

AJ-1539
M.A./M.Sc. (Previous)
Term End Examination, 2021-22
Mathematics

ADVANCED ABSTRACT ALGEBRA

Time : Three hours]

[Maximum Marks : 100

Note: Attempt any five questions. All questions carry equal marks.

1. (a) Suppose that N and M are two normal subgroups of G and $N \cap M = \{e\}$. Show that every element of N is commutative with every element of M .
(b) Show that the centre $Z(G)$ of any group G is a normal subgroup of G .
2. (a) If $f : G \rightarrow G'$ is an isomorphism, then prove that :
(i) The f -image of the identity e of G is the identity of G' is $f(e) = e'$,
(ii) The f -image of the inverse of an element ' a ' of G is the inverse of the f -image of that element i.e. $f(a^{-1}) = [f(a)]^{-1}$.
(iii) The order of the f -image of an element is the same as the order of the element.
(b) Every isomorphic image of a cyclic group is cyclic.
3. (a) Prove that let G be a finite abelian group and let p be a prime, if $p \mid O(G)$. Then There is an element $a \neq e \in G$ such that $a^p = e$
(b) Prove that if G is a group of order p^n where p and n is a prime number and n is a positive integer then $Z(G) \neq \{e\}$.
4. (a) State and prove Jordan-Holder theorem for finite groups.
(b) Define solvable group. Prove that a group G is solvable if and only if $G^{(K)} = \{e\}$ for some integer K .
5. (a) Prove that the necessary and sufficient condition for an R -module M to be a direct sum of its two sub modules N_1 and N_2 are that (i) $M = N_1 + N_2$ and (ii) $N_1 \cap N_2 = \{0\}$.
(b) State and prove the fundamental theorem on homomorphism of modules.
6. (a) Let $p(x)$ be an irreducible polynomial in $F(x)$. Then show that there exists an extension E of F in which $p(x)$ has a root.
(b) Prove that every finite extension of a field is an algebraic extension.
7. (a) Define Normal extension. Show that
(i) $\mathbb{Q}(\sqrt{2})$ is normal over $\mathbb{Q}$.
(ii) $\mathbb{Q}(\sqrt[5]{7})$ is normal over $\mathbb{Q}$
(b) State & prove fundamental theorem of Galoi's theory.
8. (a) For an R -module M the following are equivalent—
(i) M is noetherian
(ii) every sub module of M is finitely generated.
(iii) Every non empty set S of sub-modules of M has a maximal element.
(b) State and prove that Hilbert's Basis theorem.

[P.T.O.]

- 9.(a) Let G be a group and let G' be the commutator sub-group of G . Then show that:
- (i) G / G' is abelian.
 - (ii) If $N < G$ then $\frac{G}{N}$ is abelian if and only if $G' < N$.
- (b) Let G be a nilpotent group then prove that every subgroup of G and every homomorphic image of G are nilpotent.
- 10.(a) Prove that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over ϕ . Show that $x^2 - 2 \in \phi(x)$ is irreducible over ϕ . Find (if it exists) an extension K of ϕ having an roots of $x^3 - 2$ such that $[K : \phi] = 6$.
- (b) Let N be a submodule of an R . Module M . Let $f: M \rightarrow \frac{M}{N}$ be a mapping from the R - module M into the Quotient module $\frac{M}{N}$ defined by :
- $$f(x) = x + N \quad \forall \quad x \in m$$