^n}$$

$$\therefore 0 \leq y \leq \frac{1}{2^{\frac{n}{2}}}$$

$\therefore$ The maximum value of y is $\frac{1}{2^{\frac{n}{2}}}$.

$$\text{Let } \alpha_1, \alpha_2 \dots \alpha_n = \frac{\pi}{4},$$

$$\cot\alpha_1 \cot\alpha_2 \dots \cot\alpha_n = 1$$

$$\text{and } \cos\alpha_1 \cos\alpha_2 \dots \cos\alpha_n = \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{2^{\frac{n}{2}}}$$

Ans. : (A)

(60) α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$, then the minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is

(A) always positive

(B) negative

(C) 0

(D) -3

Sol. : $\alpha + \beta + \gamma = \pi$

It is true for $\alpha = -\frac{\pi}{2}, \beta = -\frac{\pi}{2}, \gamma = 2\pi$.

$$\begin{aligned} \sin\alpha + \sin\beta + \sin\gamma &= \sin\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right) + \sin 2\pi \\ &= (-1) + (-1) + 0 = -2 \end{aligned}$$

$\therefore$ The value of $\sin\alpha + \sin\beta + \sin\gamma$ is -2.

(A), (C), (D) are not possible.

$\therefore$ Value can be negative.

Ans. : (B)

[Here, (A) and (C) are clearly not true. (D) is true for $\alpha = \beta = \gamma = \frac{3\pi}{2}, \frac{7\pi}{2}$ etc. but $\alpha + \beta + \gamma = \pi$. This is not possible.]

(61) Range of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ is

(A) [4, 10]

(B) [-4, 10]

(C) [3, 10]

(D) [-3, 10]

Sol. : $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right)$

$$= 5\cos\theta + 3\left(\cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3}\right)$$

$$= 5\cos\theta + \left(\frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right)$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta. \text{ Hence, } a = \frac{13}{2}, b = -\frac{3\sqrt{3}}{2}$$

$$r^2 = a^2 + b^2 = \frac{169+27}{4} = 49. \Rightarrow r = 7$$

$\therefore$ Range of f is $[-7, 7]$.

$$\therefore -7 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \leq 7$$

$$\Rightarrow -4 \leq 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \leq 10$$

Required range is $[-4, 10]$.

Ans. : (B)

(62) Principal period of $f(\theta) = \sin^2\theta$ is

[AIEEE : 2002]

(A) π^2

(B) π

(C) 2π

(D) $\frac{\pi}{2}$

Sol. : $f(\theta) = \sin^2\theta = \sin^2(\pi + \theta) = f(\pi + \theta)$

Here smallest positive value is π . So, principal period is π .

Ans. : (B)

(63) Let α and β be numbers such that $\pi < \alpha - \beta < 3\pi$

If $\sin\alpha + \sin\beta = \frac{-21}{65}$ and $\cos\alpha + \cos\beta = \frac{-27}{65}$, then $\cos\left(\frac{\alpha - \beta}{2}\right) = \dots$ [AIEEE : 2004]

(A) $\frac{-6}{65}$

(B) $\frac{3}{\sqrt{130}}$

(C) $\frac{6}{65}$

(D) $\frac{-3}{\sqrt{130}}$

Sol. : $\sin\alpha + \sin\beta = \frac{-21}{65} \Rightarrow 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) = \frac{-21}{65}$ (1)

$\cos\alpha + \cos\beta = \frac{-27}{65} \Rightarrow 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) = \frac{-27}{65}$ (2)

On squaring and adding (1) and (2), we get

$$\therefore 4\sin^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right) + 4\cos^2\left(\frac{\alpha + \beta}{2}\right)\cos^2\left(\frac{\alpha - \beta}{2}\right) = \frac{441 + 729}{65^2}$$

$$\therefore 4\cos^2\left(\frac{\alpha - \beta}{2}\right) = \frac{18}{65}$$

$$\therefore \cos^2\left(\frac{\alpha - \beta}{2}\right) = \frac{9}{130}$$

$$\therefore \cos\left(\frac{\alpha - \beta}{2}\right) = \pm \frac{3}{\sqrt{130}}$$

$$\therefore \cos\left(\frac{\alpha - \beta}{2}\right) = \frac{-3}{\sqrt{130}} \text{ because } \frac{\pi}{2} < \frac{\alpha - \beta}{2} < \frac{3\pi}{2}$$

Ans. : (D)

- (64) The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2x + 5\sin x - 3 = 0$ is [AIEEE : 2006]

(A) 4 (B) 6 (C) 1 (D) 2

Sol. : $2\sin^2x + 5\sin x - 3 = 0$

$$\therefore \sin x = \frac{1}{2} \text{ or } \sin x = -3$$

Now, $\sin x = -3$ is not possible.

$$\therefore \sin x = \frac{1}{2} \text{ and } x \in [0, 3\pi]$$

$\therefore$ Two solutions are possible in $[0, \pi]$ and two solutions are possible in $[2\pi, 3\pi]$.

$$\therefore \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \text{ are the possible solutions.}$$

$\therefore$ The number of solutions = 4

Ans. : (A)

- (65) If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x = \dots$

(A) $\frac{1-\sqrt{7}}{4}$ (B) $3\sqrt{7}$ (C) $\frac{-4 \pm \sqrt{7}}{3}$ (D) $\frac{1+\sqrt{7}}{4}$

Sol. : $\cos x + \sin x = \frac{1}{2}$

$$\therefore 1 + \tan x = \frac{1}{2} \sec x \quad (\cos x \neq 0)$$

$$\therefore 2 + 2\tan x = \sec x$$

$$\therefore 4 + 8\tan x + 4\tan^2 x = \tan^2 x + 1$$

$$\therefore 3\tan^2 x + 8\tan x + 3 = 0$$

$$\therefore \tan x = \frac{-8 \pm \sqrt{28}}{6} = \frac{-8 \pm 2\sqrt{7}}{6} = \frac{-4 \pm \sqrt{7}}{3}$$

Ans. : (C)

- (66) If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, and P : $\cos\alpha + \cos\beta + \cos\gamma = 0$ and Q : $\sin\alpha + \sin\beta + \sin\gamma = 0$, then [AIEEE 2009]

(A) P is false and Q is true

(B) P and Q both are true.

(C) P and Q both are false.

(D) P is true and Q is false.

Sol. : $(\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2$
 $= 3 + 2[(\cos\alpha \cos\beta + \sin\alpha \sin\beta) + (\cos\beta \cos\gamma + \sin\beta \sin\gamma) + (\cos\alpha \cos\gamma + \sin\alpha \sin\gamma)]$
 $= 3 + 2[\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\alpha - \gamma)]$
 $= 3 + 2\left(-\frac{3}{2}\right) = 0$

$\therefore \cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$ both are true. **Ans. : (B)**

- (67) If $a_1, a_2 \dots a_{n+1}$ are in A.P., then $\tan^{-1} \frac{d}{1+a_1a_2} + \tan^{-1} \frac{d}{1+a_2a_3} + \dots n \text{ terms} = \dots$
 (where $d > 0, a_i > 0, \forall i$) (d is the common difference.)

(A) $\tan^{-1} \frac{nd}{1+a_1a_{n+1}}$

(B) $\tan^{-1} \left(\frac{nd}{1+a_1a_n} \right)$

(C) $\tan^{-1} \left(\frac{(n-1)d}{1+a_1a_n} \right)$

(D) None of these

Sol. : $a_1, a_2, a_3, \dots, a_{n+1}$ are in A.P.

$$\therefore a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = a_{n+1} - a_n = d \quad (1)$$

$$\begin{aligned} & \tan^{-1} \frac{d}{1+a_1 a_2} + \tan^{-1} \frac{d}{1+a_2 a_3} + \dots \text{ } n \text{ terms} \\ &= \tan^{-1} \left(\frac{a_2 - a_1}{1 + a_1 a_2} \right) + \tan^{-1} \left(\frac{a_3 - a_2}{1 + a_2 a_3} \right) + \dots \text{ } n \text{ terms} \\ &= (\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) + \dots + [\tan^{-1}(a_{n+1}) - \tan^{-1}(a_n)] \\ &= \tan^{-1}(a_{n+1}) - \tan^{-1} a_1 \\ &= \tan^{-1} \left(\frac{a_{n+1} - a_1}{1 + a_{n+1} a_1} \right) \end{aligned} \quad (2)$$

Now, from (1)

$$a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + \dots + a_{n+1} - a_n = nd$$

$$\therefore a_{n+1} - a_1 = nd$$

$$\therefore \tan^{-1} \left(\frac{a_{n+1} - a_1}{1 + a_{n+1} a_1} \right) = \tan^{-1} \left(\frac{nd}{1 + a_{n+1} a_1} \right) \quad \text{Ans. : (A)}$$

(68) The range of $\sin^2 x + \cos^4 x$ is, $\forall x \in \mathbb{R}$. [AIEEE : 2011]

(A) $\left[\frac{13}{16}, 1\right]$ (B) $[1, 2]$ (C) $\left[\frac{3}{4}, \frac{13}{16}\right]$ (D) $\left[\frac{3}{4}, 1\right]$

$$\begin{aligned} \text{Sol. : } \sin^2 x + \cos^4 x &= \left(\frac{1 - \cos 2x}{2} \right) + \left(\frac{1 + \cos 2x}{2} \right)^2 \\ &= \frac{1 - \cos 2x}{2} + \frac{1 + 2\cos 2x + \cos^2 2x}{4} \\ &= \frac{2 - 2\cos 2x + 1 + 2\cos 2x + \cos^2 2x}{4} \\ &= \frac{3 + \cos^2 2x}{4} \\ &= \frac{3}{4} + \frac{1}{4} \left(\frac{1 + \cos 4x}{2} \right) \\ &= \frac{7}{8} + \frac{1}{8} \cos 4x \end{aligned}$$

The range of $\cos 4x$ is $[-1, 1]$.

$$\therefore \text{The range of } \frac{7}{8} + \frac{1}{8} \cos 4x \text{ is } \left[\frac{7}{8} - \frac{1}{8}, \frac{7}{8} + \frac{1}{8} \right] = \left[\frac{3}{4}, 1 \right]. \quad \text{Ans. : (D)}$$

(69) Possible values of $\theta \in (0, \pi)$ such that $\sin \theta + \sin 4\theta + \sin 7\theta = 0$ are [AIEEE : 2011]

(A) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (B) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$
(C) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (D) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$

Sol. : $\sin \theta + \sin 4\theta + \sin 7\theta = 0$

$$\therefore (\sin 7\theta + \sin \theta) + \sin 4\theta = 0$$

$$\therefore 2\sin 4\theta \cos 3\theta + \sin 4\theta = 0$$

$$\therefore \sin 4\theta (2\cos 3\theta + 1) = 0$$

$$\therefore \sin 4\theta = 0 \text{ or } \cos 3\theta = -\frac{1}{2}$$

$$(i) \text{ Let } \sin 4\theta = 0,$$

$$0 < \theta < \pi \Rightarrow 0 < 4\theta < 4\pi$$

$$\therefore 4\theta = \pi, 2\pi, 3\pi. \Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \quad (1)$$

$$(ii) \text{ Let } \cos 3\theta = -\frac{1}{2},$$

$$0 < \theta < \pi \Rightarrow 0 < 3\theta < 3\pi$$

$$\therefore 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}. \Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9} \quad (2)$$

$$\therefore \theta = \frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$$

Ans. : (D)

- (70) ABCD is a trapezium such that $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \perp \overline{CD}$. If $m\angle ADB = \theta$, $BC = p$ and $CD = q$, then $AB = \dots\dots$ [JEE Main : 2013]

$$(A) \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$$

$$(B) \frac{p^2 + q^2 \cos \theta}{p \cos \theta + q \sin \theta}$$

$$(C) \frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$$

$$(D) \frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)^2}$$

Sol. :

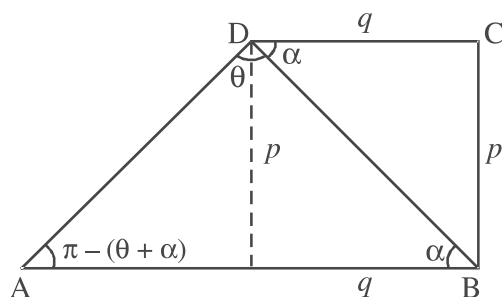


Figure 12.10

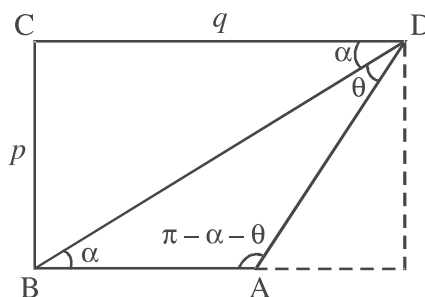


Figure 12.11

For the trapezium ABCD, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \perp \overline{CD}$ is given. From the figure BDC is a right angle triangle where $m\angle BDC = \alpha$

$$\therefore BD^2 = BC^2 + CD^2 = p^2 + q^2 \quad (1)$$

$$\text{and } \cos \alpha = \frac{CD}{BD} = \frac{q}{\sqrt{p^2 + q^2}} \text{ and } \sin \alpha = \frac{BC}{BD} = \frac{p}{\sqrt{p^2 + q^2}} \quad (2)$$

Also, in $\triangle ABD$, using *sine* rule, we get,

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin A}$$

$$\therefore \frac{AB}{\sin \theta} = \frac{\sqrt{p^2 + q^2}}{\sin(\pi - (\theta + \alpha))} \quad (\text{From (1)})$$

$$\frac{AB}{\sin \theta} = \frac{\sqrt{p^2 + q^2}}{\sin \theta \cos \alpha + \cos \theta \sin \alpha}$$

$$\therefore AB = \frac{\sqrt{p^2 + q^2} \sin \theta}{\sin \theta \left(\frac{q}{\sqrt{p^2 + q^2}} \right) + \cos \theta \left(\frac{p}{\sqrt{p^2 + q^2}} \right)} \quad (\text{From (2)})$$

$$\therefore AB = \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta} \quad \text{Ans. : (A)}$$

(71) If $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $0 < x < \frac{1}{\sqrt{3}}$, then $y = \dots$

(A) $\frac{3x-x^3}{1-3x^2}$ (B) $\frac{3x+x^3}{1-3x^2}$ (C) $\frac{3x-x^3}{1+3x^2}$ (D) $\frac{3x+x^3}{1+3x^2}$

Sol. : $\tan^{-1}y = \tan^{-1}\left(\frac{x + \frac{2x}{1-x^2}}{1 - \frac{2x^2}{1-x^2}}\right) \quad \left(0 < x < \frac{1}{\sqrt{3}}\right)$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$\therefore y = \left(\frac{3x-x^3}{1-3x^2}\right) \quad \text{Ans. : (A)}$

(72) $\sin^{-1}(\sin 10) = \dots$

(A) 10 (B) $10 - 3\pi$ (C) $3\pi - 10$ (D) None of these

Sol. : $\sin^{-1}(\sin 10) = \sin^{-1}[\sin(3\pi - 10)] = 3\pi - 10 \quad \left(-\frac{\pi}{2} < 3\pi - 10 < 0\right)$

Ans. : (C)

(73) $\sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right] = \dots \quad x \in [0, 1]$

(A) $\sin^{-1}\sqrt{x} + \sin^{-1}x$ (B) $\sin^{-1}x - \sin^{-1}\sqrt{x}$ (C) $\sin^{-1}\sqrt{x} - \sin^{-1}x$ (D) None of these

Sol. : Let $x = \sin \theta$ and $\sqrt{x} = \sin \alpha$,

Here, $x \in [0, 1] \Rightarrow \theta, \alpha \in \left[0, \frac{\pi}{2}\right]$

$\therefore \theta - \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\begin{aligned}
& \sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right) \\
&= \sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\alpha}-\sin\alpha\sqrt{1-\sin^2\theta}\right) \\
&= \sin^{-1}(\sin\theta\cos\alpha-\cos\theta\sin\alpha) \quad (\cos\alpha > 0, \cos\theta > 0) \\
&= \sin^{-1}(\sin(\theta-\alpha)) \\
&= \theta-\alpha \\
&= \sin^{-1}x-\sin^{-1}\sqrt{x} \quad \left(0 \leq \theta, \alpha \leq \frac{\pi}{2}\right) \quad \text{Ans. : (B)}
\end{aligned}$$

(74) If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in \mathbb{N}$, then the maximum value of n is

- (A) 6 (B) 7 (C) 5 (D) None of these

Sol. : $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$

$$\therefore \frac{n}{\pi} < \cot\frac{\pi}{6} \quad (\cot \text{ is a decreasing function})$$

$$\therefore \frac{n}{\pi} < \sqrt{3}$$

$$\therefore n < \sqrt{3}\pi \Rightarrow n < 6$$

$$\therefore \text{The maximum value of } n = 5.$$

Ans. : (C)

(75) $\sum_{r=1}^n \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \dots\dots$

- (A) $\tan^{-1}\sqrt{n} - \frac{\pi}{4}$ (B) $\tan^{-1}(\sqrt{n+1}) - \frac{\pi}{4}$ (C) $\tan^{-1}\sqrt{n}$ (D) $\tan^{-1}(\sqrt{n+1})$

Sol. : $\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right)$

$$= \tan^{-1}\sqrt{r} - \tan^{-1}(\sqrt{r-1})$$

$$\therefore \sum_{r=1}^n \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right) = \sum_{r=1}^n \left(\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{r-1}\right) = \tan^{-1}\sqrt{n} \quad \text{Ans. : (C)}$$

(76) $\sum_{r=1}^n \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right) = \dots\dots$

- (A) $\tan^{-1}2^n$ (B) $\tan^{-1}2^n - \frac{\pi}{4}$ (C) $\tan^{-1}2^{n+1}$ (D) $\tan^{-1}2^{n+1} - \frac{\pi}{4}$

Sol. : $\sum_{r=1}^n \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right) = \sum_{r=1}^n \tan^{-1}\left(\frac{2^r-2^{r-1}}{1+2^r \cdot 2^{r-1}}\right)$

$$= \sum_{r=1}^n \left[\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})\right]$$

$$= \tan^{-1}2^n - \tan^{-1}1 = \tan^{-1}2^n - \frac{\pi}{4} \quad \text{Ans. : (B)}$$

$$(77) \quad \sum_{m=1}^n \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2} \right) = \dots$$

$$(A) \tan^{-1} \left(\frac{n^2 + n}{n^2 + n + 2} \right) \quad (B) \tan^{-1} \left(\frac{n^2 - n}{n^2 - n + 2} \right) \quad (C) \tan^{-1} \left(\frac{n^2 + n + 2}{n^2 + n} \right) \quad (D) \text{None of these}$$

$$\begin{aligned} \text{Sol. : } \sum_{m=1}^n \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2} \right) &= \sum_{m=1}^n \tan^{-1} \left(\frac{(m^2 + m + 1) - (m^2 - m + 1)}{1 + (m^2 + m + 1)(m^2 - m + 1)} \right) \\ &= \sum_{m=1}^n \left[\tan^{-1}(m^2 + m + 1) - \tan^{-1}(m^2 - m + 1) \right] \\ &= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}7 - \tan^{-1}3) + (\tan^{-1}13 - \tan^{-1}7) + \dots \\ &\quad + (\tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1)) \\ &= \tan^{-1}(n^2 + n + 1) - \tan^{-1}1 \\ &= \tan^{-1} \left(\frac{n^2 + n + 1 - 1}{1 + (n^2 + n + 1)(1)} \right) \quad (n^2 + n + 1 > 0) \\ &= \tan^{-1} \left(\frac{n^2 + n}{n^2 + n + 2} \right) \end{aligned} \quad \text{Ans. : (A)}$$

(78) For positive real values a, b, c ,

$$\text{If } \theta = \tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}}, \text{ then } \tan \theta = \dots$$

[IIT:1981]

$$(A) 0 \quad (B) 2 \quad (C) 1 \quad (D) \sqrt{3}$$

$$\text{Sol. : Let } \tan \alpha = \sqrt{\frac{a(a+b+c)}{bc}}, \tan \beta = \sqrt{\frac{b(a+b+c)}{ca}}, \tan \gamma = \sqrt{\frac{c(a+b+c)}{ab}}$$

$$0 < \alpha, \beta, \gamma < \frac{\pi}{2}$$

$$\therefore \theta = \alpha + \beta + \gamma.$$

$$\tan \alpha + \tan \beta + \tan \gamma = \frac{\sqrt{a+b+c}(a+b+c)}{\sqrt{abc}} = \frac{(a+b+c)^{\frac{3}{2}}}{\sqrt{abc}}$$

$$\tan \alpha \tan \beta \tan \gamma = \frac{(a+b+c)^{\frac{3}{2}}}{\sqrt{abc}}$$

$$\therefore \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$$

$$\therefore \alpha + \beta + \gamma = (2k + 1)\pi; k \in \mathbb{Z}$$

$$\text{But } 0 < \alpha + \beta + \gamma < \frac{3\pi}{2}. \text{ So, } \alpha + \beta + \gamma = \pi$$

$$\therefore \tan \theta = \tan(\alpha + \beta + \gamma) = \tan \pi = 0$$

Ans. : (A)

Second method :

Let $a + b + c = u$

$$\therefore \theta = \tan^{-1} \sqrt{\frac{au}{bc}} + \tan^{-1} \sqrt{\frac{bu}{ca}} + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$\text{Now, } \sqrt{\frac{au}{bc}} \cdot \sqrt{\frac{bu}{ca}} = \frac{u}{c} = \frac{a+b+c}{c} > 1$$

$$\therefore \theta = \pi + \tan^{-1} \left(\frac{\sqrt{\frac{au}{bc}} + \sqrt{\frac{bu}{ca}}}{1 - \left(\sqrt{\frac{au}{bc}} \right) \left(\sqrt{\frac{bu}{ca}} \right)} \right) + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$= \pi + \tan^{-1} \left(\frac{\frac{(a+b)\sqrt{u}}{\sqrt{abc}}}{1 - \frac{u}{c}} \right) + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$= \pi + \tan^{-1} \left(\frac{(u-c)\sqrt{u}}{(c-u)} \cdot \frac{c}{\sqrt{abc}} \right) + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$= \pi - \tan^{-1} \sqrt{\frac{cu}{ab}} + \tan^{-1} \sqrt{\frac{cu}{ab}}$$

$$= \pi$$

$$\therefore \tan \theta = \tan \pi = 0$$

Ans. : (A)

$$(79) \quad \tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) - \frac{\pi}{4} \right] = \dots\dots$$

$$(A) \quad \frac{7}{17}$$

$$(B) \quad \frac{-7}{17}$$

$$(C) \quad \frac{17}{7}$$

$$(D) \quad \frac{15}{7}$$

$$\text{Sol. : } \tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) - \frac{\pi}{4} \right]$$

$$= \tan \left[\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{5} - \tan^{-1} 1 \right]$$

$$= \tan \left[\tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{25}} - \tan^{-1} 1 \right]$$

$$= \tan \left[\tan^{-1} \frac{5}{12} - \tan^{-1} 1 \right]$$

$$= \tan \left[\tan^{-1} \left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12}} \right) \right]$$

$$= \tan \left[\tan^{-1} \left(\frac{-7}{17} \right) \right] = \frac{-7}{17}$$

Ans. : (B)

$$(80) \quad \lim_{n \rightarrow \infty} \sum_{r=1}^n \tan^{-1} \left(\frac{1}{2r^2} \right) = \dots\dots$$

$$(A) \quad \frac{\pi}{6}$$

$$(B) \quad \frac{\pi}{4}$$

$$(C) \quad \frac{\pi}{2}$$

$$(D) \quad \frac{\pi}{3}$$

$$\text{Sol. : } \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{2r^2}\right) = \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2}{4r^2}\right)$$

$$= \sum_{r=1}^{\infty} \tan^{-1}\left[\frac{2}{1+4r^2-1}\right]$$

$$= \sum_{r=1}^{\infty} \tan^{-1}\left[\frac{(2r+1)-(2r-1)}{1+(2r-1)(2r+1)}\right]$$

$$= \sum_{r=1}^{\infty} \left[\tan^{-1}(2r+1) - \tan^{-1}(2r-1) \right]$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{r=1}^n \tan^{-1}\left(\frac{1}{2r^2}\right) = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left[\tan^{-1}(2r+1) - \tan^{-1}(2r-1) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\tan^{-1}(2n+1) - \frac{\pi}{4} \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Ans. : (B)

$$(81) \quad \text{If } \cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x, \text{ then } \sin x = \dots$$

[AIEEE : 2002]

$$(A) \tan^2\left(\frac{\alpha}{2}\right)$$

$$(B) \cot^2\left(\frac{\alpha}{2}\right)$$

$$(C) \tan \alpha$$

$$(D) \cot \frac{\alpha}{2}$$

$$\text{Sol. : } x = \cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha})$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{\cos \alpha}}\right) - \tan^{-1}(\sqrt{\cos \alpha})$$

$$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos \alpha}} - \sqrt{\cos \alpha}}{1+1}\right)$$

$$= \tan^{-1}\left(\frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}}\right)$$

$$\therefore \sin x = \sin \left[\tan^{-1}\left(\frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}}\right) \right]$$

$$\sin x = \sin \left[\sin^{-1}\left(\frac{1 - \cos \alpha}{1 + \cos \alpha}\right) \right]$$

$$\left(\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}} \right)$$

$$= \sin \left(\sin^{-1}\left(\tan^2 \frac{\alpha}{2}\right) \right)$$

$$= \tan^2 \frac{\alpha}{2}$$

Ans. : (A)

(82) The equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for [AIEEE : 2003]

- (A) $|a| \geq \frac{1}{\sqrt{2}}$ (B) $|a| \leq \frac{1}{\sqrt{2}}$ (C) $\forall a \in \mathbb{R}$ (D) $|a| < \frac{1}{2}$

Sol. : $\sin^{-1}x = 2\sin^{-1}a$

We know that, $\frac{-\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

$$\begin{array}{l} \therefore \frac{-\pi}{2} \leq 2\sin^{-1}a \leq \frac{\pi}{2} \\ \therefore \frac{-\pi}{4} \leq \sin^{-1}a \leq \frac{\pi}{4} \end{array} \quad \left| \quad \begin{array}{l} \therefore -\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}} \\ \therefore |a| \leq \frac{1}{\sqrt{2}} \end{array} \right.$$

Ans. : (B)

(83) If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2 = \dots\dots$ [AIEEE : 2005]

- (A) $2\sin 2\alpha$ (B) 4 (C) $4\sin^2\alpha$ (D) $-4\sin^2\alpha$

Sol. : Let $\cos^{-1}x = p$, $\cos^{-1}\frac{y}{2} = q$, $p \in [0, \pi]$, $q \in [0, \pi]$

$$\therefore \cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha, x = \cos p, \frac{y}{2} = \cos q$$

Now, $p - q = \alpha$

$$\therefore \cos(p - q) = \cos\alpha$$

$$\therefore \cos p \cos q + \sin p \sin q = \cos\alpha$$

$$\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha \quad (\sin p > 0)$$

$$\left(\frac{xy}{2} - \cos\alpha\right)^2 = (1-x^2)\left(1-\frac{y^2}{4}\right)$$

$$\frac{x^2y^2}{4} - xycos\alpha + \cos^2\alpha = 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$$

$$x^2 + \frac{y^2}{4} = 1 - \cos^2\alpha + xycos\alpha$$

$$\therefore 4x^2 - 4xycos\alpha + y^2 = 4\sin^2\alpha$$

Ans. : (C)

(84) If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then $x = \dots\dots$ [AIEEE : 2007]

- (A) 4 (B) 5 (C) 1 (D) 3

Sol. : $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\therefore \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$$

$$\therefore \sin^{-1}\left(\frac{x}{5}\right) + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$$

$$\therefore x = 3$$

Ans. : (D)

$$(85) \quad \cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right) = \dots\dots$$

$$(A) \quad \frac{6}{17}$$

$$(B) \quad \frac{3}{17}$$

$$(C) \quad \frac{4}{17}$$

$$(D) \quad \frac{5}{17}$$

$$\text{Sol. : } \operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3} = \tan^{-1}\left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{6}{12}}\right) = \tan^{-1}\left(\frac{17}{6}\right)$$

$$\therefore \cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right) = \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$$

Ans. : (A)

$$(86) \quad \text{For } 0 < \theta < \frac{\pi}{2}, \text{ the solution of the equation,}$$

$$\sum_{m=1}^6 \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2} \text{ is } \dots\dots$$

[IIT : 2009]

$$(A) \quad \frac{\pi}{4}$$

$$(B) \quad \frac{\pi}{6}$$

$$(C) \quad \frac{\pi}{12}$$

$$(D) \quad \frac{5\pi}{12}$$

$$\text{Sol. : } 0 < \theta < \frac{\pi}{2}$$

$$\sum_{m=1}^6 \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\therefore \sum_{m=1}^6 \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\therefore \sum_{m=1}^6 \frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{4} \sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\therefore \sum_{m=1}^6 \frac{\sin\left[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\frac{\pi}{4} \left[\sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)\right]} = 4\sqrt{2}$$

$$\therefore \sum_{m=1}^6 \left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right) \right] = 4$$

$$\therefore \cot\theta - \cot\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) - \cot\left(\theta + \frac{2\pi}{4}\right) + \dots + \cot\left(\theta + \frac{5\pi}{4}\right) - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$$\therefore \cot\theta - \cot\left(\theta + \frac{3\pi}{2}\right) = 4$$

$$\therefore \cot\theta + \tan\theta = 4$$

$$\therefore \frac{1}{\tan\theta} + \tan\theta = 4$$

$$\therefore 1 + \tan^2\theta = 4\tan\theta$$

$$\therefore \tan^2\theta - 4\tan\theta + 1 = 0$$

Second method : (from $\cot\theta + \tan\theta = 4$)

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = 4$$

$$\therefore \frac{1}{\sin\theta \cos\theta} = 4$$

$$\therefore \sin 2\theta = \frac{1}{2}$$

$$\therefore 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\therefore \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

$$\therefore \tan \theta = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$\tan \theta = 2 + \sqrt{3} \Rightarrow \theta = \frac{5\pi}{12} \text{ and } \tan \theta = 2 - \sqrt{3} \Rightarrow \theta = \frac{\pi}{12} \quad \text{Ans. : (C), (D)}$$

- (87) If x, y, z are in A.P. and $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are also in A.P., then [JEE : 2013]
($0 < x, y, z < 1$)

(A) $x = y = z$ (B) $2x = 3y = 6z$ (C) $6x = 3y = 6z$ (D) $6x = 4y = 3z$

Sol. : x, y, z are in A.P.

$$\therefore x + z = 2y \quad (1)$$

Now, $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are also in A.P.

$$\therefore \tan^{-1}x + \tan^{-1}z = 2\tan^{-1}y$$

$$\therefore \tan^{-1}\left(\frac{x+z}{1-xz}\right) = 2\tan^{-1}y = \tan^{-1}\left(\frac{y+y}{1-y^2}\right)$$

$$\therefore \tan^{-1}\left(\frac{2y}{1-xz}\right) = \tan^{-1}\left(\frac{2y}{1-y^2}\right) \quad (\text{From (1)})$$

$$\therefore 1 - xz = 1 - y^2$$

$$\therefore y^2 = xz$$

$\therefore x, y, z$ are in G.P. and x, y, z are in A.P.

$$\therefore x = y = z \quad \text{Ans. : (A)}$$

- (88) $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\frac{2}{3}\right] = \dots\dots$ [IIT : 1993]

(A) $\frac{6}{17}$ (B) $\frac{7}{16}$ (C) $\frac{16}{7}$ (D) None of these

Sol. : Now, $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3} = \tan^{-1}\left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{6}{12}}\right) = \tan^{-1}\left(\frac{17}{6}\right)$

$$\therefore \tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right] = \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6} \quad \text{Ans. : (D)}$$

- (89) $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right] = \dots\dots$ [IIT : 1994]

(A) $\frac{\sqrt{29}}{3}$ (B) $\frac{29}{3}$ (C) $\frac{\sqrt{3}}{29}$ (D) $\frac{3}{29}$

Sol. : $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right]$

$$= \tan[\tan^{-1}7 - \tan^{-1}4]$$

$$= \tan\left(\tan^{-1}\left(\frac{7-4}{1+28}\right)\right) = \frac{3}{29} \quad \text{Ans. : (D)}$$

- (90) The number of real roots of the equation $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is
[IIT : 1999]

(A) 0 (B) 1 (C) 2 (D) infinite

Sol. : $\tan^{-1}\sqrt{x(x+1)} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$

$$\therefore \tan^{-1}\sqrt{x(x+1)} = \cos^{-1}\sqrt{x^2+x+1}$$

$$\therefore \cos^{-1}\frac{1}{\sqrt{x^2+x+1}} = \cos^{-1}\sqrt{x^2+x+1}$$

$$\therefore x^2 + x + 1 = 1$$

$$\therefore x^2 + x = 0$$

$$\therefore x(x+1) = 0$$

$$\therefore x = 0 \text{ or } x = -1$$

$\therefore$ The two real roots are $x = 0$ and $x = -1$. Verifying, both the roots satisfy the given equation.

Ans. : (C)

- (91) If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^8}{4} - \dots\right) = \frac{\pi}{2}$, $0 < |x| < \sqrt{2}$, then $x = \dots$

(A) $\frac{1}{2}$ (B) 1 (C) $-\frac{1}{2}$ (D) -1 [IIT : 2015]

Sol. : $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^8}{4} - \dots\right) = \frac{\pi}{2}$

$$\therefore \left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) = \left(x^2 - \frac{x^4}{2} + \frac{x^8}{4} - \dots\right)$$

$$\therefore \frac{x}{1 - \left(\frac{-x}{2}\right)} = \frac{x^2}{1 - \left(\frac{-x^2}{2}\right)} \quad (0 < x^2 < 2, 0 < |x| < \sqrt{2})$$

$$\therefore \frac{2x}{2+x} = \frac{2x^2}{2+x^2}$$

$$\therefore x(2+x^2) = x^2(2+x)$$

$$\therefore 2x + x^3 = 2x^2 + x^3. \quad \text{So, } x^2 - x = 0. \quad \text{Hence, } x(x-1) = 0$$

$$\therefore x = 0 \text{ or } x = 1. \text{ But } 0 < |x| < \sqrt{2}$$

$$\therefore x = 1$$

Ans. : (B)

Note : $x = 0$ does satisfy the equation.

- (92) If $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$, then $x = \dots$

(A) $\frac{1}{2}$ (B) 1 (C) 0 (D) $-\frac{1}{2}$

Sol. : $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$

$$\therefore \sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$\therefore \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}. \text{ Hence, } 1+x^2 = x^2+2x+2$$

$$\therefore 2x = -1. \text{ Hence, } x = -\frac{1}{2} \left(\text{Verify putting } x = -\frac{1}{2} \right)$$

Ans. : (D)

(93) $\sqrt{1+x^2} \left[\left\{ x \cos(\cot^{-1}x) + \sin(\cot^{-1}x) \right\}^2 - 1 \right]^{\frac{1}{2}} = \dots \quad (0 < x < 1) \quad [\text{IIT : 2008}]$

(A) $\frac{x}{\sqrt{1+x^2}}$

(B) x

(C) $x\sqrt{1+x^2}$

(D) $\sqrt{1+x^2}$

Sol. : $\sqrt{1+x^2} \left[\left\{ x \cos(\cot^{-1}x) + \sin(\cot^{-1}x) \right\}^2 - 1 \right]^{\frac{1}{2}}$

$$= \sqrt{1+x^2} \left[\left\{ x \left(\cos \left(\cos^{-1} \frac{x}{\sqrt{1+x^2}} \right) \right) + \sin \left(\sin^{-1} \frac{1}{\sqrt{1+x^2}} \right) \right\}^2 - 1 \right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2} \left[\left\{ \frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right\}^2 - 1 \right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2} [1+x^2-1]^{\frac{1}{2}}$$

$$= x\sqrt{1+x^2}$$

$(x > 0)$

Ans. : (C)

(94) If the lengths of three sides of a triangle are $3x+4y$, $4x+3y$, $5x+5y$, then the triangle is where $x, y > 0$

(A) Right angled

(B) Obtuse angled

(C) Equilateral

(D) None of these

Sol. : In $\triangle ABC$, let $BC = a = 3x+4y$, $AC = b = 4x+3y$, $AB = c = 5x+5y$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(3x+4y)^2 + (4x+3y)^2 - (5x+5y)^2}{2(3x+4y)(4x+3y)} = \frac{-2xy}{2(3x+4y)(4x+3y)} < 0$$

$$\therefore C > \frac{\pi}{2}.$$

So, $\triangle ABC$ is obtuse angled triangle.

Ans. : (B)

- (95) The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side ' a ' is

(A) $\frac{a}{4} \cot \frac{\pi}{2n}$ (B) $a \cot \frac{\pi}{n}$ (C) $\frac{a}{2} \cot \frac{\pi}{2n}$ (D) $a \cot \frac{\pi}{2n}$

Sol. : $\tan \frac{\pi}{n} = \frac{a}{2r}$

$$\sin \frac{\pi}{n} = \frac{a}{2R}$$

$$\therefore r = \frac{a}{2} \cot \frac{\pi}{n}$$

$$R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$$

$$\therefore r + R = \frac{a}{2} \left[\cot \frac{\pi}{n} + \operatorname{cosec} \frac{\pi}{n} \right]$$

$$= \frac{a}{2} \left[\frac{1 + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right]$$

$$= \frac{a}{2} \left[\frac{2 \cos^2 \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} \right] = \frac{a}{2} \cot \frac{\pi}{2n}$$

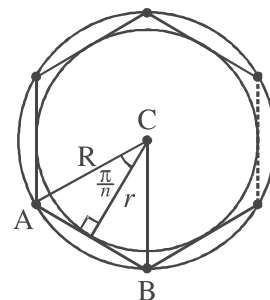


Figure 12.12

Ans. : (C)

- (96) $(2\cos\theta - 1)(2\cos2\theta - 1)(2\cos4\theta - 1) = \dots$

(A) $\frac{2\cos8\theta + 1}{2\cos\theta + 1}$ (B) $\frac{2\cos8\theta - 1}{2\cos\theta + 1}$ (C) $\frac{2\cos8\theta + 1}{2\cos\theta - 1}$ (D) None of these

Sol. : $(2\cos\theta - 1)(2\cos2\theta - 1)(2\cos4\theta - 1)$

$$= \frac{(2\cos\theta + 1)(2\cos\theta - 1)(2\cos2\theta - 1)(2\cos4\theta - 1)}{(2\cos\theta + 1)}$$

$$= \frac{(4\cos^2\theta - 1)(2\cos2\theta - 1)(2\cos4\theta - 1)}{2\cos\theta + 1}$$

$$= \frac{\left[4 \left(\frac{1 + \cos2\theta}{2} \right) - 1 \right] (2\cos2\theta - 1)(2\cos4\theta - 1)}{2\cos\theta + 1}$$

$$= \frac{(2\cos2\theta + 1)(2\cos2\theta - 1)(2\cos4\theta - 1)}{2\cos\theta + 1}$$

$$= \frac{(4\cos^2 2\theta - 1)(2\cos4\theta - 1)}{2\cos\theta + 1}$$

$$= \frac{(2(1 + \cos4\theta) - 1)(2\cos4\theta - 1)}{2\cos\theta + 1}$$

$$= \frac{4\cos^2 4\theta - 1}{2\cos\theta + 1} = \frac{2\cos8\theta + 1}{2\cos\theta + 1} \quad \text{Ans. : (A)}$$

- (97) If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then $x = \dots$

(A) -1 (B) 1 (C) 0 (D) None of these

Sol. : $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$$\therefore (\tan^{-1}x)^2 + \left(\frac{\pi}{2} - \tan^{-1}x \right)^2 = \frac{5\pi^2}{8}$$

$$\therefore (\tan^{-1}x)^2 + (\tan^{-1}x)^2 - \pi(\tan^{-1}x) + \frac{\pi^2}{4} = \frac{5\pi^2}{8}$$

$$\therefore 2(\tan^{-1}x)^2 - \pi(\tan^{-1}x) - \frac{3\pi^2}{8} = 0$$

$$\text{Let } y = \tan^{-1}x$$

$$\therefore 2y^2 - \pi y - \frac{3\pi^2}{8} = 0$$

$$\therefore 16y^2 - 8\pi y - 3\pi^2 = 0$$

$$\therefore (4y - 3\pi)(4y + \pi) = 0$$

$$\therefore y = -\frac{\pi}{4} \text{ or } y = \frac{3\pi}{4}$$

$$\therefore \tan^{-1}x = -\frac{\pi}{4} \text{ or } \tan^{-1}x = \frac{3\pi}{4}$$

$$\therefore \tan^{-1}x = -\frac{\pi}{4} \quad \left(\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$$

$$\therefore x = \tan\left(-\frac{\pi}{4}\right)$$

$$\therefore x = -1$$

Verifying, $x = -1$, satisfies given equation. So, $x = -1$ is the required solution. **Ans. : (A)**

(98) Which of the following is true for $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$?

$$(A) \cos(\log\theta) > \log(\cos\theta)$$

$$(B) \cos(\log\theta) < \log(\cos\theta)$$

$$(C) \cos(\log\theta) = \log(\cos\theta)$$

$$(D) \cos(\log\theta) = \frac{2}{3}\log(\cos\theta)$$

$$\text{Sol. : } e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$$

$$\log e^{-\frac{\pi}{2}} < \log\theta < \log\frac{\pi}{2}$$

$$\therefore -\frac{\pi}{2} < \log\theta < \frac{\pi}{2} \quad \left(\frac{\pi}{2} < e \Rightarrow \log\frac{\pi}{2} < 1 < \frac{\pi}{2} \right)$$

$$\therefore \cos(\log\theta) > 0 \quad (1)$$

$$\text{Now, } 0 < \cos\theta < 1$$

$$\therefore \log(\cos\theta) < \log 1$$

$$\therefore \log(\cos\theta) < 0 \quad (2)$$

From (1) and (2),

$$\cos(\log\theta) > \log(\cos\theta)$$

Ans. : (A)

(99) Principal period for $\cot(x + 2x + 4x + 8x + \dots n \text{ terms})$ is

- (A) $\frac{\pi}{2^n}$ (B) $\frac{\pi}{2^n - 1}$ (C) $\frac{\pi}{2^n + 1}$ (D) 2π

Sol. : For $\cot(x + 2x + 4x + 8x + \dots n \text{ terms})$, $a = x$, $r = 2$

$$\text{Given function} = \cot\left[\frac{x(2^n - 1)}{2 - 1}\right]$$

$$= \cot[x(2^n - 1)] \text{ whose principal period} = \frac{\pi}{2^n - 1} \quad \text{Ans. : (B)}$$

(100) If the measures of three angles of ΔABC are in the ratio $3 : 5 : 4$, then $a + c\sqrt{2} = \dots$

- (A) $2a$ (B) $2b$ (C) $\sqrt{2}c$ (D) c

Sol. : If the measures of the angles of ΔABC are in the ratio $3 : 5 : 4$.

$\therefore$ Let their measures be $3x, 5x, 4x$.

$$3x + 5x + 4x = \pi$$

$$\therefore x = \frac{\pi}{12}$$

$$A = 3x = \frac{\pi}{4}, B = 5x = \frac{5\pi}{12}, C = 4x = \frac{\pi}{3}$$

$$b = a \cos C + c \cos A$$

$$\therefore b = a \cos \frac{\pi}{3} + c \cos \frac{\pi}{4}$$

$$\therefore b = \frac{a}{2} + \frac{c}{\sqrt{2}}$$

$$\therefore b = \frac{a + \sqrt{2}c}{2}$$

$$\therefore a + \sqrt{2}c = 2b$$

Ans. : (B)

(101) For right angled triangle, $\frac{a^2 + b^2 + c^2}{R^2} = \dots$

- (A) 4 (B) 8 (C) 2 (D) 1

Sol. : Let $m\angle B = \frac{\pi}{2}$ in ΔABC .

$$\text{Now, } \frac{b}{\sin B} = 2R. \text{ Hence, } b = 2R$$

$$\therefore a^2 + c^2 = b^2$$

$$\therefore a^2 + c^2 = 4R^2$$

$$\therefore \frac{a^2 + b^2 + c^2}{R^2} = \frac{4R^2 + 4R^2}{R^2} = 8$$

Ans. : (B)

(102) If $a = \sqrt{3}$, $b = \frac{1}{2}(\sqrt{6} + \sqrt{2})$, $c = \sqrt{2}$, then $A = \dots$

- (A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

$$\text{Sol. : } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\left(\frac{\sqrt{6} + \sqrt{2}}{2}\right)^2 + 2 - 3}{\frac{2}{2}(\sqrt{6} + \sqrt{2})(\sqrt{2})} = \frac{\sqrt{3} + 1}{2(\sqrt{3} + 1)} = \frac{1}{2}$$

$$A = \frac{\pi}{3}$$

Ans. : (C)

(103) If the lengths of three sides of a triangle are $a^2 + 2a$, $2a + 3$, $a^2 + 3a + 8$ ($a > 0$), then $a \in$

- (A) $(3, \infty)$ (B) $(4, \infty)$ (C) $(5, \infty)$ (D) $(-\infty, 5)$

Sol. : The sum of the lengths of any two sides of a triangle is greater than the third side. So,

$$(i) (a^2 + 2a) + (2a + 3) > a^2 + 3a + 8$$

$$\therefore a^2 + 4a + 3 > a^2 + 3a + 8. \text{ So, } a > 5$$

$$\therefore a \in (5, \infty) \quad (1)$$

Similarly,

$$(ii) (2a + 3) + (a^2 + 3a + 8) > a^2 + 2a$$

$$\therefore 3a + 11 > 0 \text{ which is always true.}$$

$$(iii) a^2 + 2a + a^2 + 3a + 8 > 2a + 3$$

$$\therefore 2a^2 + 3a + 5 > 0$$

$$\Delta = b^2 - 4ac = 9 - 40 = -31 < 0 \text{ and } a > 0$$

$$\therefore \forall a \in \mathbb{R} \text{ is true.}$$

$$\therefore \text{Thus } a \in (5, \infty).$$

Ans. : (C)

(104) If $\cot A$, $\cot B$, $\cot C$ are in A.P. in $\triangle ABC$, then a^2 , b^2 , c^2 are in

- (A) A.P. (B) H.P. (C) G.P. (D) None of these

Sol. : $\cot A$, $\cot B$, $\cot C$ are in A.P.

$$\therefore \cot A + \cot C = 2\cot B$$

$$\therefore \frac{b^2 + c^2 - a^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta} = \frac{2(a^2 + c^2 - b^2)}{4\Delta}$$

$$\therefore 2b^2 = 2a^2 + 2c^2 - 2b^2$$

$$\therefore a^2 + c^2 = 2b^2$$

$$\therefore a^2, b^2, c^2 \text{ are in A.P.}$$

Ans. : (A)

(105) If the length of each side is 2 for a regular polygon of 9 sides, then the radius of the circumcircle is

- (A) $\sec 20^\circ$ (B) $\operatorname{cosec} 20^\circ$ (C) $\cot 20^\circ$ (D) None of these

Sol. : For a regular polygon of the 9 sides, the angle at centre has measure 40° (in degrees)

In $\triangle OAB$, by *sine* rule,

$$\frac{a}{\sin 70^\circ} = \frac{2}{\sin 40^\circ}$$

$$\therefore a = \frac{2\sin 70^\circ}{\sin 40^\circ}$$

$$= \frac{2\sin 70^\circ}{2\sin 20^\circ \cos 20^\circ}$$

$$= \frac{1}{\sin 20^\circ} = \operatorname{cosec} 20^\circ$$

$$(\sin 70^\circ = \cos 20^\circ)$$

Ans. : (B)

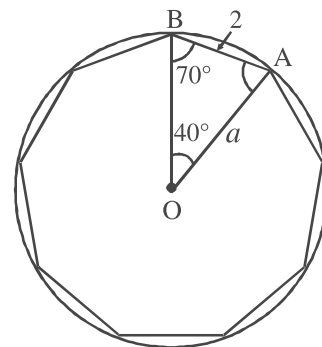


Figure 12.13

(106) The measure of two angles of a triangle are $\frac{\pi}{6}$ and $\frac{\pi}{4}$ and the included side is of length $(\sqrt{3} + 1)$. Then the area of this triangle is

- (A) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (B) $\sqrt{3} + 1$ (C) $\frac{\sqrt{3}+1}{2}$ (D) $\frac{\sqrt{3}+1}{\sqrt{2}}$

Sol. : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Here, $A = \frac{\pi}{4}$, $B = \frac{7\pi}{12}$, $C = \frac{\pi}{6}$

$$\therefore \frac{a}{\sin 45^\circ} = \frac{b}{\sin 105^\circ} = \frac{c}{\sin 30^\circ}$$

$$\therefore \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{c}{\frac{1}{2}} \quad \left(\sin 105^\circ = \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} \right)$$

$$\therefore \sqrt{2}a = \frac{\sqrt{3}+1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = 2c$$

$$\therefore \sqrt{2}a = 2\sqrt{2} = 2c$$

$$\therefore a = 2, c = \sqrt{2}$$

$$\begin{aligned} \therefore \Delta &= \frac{1}{2}ac \sin B \\ &= \frac{1}{2}(2)(\sqrt{2}) \sin 105^\circ \\ &= \sqrt{2} \cos 15^\circ \\ &= \sqrt{2} \left(\frac{\sqrt{6}+\sqrt{2}}{4} \right) = \frac{\sqrt{3}+1}{2} \end{aligned}$$

Ans. : (C)

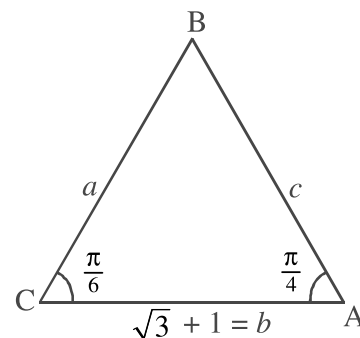


Figure 12.14

(107) If $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$ in ΔABC , then $A = \dots$

- (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Sol. : In ΔABC , let $AB = c$, $BC = a$, $AC = b$,

$$\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$

$$\therefore \frac{2\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\left(\frac{a^2+c^2-b^2}{2ac}\right)}{b} + \frac{2\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c} = \frac{a}{bc} + \frac{b}{ac}$$

$$\therefore \frac{b^2+c^2-a^2}{abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{abc} = \frac{a^2+b^2}{abc}$$

$$\therefore \frac{2b^2}{abc} + \frac{a^2+c^2-b^2}{2abc} = \frac{a^2+b^2}{abc}$$

$$\therefore \frac{a^2 + c^2 - b^2}{2} = a^2 - b^2$$

$$\therefore a^2 + c^2 - b^2 = 2a^2 - 2b^2$$

$$\therefore b^2 + c^2 = a^2$$

$$\therefore A = \frac{\pi}{2}$$

Ans. : (A)

(108) If $a : b : c = 4 : 5 : 6$ in $\triangle ABC$, then the ratio of circumradius and inradius is

(A) $8 : 7$

(B) $12 : 7$

(C) $24 : 7$

(D) $16 : 7$

Sol. : Let $\frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$

$$\therefore a = 4k, b = 5k, c = 6k$$

$$s = \frac{a+b+c}{2} = \frac{15k}{2}$$

$$s - a = \frac{15k}{2} - 4k = \frac{7k}{2}$$

$$s - b = \frac{15k}{2} - 5k = \frac{5k}{2}$$

$$s - c = \frac{15k}{2} - 6k = \frac{3k}{2}$$

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$$= \left(\frac{15k}{2}\right)\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)$$

$$\Delta = \frac{15\sqrt{7}k^2}{4}$$

$$r = \frac{\Delta}{s}$$

$$R = \frac{abc}{4\Delta}$$

$$\therefore \frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{s}{\Delta}$$

$$= \frac{abc \cdot s}{4\Delta^2}$$

$$= \frac{(4k)(5k)(6k)\left(\frac{15k}{2}\right)}{\frac{4 \times 225 \times 7 \times k^4}{16}}$$

$$= \frac{16}{7}$$

Ans. : (D)

Another method :

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\therefore \frac{r}{R} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 4 \sqrt{\frac{(s-b)(s-c)(s-a)(s-a)(s-b)}{bc \cdot ca \cdot ab}}$$

$$= \frac{4(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{4 \frac{105k^3}{8}}{120k^3} = \frac{7}{16}$$

$$\therefore \frac{R}{r} = \frac{16}{7}$$

Ans. : (D)

(109) If three sides of a triangle are of lengths 3, 5, 7, then the measure of the greatest angle is

(A) $\frac{\pi}{2}$

(B) $\frac{5\pi}{6}$

(C) $\frac{2\pi}{3}$

(D) $\frac{3\pi}{4}$

Sol. : $a = 3, b = 5, c = 7$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{9 + 25 - 49}{2(3)(5)} = \frac{-15}{30} = \frac{-1}{2}. \text{ Hence, } C = \frac{2\pi}{3}$$

Ans. : (C)

(110) If in $\triangle ABC$, $m\angle B = \frac{\pi}{3}$, $m\angle C = \frac{\pi}{4}$, D divides $\overline{BC}$ in the ratio 1 : 3 from B, internally, then

$$\frac{\sin m\angle BAD}{\sin m\angle CAD} = \dots\dots$$

(A) $\frac{1}{\sqrt{6}}$

(B) $\frac{1}{3}$

(C) $\frac{1}{\sqrt{3}}$

(D) $\sqrt{\frac{2}{3}}$

Sol. : In $\triangle ABD$, $\frac{\sin(m\angle BAD)}{\sin B} = \frac{x}{AD}$

(1)

In $\triangle ACD$, $\frac{\sin(m\angle CAD)}{\sin C} = \frac{3x}{AD}$

(2)

Taking ratio of (1) and (2),

$$\frac{\sin(m\angle BAD)}{\sin(m\angle CAD)} \cdot \frac{\sin C}{\sin B} = \frac{1}{3}$$

$$\therefore \frac{\sin(m\angle BAD)}{\sin(m\angle CAD)} = \frac{1}{3} \cdot \frac{\sin B}{\sin C}$$

$$= \frac{1}{3} \cdot \frac{\sin \frac{\pi}{3}}{\sin \frac{\pi}{4}}$$

$$= \frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{2}$$

$$= \frac{1}{\sqrt{6}}$$

Ans. : (A)

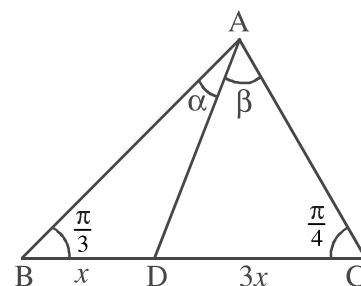


Figure 12.15

(111) If in $\triangle ABC$, the measures of angles are in the ratio 4 : 1 : 1, then the ratio of the greatest side to the perimeter of the triangle is

(A) $\sqrt{3} : (2 + \sqrt{3})$

(B) 1 : 6

(C) $1 : 2 + \sqrt{3}$

(D) 2 : 3

Sol. : The measures of the angles of $\triangle ABC$ are in the ratio 4 : 1 : 1.

$\therefore$ Let $m\angle A = 4x$, $m\angle B = x$, $m\angle C = x$

$\therefore$ In $\triangle ABC$, $m\angle A + m\angle B + m\angle C = \pi$. Hence, $6x = \pi$. So, $x = \frac{\pi}{6}$

$$\therefore m\angle A = \frac{2\pi}{3}, m\angle B = \frac{\pi}{6}, m\angle C = \frac{\pi}{6}$$

$$\text{Now, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin \frac{2\pi}{3}} = \frac{b}{\sin \frac{\pi}{6}} = \frac{c}{\sin \frac{\pi}{6}}$$

$$\text{Let } \frac{a}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{b}{\left(\frac{1}{2}\right)} = \frac{c}{\left(\frac{1}{2}\right)} = k$$

$$\therefore a = \frac{\sqrt{3}k}{2}, b = \frac{k}{2}, c = \frac{k}{2}$$

$$\therefore a + b + c = \frac{\sqrt{3}k + k + k}{2}$$

$$\frac{\text{The length of the greatest side}}{\text{perimeter}} = \frac{\sqrt{3}}{(2 + \sqrt{3})}$$

$$\therefore \text{The ratio is } \sqrt{3} : (2 + \sqrt{3}). \quad \text{Ans. : (A)}$$

(112) If $\cot(\theta - \alpha)$, $3\cot\theta$, $\cot(\theta + \alpha)$ are in A.P. $\theta \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$, then $\sin\theta \cdot \operatorname{cosec}\alpha = \dots$

$$(A) \pm\sqrt{2}$$

$$(B) \pm\frac{\sqrt{3}}{2}$$

$$(C) \pm\frac{\sqrt{2}}{3}$$

(D) None of these

Sol. : $\cot(\theta - \alpha)$, $3\cot\theta$, $\cot(\theta + \alpha)$ are in A.P.

$$\therefore \cot(\theta - \alpha) + \cot(\theta + \alpha) = 6\cot\theta$$

$$\therefore \frac{\cos(\theta - \alpha)}{\sin(\theta - \alpha)} + \frac{\cos(\theta + \alpha)}{\sin(\theta + \alpha)} = 6\cot\theta$$

$$\therefore \frac{\cos(\theta - \alpha)\sin(\theta + \alpha) + \cos(\theta + \alpha)\sin(\theta - \alpha)}{\sin(\theta + \alpha)\sin(\theta - \alpha)} = 6\cot\theta$$

$$\therefore \frac{\sin(\theta + \alpha + \theta - \alpha)}{\sin^2\theta - \sin^2\alpha} = 6\cot\theta$$

$$\therefore \frac{\sin 2\theta}{\sin^2\theta - \sin^2\alpha} = \frac{6\cos\theta}{\sin\theta}$$

$$\therefore \frac{2\sin\theta\cos\theta}{\sin^2\theta - \sin^2\alpha} = \frac{6\cos\theta}{\sin\theta}$$

$$\therefore \sin^2\theta = 3\sin^2\theta - 3\sin^2\alpha$$

$$\therefore 2\sin^2\theta = 3\sin^2\alpha$$

$$\therefore \frac{\sin^2\theta}{\sin^2\alpha} = \frac{3}{2}. \text{ Hence } \sin^2\theta \operatorname{cosec}^2\alpha = \frac{3}{2}$$

$$\therefore \sin\theta \operatorname{cosec}\alpha = \pm\frac{\sqrt{3}}{2}$$

Ans. : (B)

(113) If $P = \tan 6^\circ \tan 42^\circ$ and $Q = \cot 66^\circ \cot 78^\circ$, then

$$(A) P = 2Q$$

$$(B) P = \frac{1}{3}$$

$$(C) P = Q$$

$$(D) 3P = 2Q$$

$$\text{Sol. : } \frac{P}{Q} = \frac{\tan 6^\circ \tan 42^\circ}{\cot 66^\circ \cot 78^\circ}$$

$$= \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$$

$$= \tan 6^\circ [\tan 66^\circ] [\tan(60^\circ - 18^\circ) \tan(60^\circ + 18^\circ)]$$

$$= \frac{\tan 6^\circ [\tan 66^\circ] [\tan 18^\circ \tan(60^\circ - 18^\circ) \tan(60^\circ + 18^\circ)]}{\tan 18^\circ}$$

$$= \frac{\tan 6^\circ \tan 66^\circ \tan 54^\circ}{\tan 18^\circ}$$

$$\left(\tan\theta \tan\left(\frac{\pi}{3} - \theta\right) \tan\left(\frac{\pi}{3} + \theta\right) = \tan 3\theta \right)$$

$$= \frac{\tan 6^\circ \tan(60^\circ - 6^\circ) \tan(60^\circ + 6^\circ)}{\tan 18^\circ}$$

$$= \frac{\tan 18^\circ}{\tan 18^\circ} = 1$$

$$\therefore P = Q$$

Ans. : (C)

$$(114) \text{ In } \triangle ABC, \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \dots$$

(A) $\cos^2 A$

(B) $\sin^2 A$

(C) $1 - \cos A$

(D) $1 + \cos A$

Sol. : $\frac{(2s)(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2}$

$$= \frac{16s(s-a)(s-b)(s-c)}{4b^2c^2} = \frac{4\Delta^2}{b^2c^2} = \frac{4 \cdot \frac{1}{4} b^2 c^2 \sin^2 A}{b^2 c^2} = \sin^2 A$$

Ans. : (B)

$$(115) \text{ If } \tan \theta = \frac{1}{7}, \sin \phi = \frac{1}{\sqrt{10}}, \text{ then } \theta + 2\phi = \dots \left(\theta, \phi \in \left(0, \frac{\pi}{2} \right) \right)$$

(A) $\frac{\pi}{2}$

(B) $\frac{5\pi}{2}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

Sol. : $\tan \theta = \frac{1}{7}$. So, $\sin \theta = \frac{1}{5\sqrt{2}}$ and $\cos \theta = \frac{7}{5\sqrt{2}}$

$\sin \phi = \frac{1}{\sqrt{10}}$. So, $\cos \phi = \frac{3}{\sqrt{10}}$

$$\therefore \sin 2\phi = 2\sin \phi \cos \phi = \frac{6}{10} = \frac{3}{5}, \cos 2\phi = \cos^2 \phi - \sin^2 \phi = \frac{8}{10} = \frac{4}{5}$$

$$0 < \theta < \frac{\pi}{2}, 0 < 2\phi < \pi \Rightarrow 0 < \theta + 2\phi < \frac{3\pi}{2}$$

$$\cos(\theta + 2\phi) = \cos \theta \cos 2\phi - \sin \theta \sin 2\phi = \frac{28}{25\sqrt{2}} - \frac{3}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\cos$ is negative in second and third quadrant.

$$\therefore \theta + 2\phi = \frac{\pi}{4}$$

Ans. : (D)

$$(116) \text{ If } A + B = \frac{\pi}{3} \text{ and } \cos A + \cos B = 1, \text{ then which of the following is true ?}$$

(A) $\cos(A - B) = \frac{1}{3}$ (B) $\cos(A - B) = -\frac{1}{3}$ (C) $|\cos A - \cos B| = \frac{\sqrt{2}}{3}$ (D) $|\cos A - \cos B| = \frac{1}{\sqrt{3}}$

Sol. : $A + B = \frac{\pi}{3} \Rightarrow \cos(A + B) = \cos \frac{\pi}{3}$

$$\cos A + \cos B = 1$$

$$\therefore 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 1$$

$$\therefore 2\cos\frac{\pi}{6}\cos\left(\frac{A-B}{2}\right) = 1 \quad \left(A + B = \frac{\pi}{3}\right)$$

$$\therefore \cos\left(\frac{A-B}{2}\right) = \frac{1}{\sqrt{3}}$$

$$\cos(A-B) = 2\cos^2\left(\frac{A-B}{2}\right) - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

$\therefore$ Correct answer is (B). (A) is not correct.

$$\begin{aligned} |\cos A - \cos B| &= \left| -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \right| \\ &= \left| -2 \sin \frac{\pi}{6} \sin\left(\frac{A-B}{2}\right) \right| \\ &= \left| \sin\left(\frac{A-B}{2}\right) \right| \\ &= \sqrt{1 - \cos^2\left(\frac{A-B}{2}\right)} \\ &= \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}} \end{aligned}$$

$\therefore$ (C) and (D) are not the correct answers.

Ans. : (B)

(117) If $\sum n^2 = \lambda \sum n$, then $\sin^{-1}\left(\frac{9\lambda^2 - 4n^2}{6\lambda + 4n}\right) = \dots$

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) π

Sol. : $\lambda = \frac{\sum n^2}{\sum n} = \frac{\frac{n}{6}(n+1)(2n+1)}{\frac{n}{2}(n+1)} = \frac{2n+1}{3}$

$$\begin{aligned} \sin^{-1}\left(\frac{9\lambda^2 - 4n^2}{6\lambda + 4n}\right) &= \sin^{-1}\left(\frac{(2n+1)^2 - 4n^2}{4n+2+4n}\right) \\ &= \sin^{-1}\left(\frac{4n^2 + 4n + 1 - 4n^2}{8n+2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \end{aligned}$$

Ans. : (A)

(118) If $4n\alpha = \pi$, then $\tan\alpha \tan 2\alpha \tan 3\alpha \dots \tan(2n-1)\alpha = \dots$ ($n \in \mathbb{N}$)

(A) 1

(B) 0

(C) -1

(D) None of these

Sol. : $\tan(2n-1)\alpha = \tan(2n-1)\frac{\pi}{4n} = \tan\left(\frac{\pi}{2} - \frac{\pi}{4n}\right) = \cot\frac{\pi}{4n} = \cot\alpha$

$\therefore \tan(2n-1)\alpha \cdot \tan\alpha = 1$

Similarly, $\tan(2n-2)\alpha = \cot 2\alpha$.

So, $\tan(2n-2)\alpha \cdot \tan 2\alpha = 1$

So, $\tan\alpha \cdot \tan 2\alpha \dots \tan(n-1)\alpha \cdot \tan n\alpha \cdot \tan(n+1)\alpha \dots \tan(2n-2)\alpha \cdot \tan(2n-1)\alpha$

$= \tan n\alpha = \tan \frac{\pi}{4} = 1$

Ans. : (A)

(119) If $\sin x + \cos y = a$ and $\cos x + \sin y = b$, then $\tan\left(\frac{x-y}{2}\right) = \dots$

- (A) $a + b$ (B) $a - b$ (C) $\frac{a+b}{a-b}$ (D) $\frac{a-b}{a+b}$

Sol. : $\sin x + \cos y = a$

$$\therefore \sin x + \sin\left(\frac{\pi}{2} - y\right) = a$$

$$\therefore 2\sin\left(\frac{x + \frac{\pi}{2} - y}{2}\right)\cos\left(\frac{x - \frac{\pi}{2} + y}{2}\right) = a \quad (1)$$

$$\cos x + \sin y = b$$

$$\therefore \cos x + \cos\left(\frac{\pi}{2} - y\right) = b$$

$$\text{and } 2\cos\left(\frac{x + \frac{\pi}{2} - y}{2}\right)\cos\left(\frac{x - \frac{\pi}{2} + y}{2}\right) = b \quad (2)$$

Taking ratio of (1) and (2),

$$\therefore \tan\left(\frac{x-y}{2} + \frac{\pi}{4}\right) = \frac{a}{b}$$

$$\therefore \frac{\tan\frac{\pi}{4} + \tan\left(\frac{x-y}{2}\right)}{1 - \tan\frac{\pi}{4} \tan\left(\frac{x-y}{2}\right)} = \frac{a}{b}$$

$$\therefore \frac{1 + \tan\left(\frac{x-y}{2}\right) + 1 - \tan\left(\frac{x-y}{2}\right)}{1 + \tan\left(\frac{x-y}{2}\right) - 1 + \tan\left(\frac{x-y}{2}\right)} = \frac{a+b}{a-b}$$

$$\therefore \frac{2}{2\tan\left(\frac{x-y}{2}\right)} = \frac{a+b}{a-b}$$

$$\therefore \tan\left(\frac{x-y}{2}\right) = \frac{a-b}{a+b}$$

Ans. : (D)

(120) $(x\tan\alpha + y\cot\alpha)(x\cot\alpha + y\tan\alpha) - 4xy\cot^2 2\alpha$ is independent of

- (A) x (B) y (C) α (D) x, y and α

Sol. : $(x\tan\alpha + y\cot\alpha)(x\cot\alpha + y\tan\alpha) - 4xy\cot^2 2\alpha$

$$= x^2 + xy\tan^2\alpha + xy\cot^2\alpha + y^2 - 4xy\cot^2 2\alpha$$

$$= x^2 + y^2 + xy(\tan^2\alpha + \cot^2\alpha - 4\cot^2 2\alpha)$$

$$= x^2 + y^2 + xy\left(\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha} - 4\frac{\cos^2 2\alpha}{\sin^2 2\alpha}\right)$$

$$= x^2 + y^2 + xy\left(\frac{\sin^4\alpha + \cos^4\alpha}{\sin^2\alpha \cos^2\alpha} - \frac{4\cos^2 2\alpha}{4\sin^2\alpha \cos^2\alpha}\right)$$

$$= x^2 + y^2 + xy\left(\frac{\sin^4\alpha + \cos^4\alpha - (\cos^2\alpha - \sin^2\alpha)^2}{\sin^2\alpha \cos^2\alpha}\right)$$

$$= x^2 + y^2 + xy\left(\frac{\sin^4\alpha + \cos^4\alpha - \cos^4\alpha + 2\sin^2\alpha \cos^2\alpha - \sin^4\alpha}{\sin^2\alpha \cos^2\alpha}\right)$$

$$= x^2 + y^2 + 2xy$$

$$= (x+y)^2 \text{ which is independent of } \alpha.$$

Ans. : (C)

(121) The minimum value of $3^{\sin^6 \alpha} + 3^{\cos^6 \alpha}$ is

- (A) $2 \cdot 3^{\frac{1}{8}}$ (B) $2 \cdot 3^{\frac{7}{8}}$ (C) $3 \cdot 2^{\frac{1}{8}}$ (D) 6

Sol. : We know that, Arithmetic mean $\geq$ Geometric mean

$$\begin{aligned} \therefore \frac{3^{\sin^6 \alpha} + 3^{\cos^6 \alpha}}{2} &\geq \sqrt{3^{\sin^6 \alpha} \cdot 3^{\cos^6 \alpha}} \\ &= \sqrt{3^{\sin^6 \alpha + \cos^6 \alpha}} \end{aligned} \quad (1)$$

We know that, $\sin^6 \alpha + \cos^6 \alpha = (\sin^2 \alpha + \cos^2 \alpha)^3 - 3\sin^2 \alpha \cos^2 \alpha$

$$= 1 - \frac{3}{4} \sin^2 2\alpha \quad (2)$$

From (1) and (2),

$$\therefore 3^{\sin^6 \alpha} + 3^{\cos^6 \alpha} \geq 2\sqrt{3^{1 - \frac{3}{4} \sin^2 2\alpha}}$$

$\sin^2 2\alpha$ should be maximum to get minimum value of $3^{1 - \frac{3}{4} \sin^2 2\alpha}$.

$$\therefore \text{If } \sin^2 2\alpha = 1, \text{ then required minimum value} = 2 \cdot \sqrt{3^{\frac{1}{4}}} = 2 \cdot 3^{\frac{1}{8}}.$$

Ans. : (A)

[**Note :** We get minimum value for $\alpha = \frac{\pi}{4}$]

(122) If $\alpha = \frac{2\pi}{7}$, then $\tan \alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 4\alpha \tan \alpha = \dots$

- (A) -7 (B) -4 (C) 0 (D) 4

Sol. : $\tan \alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 4\alpha \tan \alpha$

$$= \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin 2\alpha}{\cos 2\alpha} + \frac{\sin 2\alpha}{\cos 2\alpha} \cdot \frac{\sin 4\alpha}{\cos 4\alpha} + \frac{\sin 4\alpha}{\cos 4\alpha} \cdot \frac{\sin \alpha}{\cos \alpha}$$

$$= \frac{\sin \alpha \cdot \sin 2\alpha \cdot \cos 4\alpha + \sin 2\alpha \sin 4\alpha \cos \alpha + \sin 4\alpha \sin \alpha \cos 2\alpha}{\cos \alpha \cos 2\alpha \cos 4\alpha}$$

$$= \frac{\cos \alpha \cdot \cos 2\alpha \cos 4\alpha - \cos(\alpha + 2\alpha + 4\alpha)}{\cos \alpha \cos 2\alpha \cos 4\alpha}$$

$$= 1 - \frac{\cos 7\alpha}{\cos \alpha \cos 2\alpha \cos 4\alpha}$$

$$= 1 - \frac{(2\sin \alpha) \cos 2\pi}{2\sin \alpha \cos \alpha \cos 2\alpha \cos 4\alpha} \quad (7\alpha = 2\pi)$$

$$= 1 - \frac{(1)(2\sin \alpha) \times 2}{2\sin 2\alpha \cos 2\alpha \cos 4\alpha}$$

$$= 1 - \frac{4\sin \alpha \times 2}{2\sin 4\alpha \cdot \cos 4\alpha}$$

$$= 1 - \frac{8\sin \alpha}{\sin 8\alpha}$$

$$= 1 - \frac{8 \sin \alpha}{\sin(7\alpha + \alpha)}$$

$$= 1 - \frac{8 \sin \alpha}{\sin(2\pi + \alpha)}$$

$$(7\alpha = 2\pi)$$

$$= 1 - \frac{8 \sin \alpha}{\sin \alpha} = -7$$

Ans. : (A)

(123) If $k_1 = \tan 27\theta - \tan \theta$ and $k_2 = \frac{\sin \theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$, then which of the following is true ?

(A) $k_1 = 2k_2$

(B) $k_1 = k_2$

(C) $k_1 = k_2 + 2$

(D) None of these

Sol. : $\tan 3\theta - \tan \theta = \frac{\sin 3\theta}{\cos 3\theta} - \frac{\sin \theta}{\cos \theta}$

$$= \frac{\sin 3\theta \cos \theta - \cos 3\theta \sin \theta}{\cos 3\theta \cos \theta} = \frac{\sin 2\theta}{\cos 3\theta \cos \theta}$$

$$= \frac{2 \sin \theta \cos \theta}{\cos 3\theta \cos \theta}$$

$$= \frac{2 \sin \theta}{\cos 3\theta}$$

$$\therefore \tan 3\theta - \tan \theta = \frac{2 \sin \theta}{\cos 3\theta}, \tan 9\theta - \tan 3\theta = \frac{2 \sin 3\theta}{\cos 9\theta}, \tan 27\theta - \tan 9\theta = \frac{2 \sin 9\theta}{\cos 27\theta}$$

$$\tan 27\theta - \tan \theta = 2 \left(\frac{\sin \theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} \right)$$

$$\therefore k_1 = 2k_2$$

Ans. : (A)

(124) If $4\cos 36^\circ + \cot\left(7\frac{1}{2}\right)^\circ = \sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4} + \sqrt{n_5} + \sqrt{n_6}$,

then $\sum_{i=1}^6 n_i^2 = \dots\dots$

(A) 81

(B) 71

(C) 91

(D) 61

Sol. : $\cot 7\frac{1}{2}^\circ = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 + \frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

$$= \frac{4 + \sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \times \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}}$$

$$= \frac{4\sqrt{6} + 4\sqrt{2} + 6 + \sqrt{12} + \sqrt{12} + 2}{4}$$

$$= \frac{4\sqrt{6} + 4\sqrt{2} + 8 + 4\sqrt{3}}{4}$$

$$= \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6} \quad (1)$$

$$\cos 36^\circ = \frac{\sqrt{5} + 1}{4}. \text{ So, } 4\cos 36^\circ = \sqrt{5} + 1 \quad (2)$$

Adding (1) and (2),

$$4\cos 36^\circ + \cot 7\frac{1}{2}^\circ = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$$

$$\therefore n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5, n_6 = 6$$

$$\sum_{i=1}^6 n_i^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

Ans. : (C)

(125) $0 < A < \frac{\pi}{2}$

If $\sin A + \cos A + \tan A + \cot A + \sec A + \operatorname{cosec} A = 7$ and $\sin A$ and $\cos A$ are the roots of $4x^2 - 3x + a = 0$, then $25a = \dots$ where $0 < A < \frac{\pi}{2}$

- (A) 21 (B) 24 (C) 28 (D) 30

Sol. : The two roots of $4x^2 - 3x + a = 0$ are $\sin A$ and $\cos A$.

$$\therefore \sin A + \cos A = \frac{3}{4} \text{ and } \sin A \cos A = \frac{a}{4}$$

$$\therefore (\sin A + \cos A) + (\tan A + \cot A) + (\sec A + \operatorname{cosec} A) = 7$$

$$\therefore (\sin A + \cos A) + \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} \right) + \left(\frac{1}{\cos A} + \frac{1}{\sin A} \right) = 7$$

$$\therefore (\sin A + \cos A) + \left(\frac{1}{\sin A \cos A} \right) + \left(\frac{\sin A + \cos A}{\sin A \cos A} \right) = 7$$

$$\therefore \frac{3}{4} + \frac{4}{a} + \frac{3}{a} = 7. \quad \text{So, } \frac{3a + 28}{4a} = 7$$

$$\therefore 3a + 28 = 28a. \quad \text{So, } 25a = 28$$

Ans. : (C)

(126) If $f(\theta) = (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2$, then $f(\theta)$ is not smaller than

- (A) 9 (B) 12 (C) 14 (D) 16

Sol. : $f(\theta) = (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2$
 $= \sin^2 \theta + \cos^2 \theta + \sec^2 \theta + \operatorname{cosec}^2 \theta + 4$
 $= \sin^2 \theta + \cos^2 \theta + 1 + \tan^2 \theta + 1 + \cot^2 \theta + 4$
 $= 7 + \tan^2 \theta + \cot^2 \theta$
 $= 9 + \tan^2 \theta - 2 + \cot^2 \theta$
 $= (\tan \theta - \cot \theta)^2 + 9 \geq 9$

Ans. : (A)

Note : For $\theta = \frac{\pi}{4}$, the value becomes 9.

(127) If $\sin \alpha - \sin \beta = a$, $\cos \alpha + \cos \beta = b$, then $a^2 + b^2$ can not be more than

- (A) 1 (B) 2 (C) 4 (D) 3

Sol. : $a^2 + b^2 = (\sin \alpha - \sin \beta)^2 + (\cos \alpha + \cos \beta)^2$
 $= \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta + \cos^2 \alpha + 2 \cos \alpha \cos \beta + \cos^2 \beta$
 $= 2 + 2 \cos(\alpha + \beta)$
 $= 2 \left(2 \cos^2 \left(\frac{\alpha + \beta}{2} \right) \right) \leq 4$

Ans. : (C)

Note : For $\alpha = \beta = 0$, value becomes 4.

(128) If $A - B = \frac{\pi}{2}$ and A, B are positive, then $(\sin A + \sin B) \cos \frac{\pi}{4}$ can not be more than

- (A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Sol. : $A = \frac{\pi}{2} + B$

$$\begin{aligned}\text{Now, } (\sin A + \sin B) \cos \frac{\pi}{4} \\&= \left(\sin \left(\frac{\pi}{2} + B \right) + \sin B \right) \cos \frac{\pi}{4} \\&= (\cos B + \sin B) \left(\frac{1}{\sqrt{2}} \right)\end{aligned}$$

Maximum value of $\cos B + \sin B$ is $\sqrt{2}$.

$$\therefore (\cos B + \sin B) \left(\frac{1}{\sqrt{2}} \right) \leq (\sqrt{2}) \left(\frac{1}{\sqrt{2}} \right) = 1$$

$\therefore (\sin A + \sin B) \cos \frac{\pi}{4}$ cannot be more than 1 (For $A = \frac{3\pi}{4}$, $B = \frac{\pi}{4}$ will be 1) **Ans. : (A)**

(129) If $\tan A < 0$ in $\triangle ABC$, then $\tan B \tan C$ lies in the interval

- (A) $\left(0, \frac{1}{2}\right)$ (B) $(0, 1)$ (C) $(1, 2)$ (D) None of these

Sol. : $\tan A < 0 \Rightarrow A > \frac{\pi}{2}$

$$\therefore \text{For } \triangle ABC, 0 < B + C < \frac{\pi}{2}$$

$$\therefore \tan(B + C) > 0$$

$$\therefore \frac{\tan B + \tan C}{1 - \tan B \tan C} > 0$$

$$\therefore 0 < \tan B \tan C < 1$$

Ans. : (B)

(130) If $\theta \neq n\pi$, $n \in \mathbb{Z}$, then the range of $\frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$ is

- (A) $(0, \infty)$ (B) $(-\infty, 0) \cup (2, \infty)$ (C) $(-\infty, 1) \cup (2, \infty)$ (D) $(-\infty, 2) \cup (3, \infty)$

Sol. : Let $y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$

$$\therefore y \cos^2 \theta + y \cos \theta = \cos^2 \theta - 1$$

$$\therefore y \cos^2 \theta + y \cos \theta - \cos^2 \theta + 1 = 0$$

$$\therefore y \cos \theta (\cos \theta + 1) - (\cos \theta - 1)(\cos \theta + 1) = 0$$

$$\therefore (\cos \theta + 1)(y \cos \theta - \cos \theta + 1) = 0$$

$$\therefore \cos \theta = -1 \text{ is not possible.}$$

$(\theta \neq n\pi, n \in \mathbb{Z})$

$$\therefore y \cos \theta - \cos \theta + 1 = 0$$

$$\therefore \cos \theta (y - 1) + 1 = 0$$

$$\therefore \cos \theta = \frac{1}{1 - y}$$

$$\theta \neq n\pi \Rightarrow -1 < \cos\theta < 1 \Rightarrow -1 < \frac{1}{1-y} < 1$$

$$\therefore -1 < \frac{1}{1-y} < 0 \quad \text{or} \quad 0 < \frac{1}{1-y} < 1 \quad \left(\frac{1}{1-y} \text{ cannot be } 0.\right)$$

$$\therefore -1 + y > 1 \quad \text{or} \quad 1 < 1 - y$$

$$\therefore y > 2 \quad \text{or} \quad y < 0$$

Ans. : (B)

(131) The minimum value of $\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2}$ for ΔABC is

(A) 1

(B) 0

(C) 2

(D) 3

Sol. : For ΔABC , $A + B + C = \pi$.

$$\text{Hence, } \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

$$\therefore \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} \tan \frac{C}{2} = 1 \quad (1)$$

$$\text{Now, } \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} - 1$$

$$= \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} - \left(\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} \tan \frac{C}{2} \right)$$

$$= \frac{1}{2} \left[\left(\tan \frac{A}{2} - \tan \frac{B}{2} \right)^2 + \left(\tan \frac{B}{2} - \tan \frac{C}{2} \right)^2 + \left(\tan \frac{C}{2} - \tan \frac{A}{2} \right)^2 \right] \geq 0$$

$$\therefore \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 1$$

$$\text{Now, if } \tan \frac{A}{2} = \tan \frac{B}{2} = \tan \frac{C}{2}, \text{ then we get, } A = B = C = \frac{\pi}{3} \text{ and}$$

$$\text{The minimum value of } \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \quad \text{Ans. : (A)}$$

(132) The number of solutions for the equation, $3\cos^{-1}x - \pi x - \frac{\pi}{2} = 0$

(A) only one

(B) no solution

(C) more than one

(D) one or more than one

$$\text{Sol. : } 3\cos^{-1}x = \pi x + \frac{\pi}{2}$$

$$\cos^{-1}x = \frac{\pi}{3}x + \frac{\pi}{6}$$

$$\text{Now, } \cos^{-1}x \text{ is decreasing in } (0, \pi) \text{ and } \frac{\pi}{3}x + \frac{\pi}{6} \text{ is increasing in } (0, \pi).$$

$\therefore$ Their point of intersection will be $\emptyset$ or unique point.

$$\text{For, } x = \frac{1}{2}.$$

$$\text{L.H.S.} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3} \quad \left| \quad \text{R.H.S.} = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3} \right.$$

$$\therefore \text{ There is only one solution } x = \frac{1}{2}.$$

Ans. : (A)

(133) If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} - \dots\right) + \cos^{-1}(1 + b + b^2 + \dots) = \frac{\pi}{2}$, then

- (A) $a = -3, b = 1$ (B) $a = 1, b = -\frac{1}{3}$ (C) $a = \frac{1}{6}, b = \frac{1}{2}$ (D) None of these

Sol. : $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} - \dots\right) + \cos^{-1}(1 + b + b^2 + \dots) = \frac{\pi}{2}$

$$\therefore a - \frac{a^2}{3} + \frac{a^3}{9} - \dots = 1 + b + b^2 + \dots$$

$$\therefore \frac{a}{1 - \left(\frac{-a}{3}\right)} = \frac{1}{1 - b}$$

$$\therefore \frac{3a}{3 + a} = \frac{1}{1 - b}$$

(A) $a = -3, b = 1$

$$\text{L.H.S.} = \frac{3a}{3 + a} = \text{undefined and R.H.S.} = \frac{1}{1 - b} = \text{undefined}$$

$\therefore$ (A) is not possible.

(B) $a = 1, b = -\frac{1}{3}$

$$\text{L.H.S.} = \frac{3}{3 + 1} = \frac{3}{4} \quad \text{and} \quad \text{R.H.S.} = \frac{1}{1 - b} = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4}$$

$\therefore$ (B) is possible.

(C) $a = \frac{1}{6}, b = \frac{1}{2}$

$$\text{L.H.S.} = \frac{3a}{3 + a} = \frac{\frac{1}{2}}{3 + \frac{1}{6}} = \frac{3}{19} \quad \text{and} \quad \text{R.H.S.} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$\therefore$ (C) is not possible.

Ans. : (B)

(134) If for $x \in [0, 2\pi]$, $2^{1 + |\sin x| + |\sin 2x| + |\sin 3x|} = 2$, then

- (A) 0 (B) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ (C) $\frac{2\pi}{3}, \frac{3\pi}{2}$ (D) $\frac{3\pi}{2}, n\pi, n \in \mathbb{Z}$

Sol. : $2^{1 + |\sin x| + |\sin 2x| + |\sin 3x|} = 2^1$

$$\therefore 1 + |\sin x| + |\sin 2x| + |\sin 3x| = 1$$

$$\therefore |\sin x| + |\sin 2x| + |\sin 3x| = 0$$

$$\therefore |\sin x| = |\sin 2x| = |\sin 3x| = 0$$

$$\therefore x = 0$$

Ans. : (A)

: Questions with More Than One Correct Answers :

(135) Which of the following is/are true for $\frac{\tan 3A}{\tan A} = k$, ($k \neq 1$) ?

- (A) $\frac{\cos A}{\cos 3A} = \frac{k^2 - 1}{2k}$ (B) $\frac{\sin 3A}{\sin A} = \frac{2k}{k - 1}$ (C) $k < \frac{1}{3}$ (D) $k > 3$

Sol. : $\frac{\tan 3A}{\tan A} = k$

$$\therefore \frac{\tan 3A - \tan A}{\tan A} = k - 1$$

$$\therefore \frac{\frac{\sin 3A}{\cos 3A} - \frac{\sin A}{\cos A}}{\frac{\sin A}{\cos A}} = k - 1$$

$$\therefore \frac{\sin 2A}{\sin A \cos 3A} = k - 1$$

$$\therefore \frac{2 \sin A \cos A}{\sin A \cos 3A} = k - 1$$

$$\therefore \frac{\cos A}{\cos 3A} = \frac{k - 1}{2} \quad (1)$$

$\therefore$ (A) is false. (because $k \neq 1$)

$$\left(\text{because } \frac{k^2 - 1}{2k} = \frac{k}{2} - \frac{1}{2k} \neq \frac{k}{2} - \frac{1}{2} \right)$$

Now, $\frac{\tan 3A}{\tan A} = k$

$$\therefore \frac{\sin 3A}{\cos 3A} \cdot \frac{\cos A}{\sin A} = k$$

$$\therefore \left(\frac{\cos A}{\cos 3A} \right) \left(\frac{\sin 3A}{\sin A} \right) = k$$

$$\therefore \left(\frac{\sin 3A}{\sin A} \right) \left(\frac{k - 1}{2} \right) = k \quad (\text{From (1)})$$

$$\therefore \frac{\sin 3A}{\sin A} = \frac{2k}{k - 1}$$

$\therefore$ (B) is true.

$$\text{Now, } \frac{3 \sin A - 4 \sin^3 A}{\sin A} = \frac{2k}{k - 1}$$

$$\therefore 3 - 4 \sin^2 A = \frac{2k}{k - 1}$$

$$\therefore 4 \sin^2 A = 3 - \frac{2k}{k - 1} = \frac{k - 3}{k - 1}$$

$\sin A \neq 0, 1$ ($\tan A, \cot A$ are well defined)

Also, $0 < \sin^2 A < 1$

$$\therefore 0 < \frac{k - 3}{k - 1} < 4$$

$$\therefore \left(\frac{k - 3}{k - 1} \right) > 0$$

$$\therefore k < 1 \text{ or } k > 3 \quad (2)$$

$$\frac{k - 3}{k - 1} - 4 < 0$$

$$\therefore \frac{k - 3 - 4k + 4}{k - 1} < 0$$

$$\therefore \frac{-3k + 1}{k - 1} < 0$$

$$\therefore \frac{3k - 1}{k - 1} > 0$$

$$\therefore \frac{k - \frac{1}{3}}{k - 1} > 0$$

$$\therefore k > 1 \text{ or } k < \frac{1}{3} \quad (3)$$

From (2) and (3),

$$k < \frac{1}{3} \text{ or } k > 3$$

$\therefore$ (C) and (D) are also true.

Ans. : (B), (C), (D)

(136) If the geometric mean of $\sin \alpha$ and $\cos \alpha$ is $\cos \beta$, $0 < \alpha, \beta < \frac{\pi}{2}$, then $\cos 2\beta$ is =

- (A) $-2 \sin^2 \left(\frac{\pi}{4} - \alpha \right)$ (B) $-2 \cos^2 \left(\frac{\pi}{4} + \alpha \right)$ (C) $2 \sin^2 \left(\frac{\pi}{4} + \alpha \right)$ (D) $2 \cos^2 \left(\frac{\pi}{4} - \alpha \right)$

Sol. : Geometric mean of $\sin\alpha$ and $\cos\alpha$ is $\cos\beta$.

$$\therefore \cos^2\beta = \sin\alpha \cos\alpha$$

$$\therefore 2\cos^2\beta = 2\sin\alpha \cos\alpha$$

$$\therefore 1 + \cos 2\beta = \sin 2\alpha$$

$$\therefore \cos 2\beta = \sin 2\alpha - 1$$

$$\therefore \cos 2\beta = -(1 - \sin 2\alpha) = -\left(1 - \cos\left(\frac{\pi}{2} - 2\alpha\right)\right)$$

$$\therefore \cos 2\beta = -2\sin^2\left(\frac{\pi}{4} - \alpha\right) = -2\cos^2\left(\frac{\pi}{2} - \frac{\pi}{4} + \alpha\right) = -2\cos^2\left(\frac{\pi}{4} + \alpha\right) \quad \text{Ans. : (A), (B)}$$

(137) If $81^{\sin^2\theta} + 81^{\cos^2\theta} = 30$, then $\theta = \dots\dots$ $\theta \in [0, \pi]$

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$

(D) $\frac{5\pi}{6}$

Sol. : $81^{\sin^2\theta} + 81^{1-\sin^2\theta} = 30$

$$\therefore 81^{\sin^2\theta} + \frac{81}{81^{\sin^2\theta}} = 30$$

Let $81^{\sin^2\theta} = x$

$$\therefore x + \frac{81}{x} = 30$$

$$\therefore x^2 - 30x + 81 = 0$$

$$\therefore x = 27 \text{ or } x = 3$$

$$\therefore 81^{\sin^2\theta} = 27 \text{ or } 81^{\sin^2\theta} = 3. \text{ Hence } 3^{4\sin^2\theta} = 3^3 \text{ or } 3^{4\sin^2\theta} = 3$$

$$\therefore \sin^2\theta = \frac{3}{4} \text{ or } \sin^2\theta = \frac{1}{4}$$

$$\therefore \sin\theta = \pm\frac{\sqrt{3}}{2} \text{ or } \sin\theta = \pm\frac{1}{2}$$

But $\theta \in [0, \pi]$ so $\sin\theta \nless 0$

$$\therefore \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3} \text{ and } \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{Ans. : (A), (B), (C), (D)}$$

(138) satisfies equation $x^3 - \frac{3}{4}x = -\frac{\sqrt{3}}{8}$.

(A) $\cos\frac{5\pi}{18}$

(B) $\cos\frac{7\pi}{18}$

(C) $\cos\left(\frac{23\pi}{18}\right)$

(D) $\cos\left(\frac{17\pi}{18}\right)$

Sol. : $x^3 - \frac{3}{4}x = -\frac{\sqrt{3}}{8}$

$$\therefore 4x^3 - 3x = -\frac{\sqrt{3}}{2}$$

Let $x = \cos\theta$. Hence, $4\cos^3\theta - 3\cos\theta = -\frac{\sqrt{3}}{2}$

$$\begin{aligned} \therefore \cos 3\theta &= -\frac{\sqrt{3}}{2} & \therefore 3\theta &= 2k\pi \pm \frac{5\pi}{6}, \\ \therefore \cos 3\theta &= \cos \frac{5\pi}{6} & \therefore \theta &= \frac{2k\pi}{3} \pm \frac{5\pi}{18}, k \in Z \end{aligned}$$

$$k = 0 \Rightarrow \theta = \pm \frac{5\pi}{18}$$

$$k = 1 \Rightarrow \theta = \frac{2\pi}{3} \pm \frac{5\pi}{18} \Rightarrow \theta = \frac{17\pi}{18} \text{ or } \theta = \frac{7\pi}{18}$$

$$k = 2 \Rightarrow \theta = \frac{4\pi}{3} \pm \frac{5\pi}{18} \Rightarrow \theta = \frac{29\pi}{18} \text{ or } \theta = \frac{19\pi}{18}$$

$\therefore$ For $k > 2$, $\theta = \frac{23\pi}{18}$ is not possible.

$\therefore$ (C) is not possible.

Ans. : (A), (B), (D)

(139) Which of the following numbers is/are rational ?

(A) $\sin\left(\frac{11\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right)$

(B) $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right)$

(C) $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)$

(D) $\left(1 + \cos\frac{2\pi}{9}\right)\left(1 + \cos\frac{4\pi}{9}\right)\left(1 + \cos\frac{8\pi}{9}\right)$

Sol. : (A) $\sin\left(\frac{11\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right) = \sin\left(\pi - \frac{\pi}{12}\right)\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)$

$$= \sin\frac{\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{\pi}{12}\cos\frac{\pi}{12}\right] = \frac{1}{2}\left[\sin\frac{\pi}{6}\right] = \frac{1}{4}. \text{ Hence, (A) is true.}$$

(B) $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right) = \operatorname{cosec}\left(\pi - \frac{\pi}{10}\right)\sec\left(\pi - \frac{\pi}{5}\right)$

$$= -\operatorname{cosec}\frac{\pi}{10}\sec\frac{\pi}{5} = \frac{-1}{\sin\frac{\pi}{10}\cos\frac{\pi}{5}}$$

$$= \frac{-1}{\sin 18^\circ \cos 36^\circ} = \frac{1}{-\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)} = -4 \in \mathbb{Q}$$

$\therefore$ (B) is true.

(C) $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) = 1 - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$

$$= 1 - \frac{1}{2}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 = 1 - \frac{1}{2}\left(\sin\frac{\pi}{4}\right)^2 = 1 - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4} \in \mathbb{Q}$$

(D) $\left(1 + \cos\frac{2\pi}{9}\right)\left(1 + \cos\frac{4\pi}{9}\right)\left(1 + \cos\frac{8\pi}{9}\right) = \left(2\cos^2\frac{\pi}{9}\right)\left(2\cos^2\frac{2\pi}{9}\right)\left(2\cos^2\frac{4\pi}{9}\right)$

$$= 8(\cos 20^\circ \cos 40^\circ \cos 80^\circ)^2$$

$$= 8\left[\frac{1}{2\sin 20^\circ}(2\sin 20^\circ \cos 20^\circ)\cos 40^\circ \cos 80^\circ\right]^2$$

$$= 8\left[\left\{\frac{1}{2\sin 20^\circ}2\sin 40^\circ \cos 40^\circ\right\}\frac{\cos 80^\circ}{2}\right]^2$$

$$= 8\left[\frac{1}{4\sin 20^\circ}(\sin 80^\circ \cos 80^\circ)\right]^2$$

$$= 8\left[\frac{1}{8\sin 20^\circ}(2\sin 80^\circ \cos 80^\circ)\right]^2$$

$$= \frac{1}{8}\left[\frac{\sin 160^\circ}{\sin 20^\circ}\right]^2 = \frac{1}{8} \in \mathbb{Q} \text{ Ans. : (A), (B), (C), (D)}$$

(140) For which of the following intervals, $\sin^6 x + \cos^6 x > \frac{5}{8}$ holds true ?

- (A) $\left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$ (B) $\left(\frac{3\pi}{8}, \frac{5\pi}{8}\right)$ (C) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (D) $\left(\frac{7\pi}{8}, \frac{9\pi}{8}\right)$

Sol. : $\sin^6 x + \cos^6 x > \frac{5}{8} \Leftrightarrow 1 - 3\sin^2 x \cos^2 x > \frac{5}{8}$

$$\Leftrightarrow 3\sin^2 x \cos^2 x < \frac{3}{8}$$

$$\Leftrightarrow \frac{3}{4}(4\sin^2 x \cos^2 x) < \frac{3}{8}$$

$$\Leftrightarrow \frac{3}{4}\sin^2 2x < \frac{3}{8}$$

$$\Leftrightarrow 2\sin^2 2x < 1$$

$$\Leftrightarrow 1 - 2\sin^2 2x > 0$$

$$\Leftrightarrow \cos 4x > 0$$

(A) $-\frac{\pi}{8} < x < \frac{\pi}{8} \Rightarrow -\frac{\pi}{2} < 4x < \frac{\pi}{2}$ Hence, $\cos 4x$ is positive.

(B) $\frac{3\pi}{8} < x < \frac{5\pi}{8} \Rightarrow \frac{3\pi}{2} < 4x < \frac{5\pi}{2}$.

$P(4x)$ is in first or fourth quadrant where $\cos 4x$ is positive.

(C) $\frac{\pi}{4} < x < \frac{3\pi}{4} \Rightarrow \pi < 4x < 3\pi$, where $\cos 4x$ is positive or negative.

$\therefore$ (C) is false.

(D) $\frac{7\pi}{8} < x < \frac{9\pi}{8} \Rightarrow \frac{7\pi}{2} < 4x < \frac{9\pi}{2}$

$P(4x) \in$ first or fourth quadrant where $\cos 4x$ is positive.

Ans. : (A), (B), (D)

(141) If $f(x) = \log\left(\log_{\frac{1}{3}}(\log_7(\sin x + a))\right)$ is not well defined for all $x \in \mathbb{R}$, then $a = \dots$

- (A) 3 (B) 5 (C) 6 (D) 4

Sol. : $\log_{\frac{1}{3}}(\log_7(\sin x + a)) > 0$ is necessary.

$$\Rightarrow 0 < \log_7(\sin x + a) < 1$$

$$\Rightarrow 1 < \sin x + a < 7$$

$$\Rightarrow 1 - a < \sin x < 7 - a \quad \forall x \in \mathbb{R}$$

$$\Rightarrow -1 < 1 - a < \sin x < 7 - a < 1 \text{ should be true.}$$

$$\Rightarrow a < 2 \text{ and } a > 6. \text{ Hence, value of } a \text{ is not possible.}$$

Ans. : (A), (B), (C), (D)

(142) For $0 < \phi < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty} \cos^{2n} \phi$, $y = \sum_{n=0}^{\infty} \sin^{2n} \phi$, $z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$, then

which of the following relation is/are true ?

[IIT 1992]

- (A) $xyz = xz + y$ (B) $xyz = xy + z$ (C) $xyz = x + y + z$ (D) $xyz = yz + x$

Sol. : $x = \sum_{n=0}^{\infty} \cos^{2n} \phi = 1 + \cos^2 \phi + \cos^4 \phi + \dots = \frac{1}{1 - \cos^2 \phi} = \frac{1}{\sin^2 \phi}$

$$y = \sum_{n=0}^{\infty} \sin^{2n} \phi = 1 + \sin^2 \phi + \sin^4 \phi + \dots = \frac{1}{1 - \sin^2 \phi} = \frac{1}{\cos^2 \phi}$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi = 1 + \sin^2\phi \cos^2\phi + \dots = \frac{1}{1 - \sin^2\phi \cos^2\phi}$$

$$\text{Now, } \sin^2\phi = \frac{1}{x}, \cos^2\phi = \frac{1}{y}$$

$$\therefore z = \frac{1}{1 - \frac{1}{x} \cdot \frac{1}{y}} \Rightarrow z = \frac{xy}{xy - 1} \Rightarrow xyz - z = xy \Rightarrow xyz = xy + z \quad (1)$$

Option (B) is possible.

$$x + y = \frac{1}{\sin^2\phi} + \frac{1}{\cos^2\phi} = \frac{1}{\sin^2\phi \cos^2\phi}$$

$$\therefore x + y = xy \quad (2)$$

and from (1) + (2),

$$x + y + xy + z = xy + xyz$$

$$\therefore x + y + z = xyz$$

Ans. : (B), (C)

$$(143) \quad f_n(\theta) = \left(\tan \frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 4\theta) \dots (1 + \sec 2^n\theta) \quad [\text{IIT : 1999}]$$

$$(A) \quad f_2\left(\frac{\pi}{16}\right) = 1 \quad (B) \quad f_3\left(\frac{\pi}{32}\right) = 1 \quad (C) \quad f_4\left(\frac{\pi}{64}\right) = 1 \quad (D) \quad f_5\left(\frac{\pi}{128}\right) = 1$$

$$\text{Sol. : } f_n(\theta) = \left(\tan \frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta) \dots (1 + \sec 2^n\theta)$$

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \left[\left(\frac{1 + \cos\theta}{\cos\theta}\right) \left(\frac{1 + \cos 2\theta}{\cos 2\theta}\right) \dots \left(\frac{1 + \cos 2^n\theta}{\cos 2^n\theta}\right) \right]$$

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \left[\frac{2\cos^2 \frac{\theta}{2}}{\cos\theta} \cdot \frac{2\cos^2\theta}{\cos 2\theta} \dots \frac{2\cos 2^{n-1}\theta}{\cos 2^n\theta} \right]$$

$$= \left(\frac{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos\theta}\right) \cdot \frac{2\cos^2\theta}{\cos 2\theta} \dots \frac{2\cos 2^{n-1}\theta}{\cos 2^n\theta}$$

$$= \left(\frac{\tan\theta \cdot 2\cos^2\theta}{\cos 2\theta}\right) \dots \frac{2\cos 2^{n-1}\theta}{\cos 2^n\theta}$$

$$= \tan 2\theta \dots \frac{2\cos 2^{n-1}\theta}{\cos 2^n\theta} = \tan 2^n\theta$$

$$(A) \quad f_2\left(\frac{\pi}{16}\right) = \tan \frac{4\pi}{16} = 1 \quad (B) \quad f_3\left(\frac{\pi}{32}\right) = \tan \frac{8\pi}{32} = 1$$

$$(C) \quad f_4\left(\frac{\pi}{64}\right) = \tan \frac{16\pi}{64} = 1 \quad (D) \quad f_5\left(\frac{\pi}{128}\right) = \tan \frac{\pi}{4} = 1 \quad \text{Ans. : (A), (B), (C), (D)}$$

(144) If $4\sin^4x + \cos^4x = 1$, then $x = \dots, n \in Z$

- (A) $n\pi$ (B) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$ (C) $\frac{2n\pi}{3}$ (D) $2n\pi \pm \frac{\pi}{4}$

Sol. : $4\sin^4x + \cos^4x = 1$

$$\begin{array}{l|l} \therefore 4\sin^4x = 1 - \cos^4x & \therefore \sin x = 0 \text{ or } \sin x = \pm\sqrt{\frac{2}{5}} \\ \therefore 4\sin^4x = (1 - \cos^2x)(1 + \cos^2x) & \therefore x = n\pi, n \in Z \text{ or } x = n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}} \\ \therefore 4\sin^4x = \sin^2x(2 - \sin^2x) & \\ \therefore \sin^2x(5\sin^2x - 2) = 0 & \end{array}$$

Ans. : (A), (B)

(145) If $\sin^3\theta + \sin\theta \cos\theta + \cos^3\theta = 1$, then $\theta = \dots, (n \in Z)$

- (A) $2n\pi$ (B) $2n\pi + \frac{\pi}{2}$ (C) $2n\pi - \frac{\pi}{2}$ (D) $n\pi$

Sol. : $\sin^3\theta + \cos^3\theta + \sin\theta \cos\theta - 1 = 0$

$$\therefore (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta \cos\theta + \cos^2\theta) - (1 - \sin\theta \cos\theta) = 0$$

$$\therefore (\sin\theta + \cos\theta)(1 - \sin\theta \cos\theta) - (1 - \sin\theta \cos\theta) = 0$$

$$\therefore (1 - \sin\theta \cos\theta)(\sin\theta + \cos\theta - 1) = 0$$

Let $\sin\theta \cos\theta = 1$

$$\therefore 2\sin\theta \cos\theta = 2$$

$\therefore \sin 2\theta = 2$ is not possible.

$\therefore \sin\theta + \cos\theta - 1 = 0 \Leftrightarrow \sin\theta + \cos\theta = 1$ is the only solution.

$$\therefore \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$\therefore \theta = 2n\pi, n \in Z \text{ or } \theta = 2n\pi + \frac{\pi}{2}, n \in Z$$

Ans. : (A), (B)

(146) If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, then $x = \dots, y = \dots, x \in [0, \pi]$

- (A) $y = 0$ (B) $y = 1$ (C) $x = \frac{3\pi}{4}$ (D) $x = \frac{\pi}{4}$

Sol. : $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$ is given. $x \in [0, \pi]$

$$\therefore y + \frac{1}{y} = (\sqrt{y})^2 - 2 + \frac{1}{(\sqrt{y})^2} + 2 = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 + 2 \geq 2$$

$$\therefore \sqrt{y + \frac{1}{y}} \geq \sqrt{2}$$

$$\therefore \sin x + \cos x \geq \sqrt{2}. \text{ But } \sin x + \cos x \leq \sqrt{2}$$

$$\therefore \sin x + \cos x = \sqrt{2}$$

(A) $y = 0$ is not possible.

(B) Taking $y = 1$, $\sin x + \cos x = \sqrt{2}$. Option (B) is true.

(C) Taking $x = \frac{3\pi}{4}$, $\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4} = 0 \neq \sqrt{2}$

$\therefore$ (C) is not possible.

(D) Taking $x = \frac{\pi}{4}$, we get $\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \sqrt{2}$. Option (D) is true. **Ans. : (B), (D)**

(147) For how many values of x and θ , $\sin \theta + \sqrt{3} \cos \theta = 6x - x^2 - 11$, $0 \leq \theta \leq 4\pi$, $x \in \mathbb{R}$, holds true ?

(A) Not true for any x, θ

(B) True for one value of x and two values of θ

(C) True for two values of x and two values of θ

(D) True for two ordered pairs (x, θ)

Sol. : For $f(\theta) = \sin \theta + \sqrt{3} \cos \theta$, $a = \sqrt{3}$, $b = 1$

$$r^2 = a^2 + b^2 = 4 \Rightarrow r = 2. \text{ So, Range is } [-2, 2].$$

$$\therefore -2 \leq 6x - x^2 - 11 \leq 2$$

$$\therefore x^2 - 6x + 9 \leq 0 \text{ and } x^2 - 6x + 13 \geq 0$$

$$\therefore x^2 - 6x + 13 \geq 0 \text{ and } (x - 3)^2 \leq 0$$

$$\therefore x^2 - 6x + 9 + 4 \geq 0 \text{ and } x = 3$$

$$\therefore (x - 3)^2 + 4 \geq 0 \text{ and } x = 3$$

$$\therefore x = 3$$

$$\therefore \sin \theta + \sqrt{3} \cos \theta = -2$$

$$\therefore \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta = -1$$

$$\therefore \sin \theta \sin \frac{\pi}{6} + \cos \theta \cos \frac{\pi}{6} = -1$$

$$\therefore \cos \left(\theta - \frac{\pi}{6} \right) = -1$$

$$\therefore \theta - \frac{\pi}{6} = \pi \text{ or } 3\pi$$

Hence, $x = 3$, $\theta = \frac{7\pi}{6}, \frac{19\pi}{6}$ (0 $\leq \theta \leq 4\pi$) **Ans. : (B), (D)**

(148) If four solutions of the equation $\sin^2 x - 2\sin x - 1 = 0$ are possible for $x \in [0, n\pi]$, then $n = \dots\dots$

(A) 5

(B) 3

(C) 4

(D) 6

Sol. : $\sin^2 x - 2\sin x - 1 = 0$

$$\therefore \sin^2 x - 2\sin x + 1 = 2$$

$$\therefore (\sin x - 1)^2 = 2$$

$$\therefore \sin x - 1 = \pm \sqrt{2}$$

$$\therefore \sin x = 1 \pm \sqrt{2}$$

$$\sin x = 1 + \sqrt{2} \text{ is not possible.}$$

$$\therefore \sin x = 1 - \sqrt{2} < 0$$

Two solutions are possible in $[0, 2\pi]$ and two solutions are possible in $[2\pi, 4\pi]$.

Thus, four solutions are possible in $[0, 4\pi]$.

$$\therefore n = 4$$

Now, for $[0, 5\pi]$ also, four solutions are possible. So, $n = 5$ is also possible. **Ans. : (A), (C)**

(149) If $x + y = \frac{\pi}{4}$ and $\tan x + \tan y = 1$, ($n \in \mathbb{Z}$), then remains true.

(A) $\sin x = 0, \forall x$

(B) $x = n\pi + \frac{\pi}{4} \Rightarrow y = -n\pi, n \in \mathbb{Z}$

(C) $x = n\pi \Rightarrow y = n\pi + \frac{\pi}{4}, \forall n \in \mathbb{Z}$

(D) $x = n\pi + \frac{\pi}{4} \Rightarrow y = n\pi - \frac{\pi}{4}$

Sol. : $\tan x + \tan y = 1$

$$\therefore \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = 1$$

$$\therefore \sin(x + y) = \cos x \cos y$$

$$\therefore 2\sin(x + y) = 2\cos x \cos y$$

$$\therefore 2\sin(x + y) = \cos(x + y) + \cos(x - y)$$

$$\therefore 2\sin \frac{\pi}{4} = \cos \frac{\pi}{4} + \cos(x - y)$$

$$\therefore \sqrt{2} - \frac{1}{\sqrt{2}} = \cos(x - y)$$

$$\therefore \cos(x - y) = \frac{1}{\sqrt{2}}$$

$$\therefore x - y = 2n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z} \quad (1)$$

Now, $x + y = \frac{\pi}{4}$ is given. (2)

Adding (1) and (2),

$$2x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}$$

$$\therefore 2x = 2n\pi + \frac{\pi}{2} \text{ or } 2x = 2n\pi, \forall n \in \mathbb{Z}$$

$$\therefore x = n\pi + \frac{\pi}{4} \text{ or } x = n\pi, n \in \mathbb{Z}$$

(i) If $x = n\pi + \frac{\pi}{4}$, then $\tan y = 0$

because $\tan x = 1$. Hence $y = -n\pi, \forall n \in \mathbb{Z}$

(ii) If $x = n\pi$, then $\tan y = 1$

$$\therefore y = n\pi + \frac{\pi}{4}, n \in \mathbb{Z} \quad \text{Ans. : (B), (C)}$$

(150) If $x \in \left(0, \frac{\pi}{2}\right)$, then which of the following values of x satisfies the equation

$$\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2} ?$$

(A) $\frac{\pi}{12}$

(B) $\frac{5\pi}{12}$

(C) $\frac{7\pi}{24}$

(D) $\frac{11\pi}{36}$

Sol. : $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$

$$\therefore \frac{\sqrt{3}-1}{(2\sqrt{2})\sin x} + \frac{(\sqrt{3}+1)}{\cos x(2\sqrt{2})} = 2$$

$$\therefore \frac{\sin \frac{\pi}{12}}{\sin x} + \frac{\cos \frac{\pi}{12}}{\cos x} = 2$$

$$\therefore \cos x \sin \frac{\pi}{12} + \sin x \cos \frac{\pi}{12} = 2\sin x \cos x$$

$$\therefore \sin\left(x + \frac{\pi}{12}\right) = \sin 2x$$

(Note : From here, verification is possible by substituting values of x .)

$$\therefore x + \frac{\pi}{12} = 2x \Rightarrow x = \frac{\pi}{12}$$

$$\left(x \in \left(0, \frac{\pi}{2}\right)\right)$$

Now, $\sin \theta = \sin(\pi - \theta)$

$$\therefore 2x = \pi - x - \frac{\pi}{12}$$

$$\therefore 3x = \frac{11\pi}{12}$$

$$\therefore x = \frac{11\pi}{36}$$

Ans. : (A), (D)

(151) If $\frac{\tan x}{1} = \frac{\tan y}{2} = \frac{\tan z}{3}$ ($\neq 0$) and $x + y + z = \pi$, then

(A) The maximum value of $\tan x + \tan y + \tan z$ is 6.

(B) The minimum value of $\tan x + \tan y + \tan z$ is -6.

(C) $\tan x = \pm 1$, $\tan y = \pm 2$, $\tan z = \pm 3$

(D) $\forall x, y, z \in \mathbb{R}$, $\tan x + \tan y + \tan z = 0$

Sol. : $\frac{\tan x}{1} = \frac{\tan y}{2} = \frac{\tan z}{3} = k \Rightarrow \tan x = k, \tan y = 2k, \tan z = 3k$

Now, $x + y + z = \pi$

$$\therefore \tan x + \tan y + \tan z = \tan x \tan y \tan z$$

$$\therefore k + 2k + 3k = (k)(2k)(3k)$$

$$\therefore 6k = 6k^3$$

$$\therefore 6k^3 - 6k = 0$$

$$\therefore 6k(k^2 - 1) = 0$$

$$\therefore k = 0 \text{ or } k = \pm 1. \text{ But } k \neq 0.$$

$$\therefore \tan x + \tan y + \tan z \neq 0$$

$$\therefore k = \pm 1$$

$$\therefore \tan x = \pm 1, \tan y = \pm 2, \tan z = \pm 3$$

$$\therefore \text{The maximum value of } \tan x + \tan y + \tan z = 6$$

$$\therefore \text{The minimum value of } \tan x + \tan y + \tan z = -6$$

Ans. : (A), (B), (C)

(152) $\cos \alpha = \frac{3}{5}$, $\cos \beta = \frac{5}{13}$, $0 < \alpha, \beta < \frac{\pi}{2}$

$$(A) \cos(\alpha + \beta) = \frac{33}{65} \quad (B) \sin(\alpha + \beta) = \frac{56}{65} \quad (C) \sin^2\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{65} \quad (D) \cos(\alpha - \beta) = \frac{63}{65}$$

Sol. : $\cos \alpha = \frac{3}{5} \Rightarrow \sin \alpha = \frac{4}{5}$, $\cos \beta = \frac{5}{13} \Rightarrow \sin \beta = \frac{12}{13}$ $(0 < \alpha, \beta < \frac{\pi}{2})$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = -\frac{33}{65}$$

$\therefore$ (A) is not true.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{56}{65}$$

$\therefore$ (B) is true.

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{63}{65}$$

$\therefore$ (D) is true.

$$\sin^2\left(\frac{\alpha - \beta}{2}\right) = \frac{1 - \cos(\alpha - \beta)}{2} = \frac{1 - \frac{63}{65}}{2} = \frac{1}{65}$$

$\therefore$ (C) is true.

Ans. : (B), (C), (D)

(153) If $\sin^6 x + \cos^6 x = a^2$ has a real solution, then $a \in$

(A) $[-1, 1]$

(B) $\left[-1, -\frac{1}{2}\right]$

(C) $\left[\frac{-1}{2}, \frac{1}{2}\right]$

(D) $\left[\frac{1}{2}, 1\right]$

Sol. : $\sin^6 x + \cos^6 x = a^2$

$$\therefore 1 - 3\sin^2 x \cos^2 x = a^2$$

$$\therefore \sin^2 x \cos^2 x = \frac{1-a^2}{3}$$

$$\therefore \frac{4\sin^2 x \cos^2 x}{4} = \frac{1-a^2}{3}$$

$$\therefore \sin^2 2x = \frac{4(1-a^2)}{3}. \text{ Now } 0 \leq \sin^2 2x \leq 1$$

$$\therefore 0 \leq \frac{4(1-a^2)}{3} \leq 1. \text{ Hence, } 0 \leq 4 - 4a^2 \leq 3$$

$$\therefore -4 \leq -4a^2 \leq -1. \text{ Hence, } 4 \geq 4a^2 \geq 1$$

$$\therefore \frac{1}{4} \leq a^2 \leq 1. \text{ Hence, } |a| \leq 1 \text{ and } |a| \geq \frac{1}{2}$$

$$\therefore a \in \left[\frac{1}{2}, 1\right] \text{ or } a \in \left[-1, -\frac{1}{2}\right]$$

Ans. : (B), (D)

(154) If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$ ($b \neq 0$), then

$$(A) \cos\left(\frac{\theta-\phi}{2}\right) = \pm \frac{1}{2}\sqrt{a^2+b^2}$$

$$(B) \cos\left(\frac{\theta-\phi}{2}\right) = \pm \frac{1}{2}\sqrt{a^2-b^2}$$

$$(C) \tan\left(\frac{\theta-\phi}{2}\right) = \pm \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

$$(D) \cos(\theta - \phi) = \frac{a^2+b^2-2}{2}$$

Sol. : Here $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$.

$$\therefore 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = a \text{ and } 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = b$$

$$4\sin^2\left(\frac{\theta+\phi}{2}\right)\cos^2\left(\frac{\theta-\phi}{2}\right) + 4\cos^2\left(\frac{\theta+\phi}{2}\right)\cos^2\left(\frac{\theta-\phi}{2}\right) = a^2 + b^2$$

$$\therefore 4\cos^2\left(\frac{\theta-\phi}{2}\right) = a^2 + b^2$$

$$\therefore \cos\left(\frac{\theta-\phi}{2}\right) = \pm \frac{1}{2}\sqrt{a^2+b^2}$$

$\therefore$ (A) is true.

$$\text{Now } \cos(\theta - \phi) = 2\cos^2\left(\frac{\theta-\phi}{2}\right) - 1 = \frac{2}{4}(a^2 + b^2) - 1 = \frac{a^2 + b^2 - 2}{2}$$

$\therefore$ (D) is true.

$$\text{Now } \tan^2\left(\frac{\theta-\phi}{2}\right) = \frac{1 - \cos(\theta-\phi)}{1 + \cos(\theta-\phi)} = \frac{1 - \left(\frac{a^2+b^2-2}{2}\right)}{1 + \left(\frac{a^2+b^2-2}{2}\right)} = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\therefore \tan\left(\frac{\theta-\phi}{2}\right) = \pm \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

$\therefore$ (C) is true.

Ans. : (A), (C), (D)

$$(155) \quad \frac{3 + \cot 76^\circ \cot 16^\circ}{\cot 76^\circ + \cot 16^\circ} = \dots\dots$$

- (A) $\tan 16^\circ$ (B) $\cot 76^\circ$ (C) $\tan 46^\circ$ (D) $\cot 44^\circ$

$$\begin{aligned} \text{Sol. : } \frac{3 + \cot 76^\circ \cot 16^\circ}{\cot 76^\circ + \cot 16^\circ} &= \frac{3 + \frac{\cos 76^\circ \cos 16^\circ}{\sin 76^\circ \sin 16^\circ}}{\frac{\cos 76^\circ}{\sin 76^\circ} + \frac{\cos 16^\circ}{\sin 16^\circ}} \\ &= \frac{3 \sin 76^\circ \sin 16^\circ + \cos 76^\circ \cos 16^\circ}{\sin(76^\circ + 16^\circ)} \\ &= \frac{2 \sin 76^\circ \sin 16^\circ + (\cos 76^\circ \cos 16^\circ + \sin 76^\circ \sin 16^\circ)}{\sin 92^\circ} \\ &= \frac{2 \sin 76^\circ \sin 16^\circ + \cos(76^\circ - 16^\circ)}{\sin 92^\circ} \\ &= \frac{-\cos 92^\circ + \cos 60^\circ + \cos 60^\circ}{\sin 92^\circ} \\ &= \frac{1 - \cos 92^\circ}{\sin 92^\circ} = \frac{2 \sin^2 46^\circ}{2 \sin 46^\circ \cos 46^\circ} = \tan 46^\circ = \cot 44^\circ \quad \text{Ans. : (C), (D)} \end{aligned}$$

$$(156) \quad \text{If } \tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}, \text{ then } \dots\dots$$

- (A) $\sin \alpha - \cos \alpha = \pm \sqrt{2} \sin \theta$ (B) $\sin \alpha + \cos \alpha = \pm \sqrt{2} \cos \theta$
(C) $\cos 2\theta = \sin 2\alpha$ (D) $\sin 2\theta + \cos 2\alpha = 0$

$$\text{Sol. : (i) } \tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha} = \frac{\tan \alpha - 1}{\tan \alpha + 1}$$

$$\begin{aligned} \therefore \tan \theta &= \tan \left(\alpha - \frac{\pi}{4} \right) & \left. \begin{aligned} \therefore \sin 2\theta &= \sin \left(2k\pi + 2\alpha - \frac{\pi}{2} \right) \\ \therefore \sin 2\theta &= \sin \left(2\alpha - \frac{\pi}{2} \right) \\ \therefore \sin 2\theta &= -\cos 2\alpha. \end{aligned} \right\} \text{Hence, (D) is true.} \\ \therefore \theta &= k\pi + \alpha - \frac{\pi}{4}, \quad k \in \mathbb{Z} & (1) \\ \therefore 2\theta &= 2k\pi + 2\alpha - \frac{\pi}{2} \end{aligned}$$

$$(ii) \quad \cos 2\theta = \cos \left(2k\pi + 2\alpha - \frac{\pi}{2} \right)$$

$$\therefore \cos 2\theta = \cos \left(-\frac{\pi}{2} + 2\alpha \right)$$

$$\therefore \cos 2\theta = \sin 2\alpha$$

$\therefore$ (C) is true.

$$\begin{aligned} \text{Now, } \sin \alpha - \cos \alpha &= \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin \alpha - \frac{1}{\sqrt{2}} \cos \alpha \right) = \sqrt{2} \sin \left(\alpha - \frac{\pi}{4} \right) = \sqrt{2} \sin(\theta - k\pi) \quad (\text{From (1)}) \\ &= \pm \sqrt{2} \sin \theta \end{aligned}$$

$\therefore$ (A) is true.

$$\text{Now, } \sin \alpha + \cos \alpha = \sqrt{2} \cos \left(\alpha - \frac{\pi}{4} \right) = \sqrt{2} \cos(\theta - k\pi) = \pm \sqrt{2} \cos \theta$$

$\therefore$ (B) is true.

Ans. : (A), (B), (C), (D)

- (157) If in $\triangle ABC$, $m\angle A = 90^\circ$, c , $\sin B$, $\cos B$ are rational numbers, then
- (A) a is rational (B) a is irrational (C) b is rational (D) b is irrational

Sol. : In $\triangle ABC$, $\cos B = \frac{c}{a}$

$$\therefore a = \frac{c}{\cos B}$$

Now, c and $\cos B$ are rationals.

$\therefore a$ is rational.

$$\sin B = \frac{b}{a}$$

$$\therefore a = \frac{b}{\sin B} \Rightarrow b = a \sin B$$

Now, a and $\sin B$ are rational.

$\therefore b$ is rational.

Ans. : (A), (C)

- (158) If $\cos x + \sec x = -2$, then for positive integers n , $\cos^n x + \sec^n x = \dots$

- (A) 2, $\forall n \in \mathbb{Z}$ (B) -2, $\forall n \in \mathbb{Z}$ (C) -2, n is odd (D) 2, n is even

Sol. : $\cos x + \sec x = -2$

$$\therefore \cos x + \frac{1}{\cos x} = -2 \quad \left| \quad \therefore (\cos x + 1)^2 = 0 \right.$$

$$\therefore \cos^2 x + 2\cos x + 1 = 0 \quad \left| \quad \therefore \cos x = -1 \text{ and } \sec x = (-1) \right.$$

$$\cos^n x + \sec^n x = (-1)^n + (-1)^n$$

If n is even, then $\cos^n x + \sec^n x = 2$ and n is odd, then $\cos^n x + \sec^n x = -2$ **Ans.** : (C), (D)

- (159) If $\left(\frac{\sin \theta}{\sin \phi} \right)^2 = \frac{\tan \theta}{\tan \phi} = 3$, then

- (A) $\tan \phi = \frac{1}{\sqrt{3}}$ (B) $\tan \phi = -\frac{1}{\sqrt{3}}$ (C) $\tan \theta = \sqrt{3}$ (D) $\tan \theta = -\sqrt{3}$

Sol. : $\left(\frac{\sin \theta}{\sin \phi} \right)^2 = \frac{\tan \theta}{\tan \phi}$

$$\therefore \left(\frac{\sin \theta}{\sin \phi} \right)^2 = \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \phi}{\sin \phi}$$

$$\therefore \frac{\sin \theta}{\sin \phi} = \frac{\cos \phi}{\cos \theta} \Rightarrow \sin \theta \cos \theta = \sin \phi \cos \phi$$

$$\therefore \sin 2\theta = \sin 2\phi$$

$$\therefore \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \tan \phi}{1 + \tan^2 \phi}$$

$$\therefore \frac{6 \tan \phi}{1 + 9 \tan^2 \phi} = \frac{2 \tan \phi}{1 + \tan^2 \phi}$$

$$(\tan \theta = 3 \tan \phi)$$

$$\therefore 3(1 + \tan^2 \phi) = 1 + 9 \tan^2 \phi$$

$$(\tan \phi \neq 0)$$

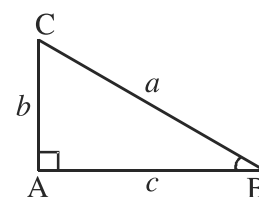


Figure 12.16

$$\therefore 6\tan^2\phi = 2. \quad \text{So, } \tan^2\phi = \frac{1}{3}$$

$$\therefore \tan\phi = \pm\frac{1}{\sqrt{3}} \text{ and } \tan\theta = \pm\sqrt{3}$$

Ans. : (A), (B), (C), (D)

(160) If $x = a\cos^3\theta \sin^2\theta$ and $y = a\sin^3\theta \cos^2\theta$ and $\frac{(x^2 + y^2)^p}{(xy)^q}$ ($p, q \in \mathbb{N}$) is independent of θ .
Then

(A) $p = 4$

(B) $p = 5$

(C) $q = 4$

(D) $q = 5$

Sol. : $x = a\cos^3\theta \sin^2\theta$ and $y = a\sin^3\theta \cos^2\theta$ is given.

$$\therefore (x^2 + y^2) = a^2\cos^6\theta \sin^4\theta + a^2\sin^6\theta \cos^4\theta = a^2\cos^4\theta \sin^4\theta$$

$$\therefore \frac{(x^2 + y^2)^p}{(xy)^q} = \frac{(a^2\cos^4\theta \sin^4\theta)^p}{(a^2\cos^5\theta \cdot \sin^5\theta)^q} \text{ is independent of } \theta.$$

$$\therefore \text{Index of } \cos = 0 \text{ and index of } \sin = 0$$

$$\therefore 4p - 5q = 0$$

$$\therefore \text{If } p = 5 \text{ and } q = 4, \text{ then } 4p - 5q = 0.$$

Ans. : (B), (C)

(161) $2^{\csc^2 x} \sqrt{\frac{1}{2}y^2 - y + 1} \leq \sqrt{2}$ for $0 \leq x \leq 2\pi$

(A) is satisfied by only one value of y .

(B) is satisfied by two values of x .

(C) holds for $\cos x = 0$.

(D) holds for $\sin x = 0$.

Sol. : $2^{\csc^2 x} \sqrt{\frac{y^2 - 2y + 2}{2}} \leq \sqrt{2}$

$$\therefore 2^{\csc^2 x} \sqrt{y^2 - 2y + 1 + 1} \leq 2$$

$$\therefore 2^{\csc^2 x} \sqrt{(y-1)^2 + 1} \leq 2 \quad (1)$$

$$\text{Now, } 2^{\csc^2 x} \geq 2 \text{ and } \sqrt{(y-1)^2 + 1} \geq 1 \text{ So,} \quad (2)$$

$$2^{\csc^2 x} \sqrt{(y-1)^2 + 1} \geq 2. \quad (3)$$

$$\therefore 2^{\csc^2 x} \sqrt{(y-1)^2 + 1} = 2 \quad (\text{From (1) and (3)})$$

Now, it is clear from (2) that,

$$\csc^2 x = 1 \text{ and } \sqrt{(y-1)^2 + 1} = 1$$

$$\therefore \sin^2 x = 1 \text{ and } (y-1)^2 + 1 = 1$$

$$\therefore \cos x = 0 \text{ and } y = 1$$

$\therefore$ Only one value of y and $\cos x = 0$ satisfies given relation.

$\therefore$ (A) and (C) are the solution.

Now, $\cos x = 0$, $0 \leq x \leq 2\pi$. Hence, $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore$ Two values of x satisfy given relation.

$\therefore$ (B) is also possible.

Ans. : (A), (B), (C)

(162) If $S_n = \cot^{-1}(3) + \cot^{-1}(7) + \cot^{-1}(13) + \cot^{-1}(21) + \dots + n$ terms, then

(A) $S_{10} = \tan^{-1} \frac{5}{6}$

(B) $S_6 = \sin^{-1} \frac{3}{5}$

(C) $S_{\infty} = \frac{\pi}{4}$

(D) $S_{20} = \cot^{-1} \left(\frac{11}{10} \right)$

Sol. : $S_n = \cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \dots + \text{upto } n \text{ terms}$

$$= \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{13} + \dots + \text{upto } n \text{ terms}$$

$$= \tan^{-1} \frac{1}{1+1 \cdot 2} + \tan^{-1} \frac{1}{1+2 \cdot 3} + \tan^{-1} \frac{1}{1+3 \cdot 4} + \dots + \text{upto } n \text{ terms}$$

$$= \tan^{-1} \left(\frac{2-1}{1+1 \cdot 2} \right) + \tan^{-1} \left(\frac{3-2}{1+2 \cdot 3} \right) + \tan^{-1} \left(\frac{4-3}{1+3 \cdot 4} \right) + \dots + \tan^{-1} \left(\frac{(n+1)-n}{1+(n+1)n} \right)$$

$$= \tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \dots + \tan^{-1}n - \tan^{-1}(n-1) + \tan^{-1}(n+1) - \tan^{-1}n$$

$$S_n = \tan^{-1}(n+1) - \frac{\pi}{4} \quad (1)$$

Now, taking $n = 10$ in (1),

$$S_{10} = \tan^{-1}11 - \tan^{-1}1 = \tan^{-1} \left(\frac{11-1}{1+11} \right) = \tan^{-1} \frac{5}{6}$$

$\therefore$ (A) is true.

Let $n \rightarrow \infty$ in (1),

$$S_{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore$ (C) is true.

Let $n = 6$ in (1),

$$\begin{aligned} S_6 &= \tan^{-1}7 - \tan^{-1}1 \\ &= \tan^{-1} \left(\frac{7-1}{1+7} \right) = \tan^{-1} \frac{3}{4} = \sin^{-1} \frac{3}{5} \end{aligned}$$

$\therefore$ (B) is true.

Let $n = 20$ in (1),

$$\begin{aligned} S_{20} &= \tan^{-1}21 - \tan^{-1}1 \\ &= \tan^{-1} \left(\frac{21-1}{1+21} \right) \\ &= \tan^{-1} \left(\frac{10}{11} \right) = \cot^{-1} \left(\frac{11}{10} \right) \end{aligned}$$

$\therefore$ (D) is true.

Ans. : (A), (B), (C), (D)

(163) $2\tan^{-1}(-2) = \dots\dots$

(A) $-\cos^{-1}\left(\frac{-3}{5}\right)$ (B) $-\pi + \cos^{-1}\left(\frac{3}{5}\right)$ (C) $-\frac{\pi}{2} + \tan^{-1}\left(\frac{-3}{4}\right)$ (D) $-\pi + \cot^{-1}\left(\frac{-3}{4}\right)$

Sol. : $2\tan^{-1}(-2) = \tan^{-1}(-2) + \tan^{-1}(-2)$
 $= -[\tan^{-1}2 + \tan^{-1}2]$
 $= -\left[\pi + \tan^{-1}\left(\frac{2+2}{1-4}\right)\right]$
 $= -\left[\pi + \tan^{-1}\left(-\frac{4}{3}\right)\right]$
 $= -\pi + \tan^{-1}\frac{4}{3}$ (1)
 $= -\pi + \cos^{-1}\frac{3}{5} = -\left(\pi - \cos^{-1}\frac{3}{5}\right) = -\cos^{-1}\left(\frac{-3}{5}\right)$

$\therefore$ (A) and (B) are true.

Now, from (1),

$$-\pi + \tan^{-1}\frac{4}{3} = -\pi + \cot^{-1}\frac{3}{4} = -\pi + \frac{\pi}{2} - \tan^{-1}\frac{3}{4} = -\frac{\pi}{2} + \tan^{-1}\left(\frac{-3}{4}\right)$$

$\therefore$ (C) is true. (Why not (D) ?)

Ans. : (A), (B), (C)

(164) Which of the following is positive ?

(A) $\cos(\tan^{-1}(\tan 4))$ (B) $\sin(\cot^{-1}(\cot 4))$ (C) $\tan(\cos^{-1}(\cos 5))$ (D) $\cot(\sin^{-1}(\sin 4))$

Sol. : $\tan 4 = \tan(4 - \pi)$

$$\therefore \cos(\tan^{-1}(\tan 4)) = \cos(\tan^{-1}(\tan(4 - \pi))) = \cos(4 - \pi) > 0, \quad 0 < 4 - \pi < \frac{\pi}{2}$$

$\therefore$ The number in (A) is positive.

Now, $\sin(\cot^{-1}(\cot 4)) = \sin(\cot^{-1}(\cot(4 - \pi))) = \sin(4 - \pi) > 0$

$\therefore$ The number in (B) is positive.

Now, $\cos 5 = \cos(2\pi - 5)$

$$\therefore \tan(\cos^{-1}(\cos 5)) = \tan(\cos^{-1}(\cos(2\pi - 5))) = \tan(2\pi - 5) > 0 \quad \left(0 < 2\pi - 5 < \frac{\pi}{2}\right)$$

The number in (C) is positive.

Now, $\cot(\sin^{-1}(\sin 4)) = \cot(\sin^{-1}(-\sin(4 - \pi)))$

$$= -\cot(4 - \pi) \text{ is negative, } 0 < 4 - \pi < \frac{\pi}{2} \quad \text{Ans. : (A), (B), (C)}$$

(165) If $z = \sec^{-1}\left(x + \frac{1}{x}\right) + \sec^{-1}\left(y + \frac{1}{y}\right)$, then possible values of z are where, $xy < 0$

(A) $\frac{8\pi}{10}$ (B) $\frac{7\pi}{10}$ (C) $\frac{9\pi}{10}$ (D) $\frac{21\pi}{20}$

Sol. : $xy < 0$

$$\therefore x + \frac{1}{x} \geq 2, y + \frac{1}{y} \leq -2 \text{ or } x + \frac{1}{x} \leq -2 \text{ and } y + \frac{1}{y} \geq 2$$

$$x + \frac{1}{x} \geq 2 \Rightarrow \sec^{-1}\left(x + \frac{1}{x}\right) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$$

$$y + \frac{1}{y} \leq -2 \Rightarrow \sec^{-1}\left(y + \frac{1}{y}\right) \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$$

$$\therefore z \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right)$$

$$\frac{4\pi}{5} < \frac{5\pi}{6}, \frac{7\pi}{10} < \frac{5\pi}{6}, \frac{5\pi}{6} < \frac{9\pi}{10}, \frac{9\pi}{10} < \frac{7\pi}{6}, \quad \frac{5\pi}{6} < \frac{21\pi}{20} < \frac{7\pi}{6}$$

Ans. : (C), (D)

(166) In $\triangle ABC$, if $c \sin A < a < c$, then (b_1, b_2 are two possible values of b .)

$$(A) b_1 + b_2 = 2c \cos A \quad (B) b_1 + b_2 = c \cos A \quad (C) b_1 b_2 = c^2 - a^2 \quad (D) b_1 b_2 = c^2 + a^2$$

$$\text{Sol. : } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore 2bccosA = b^2 + c^2 - a^2$$

$$\therefore b^2 - 2bccosA + (c^2 - a^2) = 0 \text{ is a quadratic equation in } b, \text{ whose roots are } b_1 \text{ and } b_2.$$

$$b_1 + b_2 = 2ccosA, \quad b_1 b_2 = c^2 - a^2$$

Ans. : (A), (C)

(167) If $2a^2 + 4b^2 + c^2 = 4ab + 2ca$ in $\triangle ABC$, then

(A) $\triangle ABC$ is an isosceles triangle

(B) $\triangle ABC$ is an obtuse angled triangle.

$$(C) B = \cos^{-1}\left(\frac{7}{8}\right)$$

$$(D) A = \cos^{-1}\frac{1}{4}$$

$$\text{Sol. : } 2a^2 + 4b^2 + c^2 - 4ab - 2ca = 0$$

$$\therefore a^2 - 4ab + 4b^2 + a^2 - 2ac + c^2 = 0$$

$$\therefore (a - 2b)^2 + (a - c)^2 = 0$$

$$\therefore a = 2b \text{ and } a = c$$

Now, $a = c \Rightarrow \triangle ABC$ is an isosceles.

(1)

$$\text{Taking } a = 2b, \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{4b^2 + 4b^2 - b^2}{8b^2} = \frac{7}{8} > 0$$

$$\therefore B = \cos^{-1}\left(\frac{7}{8}\right).$$

So, Angle B is acute.

(2)

$$\text{Now, } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{4}$$

So, $A = \cos^{-1}\frac{1}{4}$. So, A, C are acute angle.

(3)

$\therefore$ Options (A), (C), (D) are true.

Ans. : (A), (C), (D)

(168) If $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$ in $\triangle ABC$, then $B = \dots$

$$(A) \frac{\pi}{4}$$

$$(B) \frac{3\pi}{4}$$

$$(C) \frac{2\pi}{3}$$

$$(D) \frac{\pi}{3}$$

$$\text{Sol. : } (a^2 - b^2 + c^2)^2 = a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 + 2a^2c^2$$

$$\therefore (a^2 - b^2 + c^2)^2 = 2a^2c^2 \quad (a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 = 0)$$

$$\therefore \frac{a^2 + c^2 - b^2}{2ac} = \pm \frac{1}{\sqrt{2}} = \cos B$$

$$\therefore B = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Ans. : (A), (B)

- (169) If the lengths of the sides of a right angled triangle are in G.P., then *cosine* values of acute angles are

(A) $\frac{\sqrt{5}-1}{2}$ (B) $\frac{\sqrt{5}+1}{2}$ (C) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (D) $\sqrt{\frac{\sqrt{5}+1}{2}}$

Sol. : a, b, c are in G.P.

$$\therefore b^2 = ac \text{ and } c^2 = a^2 + b^2$$

$$\therefore c^2 = a^2 + ac$$

$$\therefore \left(\frac{a}{c}\right)^2 + \left(\frac{a}{c}\right) - 1 = 0. \text{ Hence, } \frac{a}{c} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\frac{a}{c} = \frac{\sqrt{5}-1}{2} \text{ because } a > 0, c > 0$$

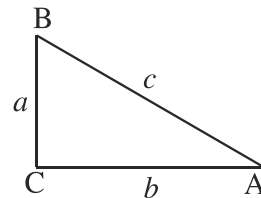


Figure 12.17

$$\cos B = \frac{\sqrt{5}-1}{2} \text{ and } \cos A = \sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{2}\right)^2} = \sqrt{\frac{\sqrt{5}-1}{2}} \quad \text{Ans. : (A), (C)}$$

- (170) If the lengths of the sides of $\triangle ABC$ are in A.P. where $a < \min \{b, c\}$, then which of the following can be a value of $\cos A$?

(A) $\frac{4c-3b}{2b}$ (B) $\frac{4c-3b}{2c}$ (C) $\frac{3c-4b}{2}$ (D) $\frac{4b-3c}{2b}$

Sol. : The lengths of the sides of $\triangle ABC$ are a, b, c .

Now, $a < \min \{b, c\}$

$\therefore$ (i) a, b, c are in A.P. or (ii) a, c, b are in A.P.

(i) If a, b, c are in A.P., then

$$a + c = 2b$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{b^2 + c^2 - (2b - c)^2}{2bc} \\ &= \frac{b^2 + c^2 - 4b^2 + 4bc - c^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c} \end{aligned}$$

(ii) If a, c, b are in A.P., then

$$a + b = 2c$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{b^2 + c^2 - (2c - b)^2}{2bc} \\ &= \frac{b^2 + c^2 - 4c^2 + 4bc - b^2}{2bc} = \frac{4bc - 3c^2}{2bc} = \frac{4b - 3c}{2b} \end{aligned}$$

Ans. : (B), (D)

(171) If $2b = a + c$ for sides a, b, c of ΔABC , then

- (A) $\frac{b}{c} > \frac{2}{3}$ (B) $\frac{b}{c} < \frac{3}{2}$ (C) $\frac{b}{c} > \frac{1}{3}$ (D) $\frac{b}{c} < 2$

Sol. : $a + c = 2b$

We know that, for ΔABC , $a + b > c$.

$$\therefore 2b - c + b > c \quad (a = 2b - c)$$

$$\therefore 3b > 2c$$

$$\therefore \frac{b}{c} > \frac{2}{3} > \frac{1}{3} \quad (1)$$

Now, $b + c > a$

$$\therefore b + c > 2b - c \quad (a = 2b - c)$$

$$\therefore 2c > b$$

$$\therefore \frac{b}{c} < 2 \quad (2)$$

If $a + c > b$, then $2b > b$ is true.

Ans. : (A), (C), (D)

(172) The solutions of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in $\theta \in [0, 2\pi]$ are

- (A) $\frac{2\pi}{3}$ (B) $\frac{4\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{3}$

Sol. : $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$

$$\therefore \cos\theta = -\frac{5}{4} \text{ or } \cos\theta = -\frac{1}{2}$$

$$\cos\theta = -\frac{5}{4} < -1 \text{ is not possible. So, } \cos\theta = -\frac{1}{2}$$

$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ and } \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{Ans. : (A), (B)}$$

(173) $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \dots$ [JEE : 2013]

- (A) $\sin A \cos A + 1$ (B) $\sec A \operatorname{cosec} A + 1$ (C) $\tan A + \cot A$ (D) $\sec A + \operatorname{cosec} A$

Sol. : $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$

$$= \frac{\frac{\sin A}{\cos A}}{1 - \frac{\cos A}{\sin A}} + \frac{\frac{\cos A}{\sin A}}{1 - \frac{\sin A}{\cos A}}$$

$$= \frac{\sin^2 A}{(\sin A - \cos A)\cos A} + \frac{\cos^2 A}{(\cos A - \sin A)\sin A}$$

$$= \frac{\sin^2 A}{(\sin A - \cos A)\cos A} - \frac{\cos^2 A}{(\sin A - \cos A)\sin A}$$

$$= \frac{\sin^3 A - \cos^3 A}{(\sin A - \cos A)\sin A \cos A}$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A \cos A + \cos^2 A)}{(\sin A - \cos A)\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A} = 1 + \sec A \operatorname{cosec} A$$

Ans. : (B)

$$(174) \quad \cot \left[\sum_{n=1}^{23} \cot^{-1} \left[1 + \sum_{k=1}^n 2k \right] \right] = \dots$$

[JEE Adv. : 2013]

(A) $\frac{23}{25}$

(B) $\frac{25}{23}$

(C) $\frac{23}{24}$

(D) $\frac{24}{23}$

Sol. : $\cot \left[\sum_{n=1}^{23} \cot^{-1} \left[1 + \sum_{k=1}^n 2k \right] \right]$

$$= \cot \left[\sum_{n=1}^{23} \cot^{-1} (1 + n^2 + n) \right] \quad \left(\sum n = \frac{n}{2}(n+1) \right)$$

$$= \cot \left[\sum_{n=1}^{23} \tan^{-1} \left(\frac{1}{1 + n(1 + n)} \right) \right] = \cot \left[\sum_{n=1}^{23} \tan^{-1} \left(\frac{(1 + n) - n}{1 + n(1 + n)} \right) \right]$$

$$= \cot \left[\sum_{n=1}^{23} \left(\tan^{-1} (1 + n) - \tan^{-1} n \right) \right]$$

$$= \cot [(\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + \dots + (\tan^{-1} 24 - \tan^{-1} 23)]$$

$$= \cot [\tan^{-1} 24 - \tan^{-1} 1]$$

$$= \cot \left[\tan^{-1} \left(\frac{24 - 1}{1 + 24} \right) \right]$$

$$= \cot \left(\tan^{-1} \frac{23}{25} \right) = \frac{25}{23}$$

Ans. : (B)

(175) If $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, $x \in \mathbb{R}$, $k \geq 1$, then $f_4(x) - f_6(x) = \dots$

[JEE : 2014]

(A) $\frac{1}{4}$

(B) $\frac{1}{12}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

Sol. : $f_4(x) - f_6(x)$

$$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\cos^6 x + \sin^6 x)$$

$$= \frac{1}{4}[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] - \frac{1}{6}[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x]$$

$$= \frac{1}{4}[1 - 2\sin^2 x \cos^2 x] - \frac{1}{6}[1 - 3\sin^2 x \cos^2 x]$$

$$= \frac{1}{4} - \frac{1}{2}\sin^2 x \cos^2 x - \frac{1}{6} + \frac{1}{2}\sin^2 x \cos^2 x = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Ans. : (B)

- (176) The number of values of x for which equation $f(x) = \sin(x^2) + \cos(x^2)$, $x \in [-\sqrt{13}, \sqrt{13}]$ is maximum are [JEE Adv. : 2014]

(A) 8 (B) 2 (C) 4 (D) 0

Sol. : $f(x) = \sin(x^2) + \cos(x^2)$

$$\begin{aligned} &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin(x^2) + \frac{1}{\sqrt{2}} \cos(x^2) \right] \\ &= \sqrt{2} \left[\sin(x^2) \sin \frac{\pi}{4} + \cos(x^2) \cos \frac{\pi}{4} \right] \\ &= \sqrt{2} \left[\cos \left(x^2 - \frac{\pi}{4} \right) \right] \end{aligned}$$

For maximum value, $\cos \left(x^2 - \frac{\pi}{4} \right) = 1$

$$x^2 = \frac{\pi}{4}, \frac{9\pi}{4} \Rightarrow x = \pm \sqrt{\frac{\pi}{4}}, \pm \frac{3\sqrt{\pi}}{2} \quad (x \in [-\sqrt{13}, \sqrt{13}])$$

$\therefore$ The number of solutions = 4

Ans. : (C)

- (177) The number of positive real solutions of $\tan^{-1} \left(\frac{1}{2x+1} \right) + \tan^{-1} \left(\frac{1}{4x+1} \right) = \tan^{-1} \frac{2}{x^2}$ are [JEE Adv. : 2014]

(A) 1 (B) 2 (C) 3 (D) 4

Sol. : $\tan^{-1} \left(\frac{1}{2x+1} \right) + \tan^{-1} \left(\frac{1}{4x+1} \right) = \tan^{-1} \frac{2}{x^2}$

$$\therefore \tan^{-1} \left(\frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \left(\frac{1}{2x+1} \right) \left(\frac{1}{4x+1} \right)} \right) = \tan^{-1} \frac{2}{x^2} \quad \left(x > 0, \frac{1}{2x+1} < 1, \frac{1}{4x+1} < 1 \right)$$

$$\therefore \tan^{-1} \left(\frac{4x+1+2x+1}{(2x+1)(4x+1)-1} \right) = \tan^{-1} \frac{2}{x^2}$$

$$\therefore \frac{6x+2}{8x^2+6x} = \frac{2}{x^2}$$

$$\therefore \frac{3x+1}{4x^2+3x} = \frac{2}{x^2}$$

$$\therefore 3x^3 + x^2 = 8x^2 + 6x$$

$$\therefore 3x^3 - 7x^2 - 6x = 0$$

$$\therefore x(3x^2 - 7x - 6) = 0$$

Since, $x > 0$, $3x^2 - 7x - 6 = 0$

$$\therefore x = 3 \text{ or } x = -\frac{2}{3}. \text{ But } x > 0$$

$$\therefore x = 3$$

$\therefore$ Only one positive solution is possible.

Ans. : (A)

: Assertion and Reason Type Questions :

Explanation : Each of these question contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

- (A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.
 (B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.
 (C) Statement 1 is true, statement 2 is false.
 (D) Statement 1 is false, statement 2 is true.

(178) **Statement 1 :** $\cos 1 < \cos 7$

Statement 2 : $1 < 7$

Sol. : $\cos 7 = \cos(2\pi - 7) = \cos(7 - 2\pi)$

$$0 < 7 - 2\pi < 1 < \frac{\pi}{2}$$

$$\therefore \cos(2\pi - 7) > \cos 1$$

($\cos$ is decreasing in first quadrant)

$$\therefore \cos 7 > \cos 1$$

$\therefore$ Statement 1 is true.

Statement 2, $1 < 7$ is true.

Thus, statements 1 and 2 both are true, but statement 2 is not the correct explanation of statement 1.

Ans. : (B)

(179) **Statement 1 :** The minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is $\frac{1}{243}$.

Statement 2 : The minimum value of $a \cos \theta + b \sin \theta$ is $-\sqrt{a^2 + b^2}$.

Sol. : Statement 1 : $27^{\cos 2x} \cdot 81^{\sin 2x}$

$$\text{Now, } 3^{3\cos 2x} \cdot 3^{4\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$$

Statement 2 : The minimum value of $f(\theta) = a \cos \theta + b \sin \theta$ is $-\sqrt{a^2 + b^2}$ is true.

$$\text{So, from statement 2, minimum value of } 3\cos 2x + 4\sin 2x = -\sqrt{a^2 + b^2} = -\sqrt{9 + 16} = -5$$

$$\therefore \text{Minimum value of } 3^{3\cos 2x + 4\sin 2x} \text{ is } = 3^{-5} = \frac{1}{243}.$$

$\therefore$ Statement 1 is true.

Thus, statements 1 and 2 both are true. Statement 2 is the correct explanation of statement 1.

Ans. : (A)

(180) **Statement 1 :** $\tan 5^\circ$ is an irrational number.

Statement 2 : $\tan 15^\circ$ is an irrational number.

Sol. : $\tan 15^\circ = 2 - \sqrt{3}$ is an irrational number. So, statement 2 is true.

$$\tan 15^\circ = \frac{3\tan 5^\circ - \tan^3 5^\circ}{1 - 3\tan^2 5^\circ}$$

If $\tan 5^\circ$ is rational, then $\tan 15^\circ$ is rational, which is not true.

$\therefore \tan 5^\circ$ is irrational. So, statement 1 is true.

$\therefore$ Statement 2 is the correct explanation of statement 1.

Ans. : (A)

(181) **Statement 1** : $\sin 3 < \sin 1 < \sin 2$

Statement 2: $\sin x$ is positive in the first and the second quadrant.

Sol. : Statement 2 : $\sin x$ is positive in the first and the second quadrant is true.

$$\text{Statement 1 : } \sin 3 = \sin(\pi - 3)$$

$$\sin 2 = \sin(\pi - 2)$$

$$0 < \pi - 3 < 1 < \pi - 2 < \frac{\pi}{2} \text{ as } 3 < \pi < 4$$

$\sin$ is increasing in the first quadrant.

$$\therefore \sin(\pi - 3) < \sin 1 < \sin(\pi - 2)$$

$$\therefore \sin 3 < \sin 1 < \sin 2$$

Statement 1 is true.

It is not sufficient that $\sin$ is positive in the Ist and IInd quadrant for statement 1 to be true.

$\therefore$ Statement 2 is not a correct explanation of statement 1.

Ans. : (B)

(182) **Statement 1** : $\sin \frac{\pi}{18}$ is one of the roots of the equation $8x^3 - 6x + 1 = 0$.

Statement 2 : For $\theta \in \mathbb{R}$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Sol. : Statement 2 : For $\theta \in \mathbb{R}$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ is true.

$$\text{Statement 1 : } 8x^3 - 6x + 1 = 0$$

$$\therefore 3x - 4x^3 = \frac{1}{2}$$

$$\therefore 3\sin\theta - 4\sin^3\theta = \frac{1}{2}$$

(Taking, $x = \sin\theta$)

$$\therefore \sin 3\theta = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\therefore \text{One of the solutions is } \theta = \frac{\pi}{18}$$

$\therefore \sin \frac{\pi}{18}$ is a solution of the given equation. Statement 1 is true.

Statement 2 is the correct explanation of statement 1.

Ans. : (A)

(183) **Statement 1** : If $\sum_{i=1}^n \sin^2\theta_i = 0$, then the distinct n -tuples $(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$ for which

$$\cos\theta_1 + \cos\theta_2 + \dots + \cos\theta_n = n - 4 \text{ is } \frac{n(n-1)}{2}.$$

Statement 2 : $\sum_{i=1}^n \sin^2\theta_i = 0 \Rightarrow \cos\theta_1, \cos\theta_2, \dots, \cos\theta_n = \pm 1$

Sol. : Statement 2 : $\sum_{i=1}^n \sin^2\theta_i = 0$

$$\therefore \sin^2\theta_1 + \sin^2\theta_2 + \dots + \sin^2\theta_n = 0$$

$$\therefore \sin\theta_1 = \sin\theta_2 = \dots = \sin\theta_n = 0$$

$$\therefore \cos^2\theta_1 = \cos^2\theta_2 = \dots = \cos^2\theta_n = 1$$

$$\therefore \cos\theta_1, \cos\theta_2, \dots, \cos\theta_n, \pm 1$$

$\therefore$ Statement 2 is true.

Now, Statement 1 : $\sum_{i=1}^n \sin^2 \theta_i = 0$

$\therefore$ We get $\cos \theta_1, \cos \theta_2, \dots, \cos \theta_n, \pm 1$

Now, $\cos \theta_1 + \cos \theta_2 + \dots + \cos \theta_n = n - 4$ is possible if values of $(n - 2) \cos \theta_i$ is 1 and remaining two values are -1.

Thus, out of n numbers selection of two -1 values can be done in ${}_nC_2 = \frac{n(n-1)}{2}$.

$\therefore$ Statement 1 is true. Statement 2 is true.

Statement 2 is the correct explanation of statement 1.

Ans. : (A)

(184) **Statement 1 :** $\tan 3.5 < \tan 8$

Statement 2 : $\tan$ is increasing in all the quadrants.

Sol. : Statement 2 : $\tan$ is increasing in all the quadrants is true.

Statement 1 : P(3.5) lies in the third quadrant, where $\tan$ is positive. $\left(\pi < 3.5 < \frac{3\pi}{2}\right)$

$$\frac{5\pi}{2} < 8 < 3\pi$$

$\therefore$ P(8) lies in second quadrant.

$$\therefore \tan 8 < 0$$

$$\therefore \tan 3.5 > \tan 8$$

$\therefore$ Statement 2 is true, Statement 1 is false.

Ans. : (D)

(185) **Statement 1 :** Range of $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is $(0, \pi)$.

Statement 2 : $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x = \frac{\pi}{2} + \tan^{-1}x, \forall x \in [-1, 1]$

Sol. : Statement 2 : $\forall x \in [-1, 1], \sin^{-1}x + \cos^{-1}x + \tan^{-1}x = \frac{\pi}{2} + \tan^{-1}x$ is true.

Statement 1 : For $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ to be defined, $-1 \leq x \leq 1$ is necessary.

$$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \text{ and } -\frac{\pi}{4} \leq \tan^{-1}x \leq \frac{\pi}{4}$$

$$\therefore \frac{\pi}{4} \leq \sin^{-1}x + \cos^{-1}x + \tan^{-1}x \leq \frac{3\pi}{4}$$

$$\therefore \text{Range of } \sin^{-1}x + \cos^{-1}x + \tan^{-1}x \text{ is } \left[\frac{\pi}{4}, \frac{3\pi}{4}\right].$$

$\therefore$ Statement 1 is false. Statement 2 is true.

Ans. : (D)

(186) **Statement 1 :** The number of solutions of $\cos^{-1}x + \cos^{-1}2x + \pi = 0$ is 0.

Statement 2 : Range of $\cot^{-1}x$ is $(0, \pi)$ and range of $\cos^{-1}x$ is $[0, \pi]$.

Sol. : Statement 2 : Range of $\cot^{-1}x$ is $(0, \pi)$ and range of $\cos^{-1}x$ is $[0, \pi]$. This Statement is true.

Statement 1 : From Statement 2, it is clear that,

$$\cos^{-1}x + \cos^{-1}2x = -\pi \text{ is not possible.}$$

$\therefore$ The number of solutions of this equation is 0.

So, Statement 1 is true. Statement 2 is the correct explanation of Statement 1. **Ans. : (A)**

(187) **Statement 1** : $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement 2 : $\sin^{-1}x > \tan^{-1}y$, where $x > y$, $\forall x, y \in (0, 1)$

Sol. : Statement 2 : $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$

$\therefore$ Statement 2 is true.

Statement 1 : $e < \pi \Rightarrow \frac{1}{\sqrt{\pi}} < \frac{1}{\sqrt{e}}$

$\therefore \sin^{-1}\frac{1}{\sqrt{e}} > \tan^{-1}\frac{1}{\sqrt{\pi}}$, where $0 < \frac{1}{\sqrt{\pi}} < \frac{1}{\sqrt{e}} < 1$. Statement 1 is true.

$\therefore$ Statement 2 is the correct explanation of Statement 1.

Ans. : (A)

(188) **Statement 1** : $\operatorname{cosec}^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$

Statement 2 : $\operatorname{cosec}^{-1}x < \sec^{-1}x$, where $1 < x < \sqrt{2}$

Sol. : Statement 2 : We have $1 < x < \sqrt{2}$.

$$\operatorname{cosec}^{-1}x < \sec^{-1}x \Leftrightarrow \operatorname{cosec}^{-1}x < \frac{\pi}{2} - \operatorname{cosec}^{-1}x$$

$$\Leftrightarrow 2\operatorname{cosec}^{-1}x < \frac{\pi}{2}$$

$$\Leftrightarrow \operatorname{cosec}^{-1}x < \frac{\pi}{4}$$

$$\Leftrightarrow 1 < x < \sqrt{2}$$

$\therefore$ Statement 2 is true.

$$\frac{1}{2} + \frac{1}{\sqrt{2}} \in (1, \sqrt{2})$$

$\therefore$ It follows that $\operatorname{cosec}^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) < \sec^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$.

$\therefore$ Statement 1 is false.

Statement 2 is true, Statement 1 is false.

Ans. : (D)

(189) For $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$,

Statement 1 : $f'(2) = -\frac{2}{5}$

Statement 2 : $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x$, $\forall x > 1$

Sol. : Statement 2 : Let $x = \tan\theta$. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We get $\theta = \tan^{-1}x$.

Since $x > 1$ we get, $\tan\theta > 1$

$$\therefore \frac{\pi}{4} < \theta < \frac{\pi}{2}$$

$$\therefore \frac{\pi}{2} < 2\theta < \pi$$

$$\therefore -\frac{\pi}{2} < 2\theta - \pi < 0$$

$$\sin(2\theta - \pi) = -\sin 2\theta$$

$$\begin{aligned} \therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) &= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = -\sin^{-1}(-\sin 2\theta) = -\sin^{-1}(\sin(2\theta - \pi)) \\ &= \pi - 2\theta = \pi - 2\tan^{-1}x \end{aligned}$$

$\therefore$ Statement 2 is true.

$$\text{Statement 1 : } f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$x = 2 \Rightarrow x > 1$$

$$\therefore f(x) = \pi - 2\tan^{-1}x$$

$$f'(x) = \frac{-2}{1+x^2}$$

$$f'(2) = -\frac{2}{5}$$

Statement 1 is true.

Statement 1 and 2 both are true. Statement 2 is the correct explanation of Statement 1.

Ans. : (A)

$$(190) \text{ Statement 1 : } \cos^{-1}(\cos(30)) = 30 - 9\pi$$

$$\text{Statement 2 : } 30 - 9\pi \in [0, \pi]$$

$$\text{Sol. : Statement 2 : } 0 < 30 - 9\pi < \pi \text{ as } 3 < \pi < \frac{10}{3}$$

$\therefore$ Statement 2 is true.

$$\text{Statement 1 : } \cos(30 - 9\pi) = -\cos 30$$

$$\begin{aligned} \therefore \cos^{-1}(\cos 30) &= \cos^{-1}(-\cos(30 - 9\pi)) \\ &= \pi - \cos^{-1}(\cos(30 - 9\pi)) \\ &= \pi - 30 + 9\pi \\ &= 10\pi - 30 \end{aligned}$$

$\therefore$ Statement 2 is true, Statement 1 is false.

Ans. : (D)

$$(191) \text{ Statement 1 : } \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}$$

$$\text{Statement 2 : } \prod_{i=0}^{n-1} \cos 2^i \theta = -\frac{1}{2^n} \text{ where } \theta = \frac{\pi}{2^n - 1}$$

$$\text{Sol. : } \cos \theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin(2^n \theta)}{2^n \sin \theta}$$

$$= \frac{\sin\left(2^n \cdot \frac{\pi}{2^n - 1}\right)}{2^n \sin\left(\frac{\pi}{2^n - 1}\right)} = \frac{\sin\left(\pi + \frac{\pi}{2^n - 1}\right)}{2^n \sin\left(\frac{\pi}{2^n - 1}\right)} = -\frac{1}{2^n}$$

∴ Statement 2 is true.

Now, let $n = 3$ in the Statement 2,

$$\cos\theta\cos2\theta\cos2^2\theta = \cos\theta\cos2\theta\cos4\theta = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}. \text{ where } \theta = \frac{\pi}{2^3-1} = \frac{\pi}{7}$$

$$\therefore \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$$

∴ Statement 1 is true.

Statement 2 is the correct explanation of Statement 1.

Ans. : (A)

(192) **Statement 1 :** $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is not true for real $x, y, x \neq y$, where $\theta \in \mathbb{R}$

Statement 2 : $\forall \theta, \sec^2\theta \geq 1$

Sol. : Statement 2 : $\forall \theta, \sec^2\theta \geq 1$ is true.

Statement 1 : Now, let $\sec^2\theta = \frac{4xy}{(x+y)^2}$

$$\text{Also, } \sec^2\theta \geq 1 \Leftrightarrow \frac{4xy}{(x+y)^2} \geq 1$$

$$\Leftrightarrow 4xy \geq x^2 + 2xy + y^2$$

$$\Leftrightarrow x^2 - 2xy + y^2 \leq 0$$

$$\Leftrightarrow (x - y)^2 \leq 0$$

$$\Leftrightarrow x - y = 0$$

$$\Leftrightarrow x = y \neq 0$$

∴ If $(x = y) \neq 0$, then $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is possible. Hence, different x, y are not possible.

∴ Statement 1 is true.

Statement 2 is the correct explanation of Statement 1.

Ans. : (A)

(193) **Statement 1 :** The maximum and the minimum values of $f(x) = \frac{1}{6\sin x - 8\cos x + 5}$ are not possible.

Statement 2 : The function $f(x)$ is unbounded.

Sol. : Statement 1 : $f(x) = \frac{1}{6\sin x - 8\cos x + 5}$

$$\therefore g(x) = 6\sin x - 8\cos x$$

$$\text{Range of } g(x) = 6\sin x - 8\cos x \text{ is } \left[-\sqrt{36+64}, \sqrt{36+64}\right] = [-10, 10]$$

$$-10 \leq 6\sin x - 8\cos x \leq 10$$

$$-5 \leq 6\sin x - 8\cos x + 5 \leq 15$$

$$\therefore -5 \leq \frac{1}{f(x)} \leq 15$$

$$\therefore f(x) \leq -\frac{1}{5} \text{ or } f(x) \geq \frac{1}{15}$$



$\therefore$ Range of $f(x)$ is $\mathbb{R} - \left(-\frac{1}{5}, \frac{1}{15}\right)$.

$\therefore f$ is an unbounded function.

$\therefore$ Statement 2 is true.

If f is unbounded, maximum or minimum values are not possible.

$\therefore$ Statement 1 is true. Statement 2 is the correct explanation of Statement 1. **Ans. : (A)**

(194) **Statement 1 :** $\tan\alpha + 2\tan2\alpha + 4\tan4\alpha + 8\tan8\alpha + 16\cot16\alpha = \cot\alpha$

Statement 2 : $\cot\alpha - \tan\alpha = 2\cot2\alpha$

Sol. : Statement 2 : $\cot\alpha - \tan\alpha = \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha \cos\alpha} = \frac{\cos2\alpha}{\frac{1}{2} \cdot \sin2\alpha} = 2\cot2\alpha$

Statement 2 is true.

Statement 1 : L.H.S. $= \tan\alpha + 2\tan2\alpha + 4\tan4\alpha + 8\tan8\alpha + 16\cot16\alpha$
 $= \tan\alpha + 2\tan2\alpha + 4\tan4\alpha + 8\tan8\alpha + 8(\cot8\alpha - \tan8\alpha)$
(From (1), $2\cot16\alpha = \cot8\alpha - \tan8\alpha$)
 $= \tan\alpha + 2\tan2\alpha + 4\tan4\alpha + 8\cot8\alpha$
 $= \tan\alpha + 2\tan2\alpha + 4\tan4\alpha + 4(\cot4\alpha - \tan4\alpha)$
 $= \tan\alpha + 2\tan2\alpha + 4\cot4\alpha$
 $= \tan\alpha + 2\tan2\alpha + 2(\cot2\alpha - \tan2\alpha)$
 $= \tan\alpha + 2\cot2\alpha$
 $= \tan\alpha + (\cot\alpha - \tan\alpha)$
 $= \cot\alpha$

$\therefore$ Statement 1 is true, Statement 2 is the correct explanation of Statement 1 **Ans. : (A)**

(195) **Statement 1 :** If $\square ABCD$ is cyclic, then $\sin A + \sin B + \sin C + \sin D = 0$

Statement 2 : If $\square ABCD$ is cyclic, then $\cos A + \cos B + \cos C + \cos D = 0$

Sol. : $\square ABCD$ is cyclic.

$A + C = \pi$ and $B + D = \pi$	
$A = \pi - C$	$B = \pi - D$
$\cos A = \cos(\pi - C)$	$\cos B = \cos(\pi - D)$
$\therefore \cos A = -\cos C$	$\cos B = -\cos D$
$\therefore \cos A + \cos C = 0$	$\Rightarrow \cos B + \cos D = 0$
$\therefore \cos A + \cos B + \cos C + \cos D = 0$	

$\therefore$ Statement 2 is true.

Also, $\sin A = \sin(\pi - C)$	$\sin B = \sin(\pi - D)$
$\therefore \sin A = \sin C$	$\sin B = \sin D$
$\therefore \sin A - \sin C = 0$ and $\sin B - \sin D = 0$	
$\therefore \sin A + \sin B - \sin C - \sin D = 0$	

Statement 1 is false.

Ans. : (D)

- (196) A circle is drawn inside an equilateral triangle whose side has length a . A square is drawn inside the circle.

Statement 1 : The area of the square = $\frac{a^2}{6}$ sq. units

Statement 2 : The radius of the circle and length of the side of the square are same.

Sol. : Statement 2 : Let the length of the side of the square be x unit.

$$x^2 + x^2 = 4r^2$$

$$\therefore 2x^2 = 4r^2. \text{ Hence, } x = \sqrt{2} r$$

$\therefore$ Statement 2 is false. (So answer has to be (C).)

$$\begin{aligned} \text{Statement 1 : } r &= \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2}} \quad (\Delta ABC \text{ is equilateral.}) \\ &= \frac{a}{2\sqrt{3}} \end{aligned}$$

$$\therefore x^2 = \frac{2a^2}{12}. \Rightarrow x^2 = \frac{a^2}{6}$$

$$\therefore \text{ The area of the square } = \frac{a^2}{6}.$$

Statement 1 is true.

Ans. : (C)

- (197) A circle is inscribed in a right angled triangle, where B is right angle.

Statement 1 : The diameter of the circle is $AB + BC - AC$.

Statement 2 : The radius of the circle = $\frac{\text{The area of the triangle}}{\text{Semi perimeter of the triangle}}$

Sol. : Statement 2 : The radius of the circle = $\frac{\text{The area of the triangle}}{\text{Semi perimeter of the triangle}}$

$$\therefore r = \frac{\Delta}{s} \text{ is true.}$$

$\therefore$ Statement 2 is true.

ΔABC is right angled.

$$\text{So, } \Delta = \frac{1}{2}ac$$

$$\therefore r = \frac{\Delta}{s} = \frac{\frac{1}{2}ac}{\frac{2s}{2}} = \frac{ac}{a+b+c} \quad (2s = a + b + c)$$

$$\therefore r = \frac{ac(a+c-b)}{(a+b+c)(a+c-b)}$$

$$= \frac{ac(a+c-b)}{(a+c)^2 - b^2} = \frac{ac(a+c-b)}{a^2 + 2ac + c^2 - b^2} = \frac{a+c-b}{2}$$

$$(a^2 + c^2 = b^2)$$

$\therefore$ The diameter = $a + c - b = BC + AB - AC$.

$\therefore$ Statement 2 is the correct explanation of Statement 1.

Ans. : (A)

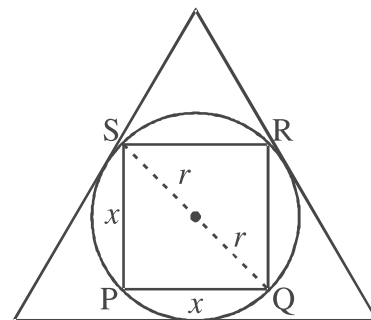


Figure 12.18

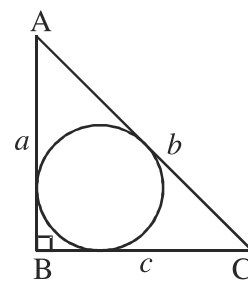


Figure 12.19

(198) **Statement 1 :** $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}, \forall x \in \mathbb{R}$

Statement 2 : $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in \mathbb{R}$

Sol. : Statement 2 : $\forall x \in \mathbb{R}, \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ is true.

Statement 1 : We know that,

$$\tan^{-1}\frac{1}{x} = \begin{cases} \cot^{-1}x; & x > 0 \\ -\pi + \cot^{-1}x; & x < 0 \end{cases}$$

$$\begin{aligned} \text{Now, taking } x < 0, \tan^{-1}\frac{1}{x} + \tan^{-1}x &= \tan^{-1}x - \pi + \cot^{-1}x \\ &= (\tan^{-1}x + \cot^{-1}x) - \pi \\ &= -\frac{\pi}{2} \end{aligned}$$

$\therefore$ Statement 1 is false. So, Statement 2 is true and Statement 1 is false. **Ans. : (D)**

: 'Match the Following' Type Questions :

(199) Match I and II :

[JEE Adv. : 2013]

I	II
(P) $\left[\frac{1}{y^2} \left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)} \right)^2 + y^4 \right]^{\frac{1}{2}}$ where $0 < y < 1$ (Q) If $\cos x + \cos y + \cos z = 0$ $\sin x + \sin y + \sin z = 0$, then a possible value of $\cos\left(\frac{x-y}{2}\right)$ (R) If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \cdot \sin 2x \sec x$ $= \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$, then a possible value of $\sec x$ (S) $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}(x\sqrt{6})\right)$, $0 < x < 1$	(1) $\frac{1}{2}\sqrt{3}$ (2) $\sqrt{2}$ (3) $\frac{1}{2}$ (4) 1

Codes :

- | | | | |
|---------|-----|-----|-----|
| P | Q | R | S |
| (A) (4) | (3) | (1) | (2) |
| (B) (4) | (3) | (2) | (1) |
| (C) (3) | (4) | (2) | (1) |
| (D) (3) | (4) | (1) | (2) |

$$\begin{aligned}
\text{Sol. : (P)} \quad & \left[\frac{1}{y^2} \left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)} \right)^2 + y^4 \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{y^2} \left\{ \frac{\cos\left(\cos^{-1}\frac{1}{\sqrt{1+y^2}}\right) + y\sin\left(\sin^{-1}\frac{y}{\sqrt{1+y^2}}\right)}{\cot\left(\cot^{-1}\frac{\sqrt{1-y^2}}{y}\right) + \tan\left(\tan^{-1}\frac{y}{\sqrt{1-y^2}}\right)} \right\}^2 + y^4 \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{y^2} \left\{ \frac{\frac{1+y^2}{\sqrt{1+y^2}}}{\frac{1-y^2+y^2}{y\sqrt{1-y^2}}} \right\}^2 + y^4 \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{y^2} \left\{ \sqrt{1+y^2} \cdot y\sqrt{1-y^2} \right\}^2 + y^4 \right]^{\frac{1}{2}} = \left[1 - y^4 + y^4 \right]^{\frac{1}{2}} = 1 \quad (\text{P}) \rightarrow (4)
\end{aligned}$$

$$(Q) \cos x + \cos y + \cos z = 0 \text{ and } \sin x + \sin y + \sin z = 0$$

$$\therefore \cos x + \cos y = -\cos z \text{ and } \sin x + \sin y = -\sin z$$

$$(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 1$$

$$\therefore 2 + 2\cos x \cos y + 2\sin x \sin y = 1$$

$$\therefore 2 + 2\cos(x - y) = 1$$

$$\therefore \cos(x - y) = -\frac{1}{2}$$

$$\cos^2\left(\frac{x-y}{2}\right) = \frac{1 + \cos(x-y)}{2} = \frac{1 - \frac{1}{2}}{2} = \frac{1}{4}$$

$$\therefore \cos\left(\frac{x-y}{2}\right) = \pm \frac{1}{2}$$

$$\text{Here, } \frac{1}{2} \text{ is given in option (3).} \quad (Q) \rightarrow (3)$$

$$(R) \cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$$

$$\therefore \cos 2x \left[\cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{\pi}{4} + x\right) \right] = \sin 2x \sec x (\cos x - \sin x)$$

$$\therefore 2\sin \frac{\pi}{4} \sin x \cos 2x = 2\sin x (\cos x - \sin x)$$

$$\therefore 2\sin x \left[\frac{1}{\sqrt{2}} (\cos^2 x - \sin^2 x) - (\cos x - \sin x) \right] = 0$$

$$\therefore 2\sin x (\cos x - \sin x) \left[\frac{\cos x + \sin x}{\sqrt{2}} - 1 \right] = 0$$

$$\therefore 2\sin x(\cos x - \sin x)(\cos x + \sin x - \sqrt{2}) = 0$$

$$\therefore \sin x = 0 \text{ or } \tan x = 1 \text{ or } \cos x + \sin x = \sqrt{2}$$

$$\therefore \cos x = \pm 1 \text{ or } \sec x = \pm \sqrt{2} \text{ or } 1 + \tan x = \sqrt{2} \sec x$$

$$\therefore \sec x = \pm 1 \text{ or } \pm \sqrt{2} \text{ or } \sec x = \sqrt{2} \quad (\text{R}) \rightarrow (2)$$

$$(S) \cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}(x\sqrt{6})\right)$$

$$\therefore \cot\left(\cot^{-1}\frac{x}{\sqrt{1-x^2}}\right) = \sin\left(\sin^{-1}\frac{x\sqrt{6}}{\sqrt{1+6x^2}}\right)$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{1+6x^2}}$$

$$\therefore 1 + 6x^2 = 6(1 - x^2)$$

$$\therefore 12x^2 = 5 \Rightarrow x = \pm \frac{\sqrt{5}}{2\sqrt{3}} \quad (\text{S}) \rightarrow (1)$$

Thus, (P) $\rightarrow$ (4), (Q) $\rightarrow$ (3), (R) $\rightarrow$ (2), (S) $\rightarrow$ (1)

Note : Here, calculation of Q is easy and we get (Q) $\rightarrow$ (3). So, answer can be (A) or (B).

Now, calculation of (S) gives (S) $\rightarrow$ (1). So, Ans. is (B).

Ans. : (B)

(200) Let (x, y) be such that $\sin^{-1}ax + \cos^{-1}y + \cos^{-1}(bxy) = \frac{\pi}{2}$. Match the following :

I		II	
(A)	$a = 1, b = 0$	(P)	(x, y) lies on circle, $x^2 + y^2 = 1$.
(B)	$a = 1, b = 1$	(Q)	(x, y) lies on $(x^2 - 1)(y^2 - 1) = 0$.
(C)	$a = 1, b = 2$	(R)	(x, y) lies on a line $y = x$.
(D)	$a = 2, b = 2$	(S)	(x, y) lies on $(4x^2 - 1)(y^2 - 1) = 0$.

(A) (A) $\rightarrow$ (P) (B) $\rightarrow$ (Q) (C) $\rightarrow$ (S) (D) $\rightarrow$ (R)

[IIT : 2007]

(B) (A) $\rightarrow$ (Q) (B) $\rightarrow$ (P) (C) $\rightarrow$ (S) (D) $\rightarrow$ (R)

(C) (A) $\rightarrow$ (P) (B) $\rightarrow$ (Q) (C) $\rightarrow$ (P) (D) $\rightarrow$ (S)

(D) (A) $\rightarrow$ (Q) (B) $\rightarrow$ (P) (C) $\rightarrow$ (R) (D) $\rightarrow$ (S)

Sol. : $\cos^{-1}y + \cos^{-1}(bxy) = \frac{\pi}{2} - \sin^{-1}(ax)$

$$\therefore \cos^{-1}y + \cos^{-1}(bxy) = \cos^{-1}(ax)$$

Let $\cos^{-1}y = \alpha$, $\cos^{-1}(bxy) = \beta$, $\cos^{-1}(ax) = \gamma$, $0 \leq \alpha, \beta, \gamma \leq \pi$

$$\therefore \alpha + \beta = \gamma$$

$$\therefore \cos(\alpha + \beta) = \cos\gamma$$

$$\therefore \cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos\gamma$$

$$\therefore y(bxy) - \sqrt{1-y^2} \sqrt{1-b^2x^2y^2} = ax$$

$$\therefore -\sqrt{(1-y^2)} \sqrt{1-b^2x^2y^2} = ax - bxy^2$$

$$\begin{aligned}\therefore (1 - y^2)(1 - b^2x^2y^2) &= (ax - bxy^2)^2 \\ \therefore 1 - b^2x^2y^2 - y^2 + b^2x^2y^4 &= a^2x^2 - 2bax^2y^2 + b^2x^2y^4 \\ \therefore a^2x^2 + y^2 + b^2x^2y^2 - 2abx^2y^2 - 1 &= 0 \\ \therefore a^2x^2 + y^2 + (b^2 - 2ab)x^2y^2 - 1 &= 0\end{aligned}$$

$$(A) \ a = 1, \ b = 0 \Rightarrow x^2 + y^2 = 1 \quad \text{So, (A) } \rightarrow (P)$$

$$(B) \ a = 1, \ b = 1$$

$$\therefore x^2 + y^2 - x^2y^2 - 1 = 0 \quad \text{So, (B) } \rightarrow (Q)$$

$$\therefore (y^2 - 1)(x^2 - 1) = 0$$

$$(C) \ a = 1, \ b = 2$$

$$\therefore x^2 + y^2 = 1 \quad \text{So, (C) } \rightarrow (P)$$

$$(D) \ a = 2, \ b = 2$$

$$4x^2 + y^2 - 4x^2y^2 - 1 = 0$$

$$\Rightarrow (4x^2 - 1)(y^2 - 1) = 0 \quad \text{So, (D) } \rightarrow (S)$$

$$(A) \rightarrow (P), (B) \rightarrow (Q), (C) \rightarrow (P), (D) \rightarrow (S)$$

From (A) $\rightarrow$ (P), (C) $\rightarrow$ (P) we will get the answer.

Ans. : (C)

- (201) From three points A, B and C on ground, a balloon is observed. The angle of elevation from B is double the angle of elevation observed from A. The angle of elevation from C is thrice the angle of elevation from A. If the distance from A to B is a , B to C is b , then the height of the balloon is

$$(A) \ \frac{a}{2b} \sqrt{(a+b)(3b+a)}$$

$$(B) \ \frac{a}{2b} \sqrt{(a-b)(3b-a)}$$

$$(C) \ \frac{2a}{b} \sqrt{(a+b)(3b-a)}$$

$$(D) \ \frac{a}{2b} \sqrt{(a+b)(3b-a)}$$

Sol. : In $\triangle PBC$, using *sin* rule,

$$\frac{a}{\sin(\pi - 3\alpha)} = \frac{b}{\sin\alpha} = \frac{PC}{\sin 2\alpha} \quad (PB = AB = a) \quad (1)$$

$$\therefore \frac{a}{3\sin\alpha - 4\sin^3\alpha} = \frac{b}{\sin\alpha} = \frac{PC}{2\sin\alpha \cos\alpha}$$

$$\therefore \frac{a}{3 - 4\sin^2\alpha} = \frac{b}{1} = \frac{PC}{2\cos\alpha}$$

$$\therefore 3 - 4\sin^2\alpha = \frac{a}{b}$$

$$\therefore 4\sin^2\alpha = 3 - \frac{a}{b} = \frac{3b - a}{b}$$

$$\therefore \sin^2\alpha = \frac{3b - a}{4b}, \quad \cos^2\alpha = \frac{b + a}{4b}$$

$$\text{In } \triangle PCD, \sin 3\alpha = \frac{h}{PC}$$

$$\therefore h = PC \sin 3\alpha$$

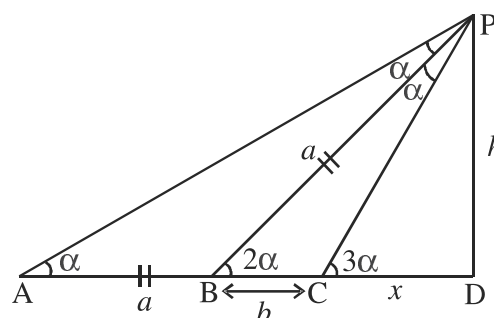


Figure 12.20

$$= a \sin 2\alpha$$

$$\text{(From (1), } PC \sin 3\alpha = a \sin 2\alpha)$$

$$= a(2 \sin \alpha \cos \alpha)$$

$$= a(2) \sqrt{\frac{3b-a}{4b}} \sqrt{\frac{b+a}{4b}}$$

$$h = \frac{a}{2b} \sqrt{(a+b)(3b-a)}$$

Ans. : (D)

- (202) For a vertical tower $\overline{PQ}$, Q lies on the ground. A and B are two different points on the ground at a distance d from each other. The angles of elevation from points A and B to the top of the tower are α and β respectively. If $\overline{AB}$ makes angle of measure γ at Q, then the height of the tower is

$$(A) \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma}}$$

$$(B) \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta + 2 \cot \alpha \cot \beta \cos \gamma}}$$

$$(C) d \sqrt{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma}$$

(D) None of these

Sol. : In $\triangle APQ$, $\tan \alpha = \frac{h}{AQ}$

$$\therefore AQ = \frac{h}{\tan \alpha}$$

In $\triangle BPQ$, $\tan \beta = \frac{h}{BQ}$

$$\therefore BQ = \frac{h}{\tan \beta}$$

In $\triangle ABQ$, $\cos \gamma = \frac{AQ^2 + BQ^2 - AB^2}{2AQ \cdot BQ}$

$$\cos \gamma = \frac{h^2 \cot^2 \alpha + h^2 \cot^2 \beta - d^2}{2h^2 \cot \alpha \cot \beta}$$

$$\therefore 2h^2 \cot \alpha \cot \beta \cos \gamma = h^2 \cot^2 \alpha + h^2 \cot^2 \beta - d^2$$

$$\therefore h = \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma}}$$

Ans. : (A)

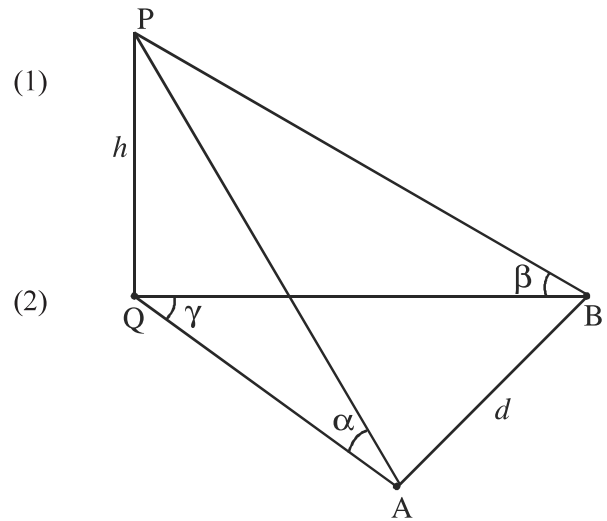


Figure 12.21

- (203) Tower $\overline{AB}$ stands vertical on the ground. A and B are points on the tower. C is the midpoint of $\overline{AB}$. $\overline{CB}$ makes angle of measure β at P. If $AP = nAB$, then $\tan \beta = \dots$

$$(A) \frac{n}{2n+1}$$

$$(B) \frac{n^2}{2n+1}$$

$$(C) \frac{n}{2n^2+1}$$

$$(D) \frac{n^2}{2n^2+1}$$

Sol. : In $\triangle PAB$, $\tan\theta = \frac{AB}{AP} = \frac{1}{n}$ (1)

Now, in $\triangle PAC$, $\tan(\theta - \beta) = \frac{\tan\theta - \tan\beta}{1 + \tan\theta \tan\beta}$

$$\therefore \frac{AC}{AP} = \frac{\tan\theta - \tan\beta}{1 + \tan\theta \tan\beta}$$

$$\therefore \frac{1}{2} \left(\frac{AB}{AP} \right) = \frac{\frac{1}{n} - \tan\beta}{1 + \frac{1}{n} \tan\beta} \quad (\text{From (1)})$$

$$\therefore \frac{1}{2} \left(\frac{1}{n} \right) = \frac{1 - n \tan\beta}{n + \tan\beta}$$

$$\therefore n + \tan\beta = 2n - 2n^2 \tan\beta$$

$$\therefore (2n^2 + 1) \tan\beta = n$$

$$\therefore \tan\beta = \frac{n}{2n^2 + 1}$$

Ans. : (C)

- (204) A ladder rests against a wall at an angle of measure β with the ground. Now, its foot is made to slide by a meter towards the wall and the upper part of ladder rises by b meter and the measure of the angle made by ladder with the ground at that time is α , then $\frac{a}{b} = \dots\dots$

(A) $\tan\left(\frac{\alpha - \beta}{2}\right)$ (B) $\tan\left(\frac{\alpha + \beta}{2}\right)$ (C) $\tan\alpha + \tan\beta$ (D) $1 + \tan\alpha \tan\beta$

Sol. : Let the length of the ladder be x meter.

Now, in right angle $\triangle OBQ$, $\cos\beta = \frac{OB}{BQ}$

$$\therefore \cos\beta = \frac{OB}{x}$$

$$\therefore OB = x \cos\beta, \sin\beta = \frac{OQ}{BQ} = \frac{OQ}{x}$$

$$\therefore OQ = x \sin\beta$$

In right angled $\triangle OPA$, $\cos\alpha = \frac{OA}{PA} \quad \left| \quad \sin\alpha = \frac{OP}{AP} = \frac{OP}{x} \right.$

$$\therefore OA = x \cos\alpha \quad \left| \quad \therefore OP = x \sin\alpha \right.$$

$$a = OB - OA = x(\cos\beta - \cos\alpha)$$

$$b = OP - OQ = x(\sin\alpha - \sin\beta)$$

$$\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2 \sin \frac{\beta + \alpha}{2} \sin \left(\frac{-\beta + \alpha}{2} \right)}{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}$$

$$\therefore \frac{a}{b} = \tan\left(\frac{\alpha + \beta}{2}\right)$$

Ans. : (B)

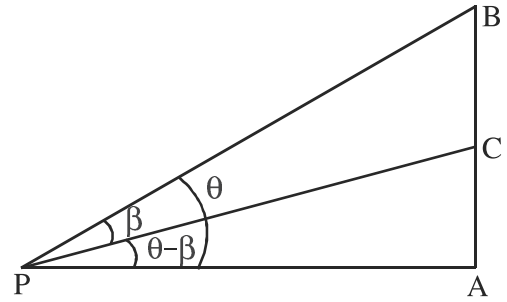


Figure 12.22

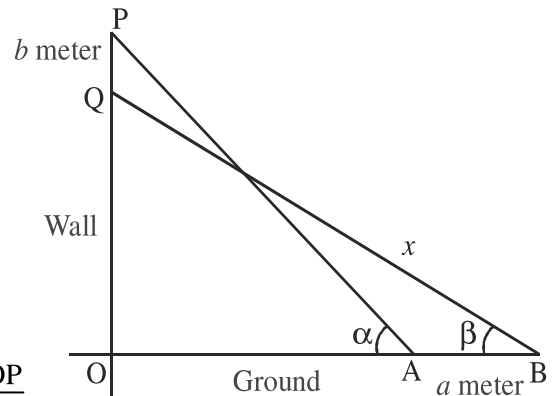


Figure 12.23