

Some Useful Results :**Integration as a limit of a sum :**

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Let $h = \frac{b-a}{n}$ for positive integer n . If we partition $[a, b]$ into n sub-intervals of equal length, then the division points are $a, a+h, a+2h, \dots, a+nh = b$.

$$\begin{aligned} \therefore \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} \left(\frac{b-a}{n} \right) \sum_{i=1}^n f(a+ih), \quad h = \frac{b-a}{n} \\ &= \lim_{h \rightarrow 0} h \sum_{i=1}^n f(a+ih) \end{aligned}$$

Fundamental Principle of Definite Integration :

If the function f is continuous on $[a, b]$ and F is a differentiable function on (a, b) such that, $F'(x) = f(x)$, $\forall x \in (a, b)$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Here, $F(x)$ is a primitive of $f(x)$. $F(b) - F(a)$ is expressed as $[F(x)]_a^b$,

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

$$(1) \quad \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$(2) \quad \text{If } a > b, \text{ then } \int_a^b f(x) dx = -\int_b^a f(x) dx$$

$$(3) \quad \text{If } a = b, \text{ then } \int_a^b f(x) dx = \int_a^a f(x) dx = 0$$

Working Rules for Definite Integration :

If functions f and g are continuous on $[a, b]$, then

$$(1) \quad \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$(2) \quad \int_a^b kf(x) dx = k \int_a^b f(x) dx, \quad k \in \mathbb{R}$$

$$(3) \quad \text{If } a < c < b, \text{ then } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Rule of Substitution for definite integration :

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $g : [\alpha, \beta] \rightarrow [a, b]$ be an increasing or decreasing (monotonic) function. $x = g(t)$ is continuous in $[\alpha, \beta]$ and differentiable in (α, β) . $g'(t)$ is continuous in (α, β) and $g'(t) \neq 0$, $\forall t \in (\alpha, \beta)$. $a = g(\alpha)$ and $b = g(\beta)$. Then

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) \cdot g'(t) dt$$

Method of integration by parts for definite integration :

If $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ all are continuous in $[a, b]$, then

$$\int_a^b f(x) g'(x) dx = [f(x) g(x)]_a^b - \int_a^b f'(x) g(x) dx$$

$$\therefore \int_a^b f(x) g'(x) dx = [f(b) g(b) - f(a) g(a)] - \int_a^b f'(x) g(x) dx$$

If $f(x) = u$ and $g'(x) = v$, then

$$\therefore \int_a^b uv dx = \left[u \int v dx \right]_a^b - \int_a^b \left[\frac{du}{dx} \int v dx \right] dx$$

Some Useful Results about Definite Integration :

- (1) If f is continuous on $[0, a]$, then $\int_0^a f(x) dx = \int_0^a f(a-x) dx$
- (2) If f is continuous on $[a, b]$, then $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$
- (3) If f is continuous on $[0, 2a]$, then

$$\int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx$$

$$\int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x), \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

- (4) If f is continuous over $[-a, a]$, then

$$\int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx$$

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f \text{ is even,} \\ 0, & \text{if } f \text{ is odd.} \end{cases}$$

(5) If the function $f(x)$ is a periodic function with period T , then

$$\int_{\alpha}^{\alpha+nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}$$

(6) If $\alpha = 0$, then $\int_{\alpha}^{\alpha+nT} f(x) dx = \int_0^{nT} f(x) dx = n \int_0^T f(x) dx$

(7) If $n = 1$, then $\int_{\alpha}^{\alpha+nT} f(x) dx = \int_{\alpha}^{\alpha+T} f(x) dx = \int_0^T f(x) dx$

(8) $\int_{\alpha+nT}^{\beta+nT} f(x) dx = \int_{\alpha}^{\beta} f(x) dx$

(9) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$

: Multiple Choice Questions :

(1) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx = \dots\dots$

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) $\frac{3\pi}{2}$

(D) π

Sol. : $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ (1)

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cot\left(\frac{\pi}{2} - x\right)} + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$
 (2)

Adding (1) and (2), we get,

$$2I = \int_0^{\frac{\pi}{2}} 1 dx = [x]_0^{\frac{\pi}{2}}$$

$\therefore 2I = \frac{\pi}{2}.$

So, $I = \frac{\pi}{4}$

Ans. : (B)

$$(2) \int_{-4}^4 |x+2| dx = \dots\dots$$

- (A) 30 (B) 20 (C) 50 (D) 10

$$\text{Sol. : } |x+2| = \begin{cases} x+2 & \text{where } x \geq -2 \\ -(x+2) & \text{where } x < -2 \end{cases}$$

$$\begin{aligned} \therefore I &= \int_{-4}^4 |x+2| dx = \int_{-4}^{-2} |x+2| dx + \int_{-2}^4 |x+2| dx \\ &= -\int_{-4}^{-2} (x+2) dx + \int_{-2}^4 (x+2) dx \\ &= -\left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^4 \\ &= -[(2-4) - (8-8)] + [(8+8) - (2-4)] \\ &= -(-2) + (16+2) = 20 \end{aligned}$$

Ans. : (B)

$$(3) \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx = \dots\dots$$

- (A) $\frac{-\pi a}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi a}{4}$ (D) πa

$$\text{Sol. : } I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$$

$$\text{Let } x = a \cos 2\theta. \quad \text{So, } dx = -2a \sin 2\theta d\theta \quad (0 \leq \theta \leq \frac{\pi}{2})$$

$$\text{Now, } x = -a \Rightarrow -a = a \cos 2\theta \Rightarrow \cos 2\theta = -1$$

$$\therefore 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$$

$$x = a \Rightarrow a = a \cos 2\theta \Rightarrow \cos 2\theta = 1. \quad \text{So, } \theta = 0$$

$$\therefore I = \int_{\frac{\pi}{2}}^0 \sqrt{\frac{a-a \cos 2\theta}{a+a \cos 2\theta}} (-2a \sin 2\theta) d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} \tan \theta \cdot \sin 2\theta d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{\cos \theta} \cdot 2 \sin \theta \cdot \cos \theta d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} 2 \sin^2 \theta \cdot d\theta$$

$$= 2a \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= 2a \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 2a \left[\left(\frac{\pi}{2} - 0 \right) \right] = \pi a$$

Ans. : (D)

$$(4) \int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{\sin x + \cos x} = \dots\dots$$

(A) $\sqrt{2} \log (\sqrt{2} - 1)$ (B) $\sqrt{2} \log (\sqrt{2} + 1)$ (C) $\frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$ (D) $\frac{1}{\sqrt{2}} \log \frac{(\sqrt{2} - 1)}{(\sqrt{2} + 1)}$

$$\text{Sol. : I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{\sin x + \cos x} \quad (1)$$

$$\therefore \text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx \quad (2)$$

Adding (1) and (2), we get,

$$\begin{aligned} 2\text{I} &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx \\ &= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)} dx \\ &= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{\cos \left(x - \frac{\pi}{4} \right)} \\ &= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec \left(x - \frac{\pi}{4} \right) dx \\ &= \frac{1}{\sqrt{2}} \left[\log \left| \sec \left(x - \frac{\pi}{4} \right) + \tan \left(x - \frac{\pi}{4} \right) \right| \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{\sqrt{2}} \left[\log (\sqrt{2} + 1) - \log (\sqrt{2} - 1) \right] \\ &= \frac{1}{\sqrt{2}} \log \left[\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right] \end{aligned}$$

$$\therefore 2I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)^2$$

$$\therefore 2I = \frac{2}{\sqrt{2}} \log (\sqrt{2} + 1)$$

$$\therefore I = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$$

Ans. : (C)

$$(5) \int_0^1 \left| \sin \frac{3\pi x}{2} \right| dx = \dots\dots$$

$$(A) \frac{3\pi}{2}$$

$$(B) \frac{2\pi}{3}$$

$$(C) \frac{2}{\pi}$$

$$(D) \frac{\pi}{3}$$

$$\text{Sol. : } I = \int_0^1 \left| \sin \frac{3\pi x}{2} \right| dx$$

$$\text{Now, } \left| \sin \frac{3\pi x}{2} \right| = \sin \frac{3\pi x}{2}, \text{ where, } 0 \leq \frac{3\pi x}{2} \leq \pi, \text{ i.e. } x \leq \frac{2}{3} \quad \text{i.e. } 0 \leq x \leq \frac{2}{3}$$

$$\text{and } \left| \sin \frac{3\pi x}{2} \right| = -\sin \frac{3\pi x}{2}, \text{ where, } \pi \leq \frac{3\pi x}{2} \leq \frac{3\pi}{2} \quad \text{i.e. } \frac{2}{3} \leq x \leq 1$$

$$\therefore I = \int_0^{\frac{2}{3}} \sin \frac{3\pi x}{2} dx + \int_{\frac{2}{3}}^1 \left(-\sin \frac{3\pi x}{2} \right) dx$$

$$= \frac{2}{3\pi} \left[-\cos \frac{3\pi x}{2} \right]_0^{\frac{2}{3}} + \frac{2}{3\pi} \left[\cos \frac{3\pi x}{2} \right]_{\frac{2}{3}}^1$$

$$= \frac{2}{3\pi} [1 - (-1)] + \frac{2}{3\pi} [0 - (-1)] = \frac{2}{3\pi} (2 + 1) = \frac{2}{\pi}$$

Ans. : (C)

$$(6) \int_{-1}^1 (x - [x]) dx = \dots\dots$$

$$(A) 0$$

$$(B) 1$$

$$(C) 2$$

$$(D) \frac{1}{2}$$

(IIT : 1998)

$$\text{Sol. : } I = \int_{-1}^1 (x - [x]) dx$$

$$\therefore I = \int_{-1}^1 x dx - \int_{-1}^1 [x] dx$$

$$= 0 - \left\{ \int_{-1}^0 [x] dx + \int_0^1 [x] dx \right\}$$

($f(x) = x$, is an odd function)

$$= -[-1 + 0]$$

$$(-1 < x < 0, [x] = -1, 0 < x < 1, [x] = 0)$$

$$I = 1$$

Ans. : (B)

$$(7) \int_0^{\frac{\pi}{2}} \log \cos x \, dx = \dots$$

$$(A) \frac{-\pi}{2} \log 2$$

$$(B) \frac{\pi}{2} \log 2$$

$$(C) \pi \log \frac{1}{2}$$

$$(D) -\pi \log \frac{1}{2}$$

$$\text{Sol. : } I = \int_0^{\frac{\pi}{2}} \log \cos x \, dx \quad (1)$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\cos \left(\frac{\pi}{2} - x \right) \right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \cos x + \log \sin x) \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{(2 \sin x \cos x)}{2} \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Taking $2x = t$, $2dx = dt$

$$x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = \pi$$

$$\therefore 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \log 2 [x]_0^{\frac{\pi}{2}}$$

$$2I = \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \cdot \log 2$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2$$

$$\therefore 2I = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2$$

Ans. : (A)

Note : Infact such integrals are called improper integrals.

(8) For $n \in \mathbb{N}$, $\int_0^{2\pi} \frac{x \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = \dots$ (IIT : 1996)

(A) 0 (B) $\frac{\pi^2}{2}$ (C) π^2 (D) $2\pi^2$

Sol. : $I = \int_0^{2\pi} \frac{x \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x}$ (1)

$$\therefore I = \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n}(2\pi - x) \, dx}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}$$

$$= \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x}$$

$$= 2\pi \int_0^{2\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} - I$$

$$\therefore 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = 4\pi \int_0^{\pi} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} = 8\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x \, dx}{\sin^{2n} x + \cos^{2n} x} \quad (2)$$

$$\therefore 2I = 8\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\left(\frac{\pi}{2} - x\right) \, dx}{\sin^{2n}\left(\frac{\pi}{2} - x\right) + \cos^{2n}\left(\frac{\pi}{2} - x\right)}$$

$$2I = 8\pi \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} x \, dx}{\cos^{2n} x + \sin^{2n} x} \quad (3)$$

Adding (2) and (3), we get,

$$4I = 8\pi \int_0^{\frac{\pi}{2}} 1 \, dx = 8\pi [x]_0^{\frac{\pi}{2}} = 4\pi^2. \quad \text{So, } I = \pi^2 \quad \text{Ans. : (C)}$$

(9) $\int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta = \dots$

(A) $\frac{\pi}{4} \log 2$ (B) $-\frac{\pi}{8} \log \frac{1}{2}$ (C) $\frac{\pi}{8} \log \frac{1}{2}$ (D) $\frac{\pi}{4} \log \frac{1}{2}$

Sol. : $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan \theta}{1 + \tan \theta}\right) \, d\theta$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \log \left(\frac{2}{1+\tan\theta} \right) d\theta \\
&= \int_0^{\frac{\pi}{4}} \log 2 \, d\theta - \int_0^{\frac{\pi}{4}} \log (1+\tan\theta) \, d\theta
\end{aligned}$$

$$I = \log 2 \int_0^{\frac{\pi}{4}} d\theta - I$$

$$\therefore 2I = \left(\frac{\pi}{4} \right) \log 2. \quad \text{So, } I = \frac{\pi}{8} \log 2 = -\frac{\pi}{8} \log \left(\frac{1}{2} \right) \quad \text{Ans. : (B)}$$

$$(10) \quad \int_{e^{-1}}^{e^2} \left| \frac{\log t}{t} \right| dt = \dots\dots$$

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) 5 (D) 3

$$\text{Sol. : } I = \int_{e^{-1}}^{e^2} \left| \frac{\log t}{t} \right| dt$$

$$= - \int_{e^{-1}}^1 \frac{\log t}{t} dt + \int_1^{e^2} \frac{\log t}{t} dt$$

$$= -\frac{1}{2} [(\log t)^2]_{e^{-1}}^1 + \frac{1}{2} [(\log t)^2]_1^{e^2} = -\frac{1}{2} (0 - 1) + \frac{1}{2} (4 - 0) = \frac{1}{2} + 2 = \frac{5}{2} \quad \text{Ans. : (B)}$$

$$(11) \quad \text{If } f(t) + f\left(\frac{1}{t}\right) = F(t), \text{ where } f(t) = \int_1^t \frac{\log x}{1+x} dx, \text{ then } F(e) = \dots\dots$$

(A) $\frac{1}{2}$ (B) 1 (C) 0 (D) 2 (IIT : 2000)

$$\text{Sol. : } F(e) = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log x}{1+x} dx + \int_1^{\frac{1}{e}} \frac{\log x}{1+x} dx$$

Substituting, $x = \frac{1}{u}$ in the second integrand, we get $dx = -\frac{1}{u^2} du$ $\left\{ \begin{array}{l} x=1 \Rightarrow u=1 \\ x=\frac{1}{e} \Rightarrow u=e \end{array} \right\}$

$$\begin{aligned}
\therefore F(e) &= \int_1^e \frac{\log x \, dx}{1+x} + \int_1^e \frac{\log u \, du}{u(1+u)} \\
&= \int_1^e \frac{\log x \, dx}{1+x} + \int_1^e \frac{\log x \, dx}{x(1+x)} \\
&= \int_1^e \frac{\log x}{1+x} \left(1 + \frac{1}{x}\right) dx \\
&= \int_1^e \frac{\log x}{x} dx = \frac{1}{2} [(\log x)^2]_1^e = \frac{1}{2}
\end{aligned}$$

Ans. : (A)

$$(12) \quad \int_0^{\frac{\pi}{2}} (x - [\sin x]) \, dx = \dots$$

- (A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{8} - 2$ (C) $\frac{\pi^2}{8} - 1$ (D) $\frac{\pi^2}{4} - 1$

$$\text{Sol. : I} = \int_0^{\frac{\pi}{2}} (x - [\sin x]) \, dx$$

$$= \int_0^{\frac{\pi}{2}} x \, dx - \int_0^{\frac{\pi}{2}} [\sin x] \, dx$$

$$= \left[\frac{x^2}{2} \right]_0^{\frac{\pi}{2}} - 0$$

$$[0 < x < \frac{\pi}{2}; \quad 0 < \sin x < 1; \quad \text{So, } [\sin x] = 0]$$

$$= \frac{\pi^2}{8}$$

Ans. : (A)

$$(13) \quad \text{If } \int_{\log 2}^x \frac{du}{(e^u - 1)^{\frac{1}{2}}} = \frac{\pi}{6}, \text{ then } e^x = \dots$$

- (A) 1 (B) 2 (C) 4 (D) 5

$$\text{Sol. : } \int_{\log 2}^x \frac{du}{(e^u - 1)^{\frac{1}{2}}} = \frac{\pi}{6}$$

Let, $e^u - 1 = t^2$. ($t > 0$) So, $e^u du = 2t dt$

$\therefore e^u = t^2 + 1$. So, $du = \frac{2t}{t^2 + 1} dt$

Now, $u = \log 2 \Rightarrow t^2 = 2 - 1 = 1$

$u = x \Rightarrow t^2 = e^x - 1 \Rightarrow t = \sqrt{e^x - 1}$

$\therefore \int_1^{\sqrt{e^x - 1}} \frac{2t}{t(t^2 + 1)} dt = \frac{\pi}{6}$

$\therefore 2 [\tan^{-1} t]_1^{\sqrt{e^x - 1}} = \frac{\pi}{6}$

$\therefore \tan^{-1} \sqrt{e^x - 1} - \frac{\pi}{4} = \frac{\pi}{12}$

$\therefore \tan^{-1} \sqrt{e^x - 1} = \frac{\pi}{3}$

$\therefore \sqrt{e^x - 1} = \tan \frac{\pi}{3} = \sqrt{3}$

$\therefore e^x - 1 = 3$

$\therefore e^x = 4$

Ans. : (C)

(14) If $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}$, $x > 0$ and $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1)$, then a possible value of k is

(A) 15

(B) 16

(C) 63

(D) 64

Sol. : $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x} \Rightarrow F(x) = \int \frac{e^{\sin x}}{x} dx$

Now, $I = \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$

Let, $x^3 = t$.

So, $3x^2 dx = dt$

$x = 1 \Rightarrow t = 1$ and $x = 4 \Rightarrow t = 64$

$\therefore I = \int_1^{64} \frac{e^{\sin t}}{t} dt = [F(t)]_1^{64} = F(64) - F(1)$

$\therefore k = 64$

Ans. : (D)

(15) If $I_n = \int_0^1 e^x (x-1)^n dx$, where, $n \leq 5$, $n \in \mathbb{N} \cup \{0\}$, then, $I_3 = \dots$

- (A) $2e - 5$ (B) $8e - 16$ (C) $2 - e$ (D) $16 - 6e$

Sol. : $I_n = \int_0^1 e^x (x-1)^n dx$

$$I_n = [(x-1)^n \cdot e^x]_0^1 - n \int_0^1 (x-1)^{n-1} e^x dx$$

$$\therefore I_n = (-1)^{n+1} - nI_{n-1}$$

$$\therefore I_0 = \int_0^1 e^x dx = e - 1$$

If $n = 1$, then $I_1 = (-1)^2 - 1I_0 = 1 - (e - 1) = 2 - e$

If $n = 2$, then $I_2 = (-1)^3 - 2I_1 = -1 - 2(2 - e) = 2e - 5$

If $n = 3$, then $I_3 = 1 - 3I_2 = 1 - 3(2e - 5) = 16 - 6e$

Ans. : (D)

(16) $\int_0^\pi \ln(1 + \cos x) dx = \dots$ ($\ln = \text{natural logarithm} = \log_e$)

- (A) $\frac{\pi}{2} \ln 2$ (B) $\pi \ln 2$ (C) $-\frac{\pi}{2} \ln 2$ (D) $-\pi \ln 2$

Sol. : $I = \int_0^\pi \ln(1 + \cos x) dx$ (1)

$$= \int_0^\pi \ln(1 + \cos(\pi - x)) dx = \int_0^\pi \ln(1 - \cos x) dx$$
 (2)

Adding (1) and (2), we get

$$2I = \int_0^\pi \ln(1 + \cos x)(1 - \cos x) dx = \int_0^\pi \ln \sin^2 x dx$$

$$\therefore 2I = 2 \int_0^\pi \ln(\sin x) dx$$

$$= 4 \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = 4 \left(-\frac{\pi}{2} \ln 2 \right)$$

$$\therefore 2I = -2\pi \ln 2$$

$$\therefore I = -\pi \ln 2$$

Ans. : (D)

$$(17) \int_0^1 \tan^{-1} (1 - x + x^2) dx = \dots\dots$$

$$(A) \frac{\pi}{2}$$

$$(B) \frac{\pi}{4}$$

$$(C) \ln 2$$

$$(D) \frac{\pi}{2} - \ln 2$$

$$\text{Sol. : I} = \int_0^1 \tan^{-1} (1 - x + x^2) dx$$

$$= \int_0^1 \left[\frac{\pi}{2} - \cot^{-1} (1 - x + x^2) \right] dx$$

$$= \int_0^1 \frac{\pi}{2} dx - \int_0^1 \tan^{-1} \left(\frac{1}{1 - x + x^2} \right) dx \quad (1 - x + x^2 > 0)$$

$$= \frac{\pi}{2} [x]_0^1 - \int_0^1 \tan^{-1} \left(\frac{(x) + (1 - x)}{1 - x(1 - x)} \right) dx$$

$$= \frac{\pi}{2} - \int_0^1 (\tan^{-1} x + \tan^{-1} (1 - x)) dx \quad (0 < x < 1, 0 < 1 - x < 1); (x(1 - x) < 1)$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1} (1 - x) dx$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1} (1 - (1 - x)) dx$$

$$= \frac{\pi}{2} - 2 \int_0^1 \tan^{-1} x dx$$

$$= \frac{\pi}{2} - 2 \left\{ [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx \right\}$$

$$= \frac{\pi}{2} - 2 \left(1 \cdot \frac{\pi}{4} - 0 \right) + \int_0^1 \frac{2x}{1 + x^2} dx = \frac{\pi}{2} - \frac{\pi}{2} + [\ln (1 + x^2)]_0^1 = \ln 2$$

Ans. : (C)

$$(18) \text{ If } g(1) = g(2), \text{ then } \int_1^2 [f\{g(x)\}]^{-1} \cdot f'(g(x)) \cdot g'(x) dx = \dots\dots$$

$$(A) 1$$

$$(B) 2$$

$$(C) 0$$

$$(D) -1$$

$$\text{Sol. : I} = \int_1^2 [f\{g(x)\}]^{-1} \cdot f'(g(x)) \cdot g'(x) dx$$

$$\text{Let, } f(g(x)) = t.$$

$$\text{So, } f'(g(x)) g'(x) dx = dt$$

When, $x = 1 \Rightarrow t = f(g(1))$ and $x = 2 \Rightarrow t = f(g(2))$

$$\begin{aligned}\therefore I &= \int_{f(g(1))}^{f(g(2))} \frac{1}{t} dt = [\log t]_{f(g(1))}^{f(g(2))} \\ &= \log |f(g(2))| - \log |f(g(1))| \\ &= 0 \quad (g(1) = g(2)) \quad \text{Ans. : (C)}\end{aligned}$$

(19) If $I_1 = \int_0^1 (1 - x^{50})^{100} dx$ and $I_2 = \int_0^1 (1 - x^{50})^{101} dx$, then $\frac{I_1}{I_2} = \dots$

(A) $\frac{5051}{5050}$ (B) $\frac{5051}{5049}$ (C) $\frac{51}{50}$ (D) $\frac{101}{100}$

$$\begin{aligned}\text{Sol. : } I_2 &= \int_0^1 (1 - x^{50})^{101} dx \\ &= \int_0^1 (1 - x^{50})(1 - x^{50})^{100} dx \\ &= \int_0^1 (1 - x^{50})^{100} dx - \int_0^1 x^{50} (1 - x^{50})^{100} dx \\ I_2 &= I_1 - \int_0^1 x(1 - x^{50})^{100} \cdot x^{49} dx \\ I_2 &= I_1 + \left[\frac{x(1 - x^{50})^{101}}{5050} \right]_0^1 - \frac{1}{5050} \int_0^1 (1 - x^{50})^{101} dx \\ I_2 &= I_1 + 0 - \frac{1}{5050} I_2 \\ \therefore I_1 &= I_2 \left(1 + \frac{1}{5050} \right) \\ \therefore I_1 &= \frac{5051}{5050} I_2 \\ \therefore \frac{I_1}{I_2} &= \frac{5051}{5050} \quad \text{Ans. : (A)}\end{aligned}$$

(20) $\int_0^{1000} e^{x - [x]} dx = \dots$

(A) $e^{1000} - 1$ (B) $\frac{e^{1000} - 1}{e - 1}$ (C) $1000(e - 1)$ (D) $\frac{e - 1}{1000}$

$$\text{Sol. : } I = \int_0^{1000} e^{x - [x]} dx$$

$e^{x - [x]}$ is a periodic function with period 1.

$$\therefore I = \int_0^{1000} e^{x - [x]} dx$$

$$= 1000 \int_0^1 e^x - [x] \, dx; \text{ Now, for } 0 < x < 1, [x] = 0$$

$$= 1000 \int_0^1 e^x \, dx$$

$$= 1000 [e^x]_0^1$$

$$= 1000 [e - 1]$$

Ans. : (C)

$$(21) \quad \lim_{n \rightarrow \infty} \left[\frac{n}{n^2 + 1} + \frac{n}{n^2 + 4} + \frac{n}{n^2 + 9} + \dots + \frac{1}{2n} \right] = \dots$$

(A) 1

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$

(D) 0

$$\text{Sol. : } \lim_{n \rightarrow \infty} \left[\frac{n}{n^2 + 1} + \frac{n}{n^2 + 4} + \frac{n}{n^2 + 9} + \dots + \frac{1}{2n} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 + r^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n^2} \left[\frac{n}{1 + \left(\frac{r}{n}\right)^2} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\frac{1}{1 + \left(\frac{r}{n}\right)^2} \right]$$

$$= \int_0^1 \frac{1}{1 + x^2} \, dx$$

$$= [\tan^{-1} x]_0^1$$

$$= \frac{\pi}{4}$$

Ans. : (C)

$$(22) \quad \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} \, dx = \dots$$

(A) 3

(B) 2

(C) 1

(D) 0

$$\text{Sol. : I} = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x + \cos x) dx \quad (0 < \theta < \frac{\pi}{2})$$

$$= [-\cos x + \sin x]_0^{\frac{\pi}{2}} = (0 + 1) - (-1 + 0) = 2 \quad \text{Ans. : (B)}$$

(23) If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^t f(t-y) g(y) dy$, then $F(t) = \dots\dots$

- (A) $-t - 1$ (B) $t \cdot e^{-t}$ (C) $e^t - (1 + t)$ (D) $t \cdot e^{-t} + (1 - t)$
(AIEEE : 2004)

Sol. : $F(t) = \int_0^t f(t-y) g(y) dy$

$$= \int_0^t e^{t-y} \cdot y dy$$

$$= e^t \int_0^t e^{-y} \cdot y dy$$

$$= e^t \left\{ [-y e^{-y}]_0^t - \int_0^t 1 \cdot (-e^{-y}) dy \right\}$$

$$= e^t \{ -t \cdot e^{-t} + [-e^{-y}]_0^t \}$$

$$= e^t \{ -t \cdot e^{-t} - e^{-t} + 1 \}$$

$$= -t e^0 - e^0 + e^t$$

$$= -t - 1 + e^t$$

$$= e^t - (t + 1)$$

Ans. : (C)

(24) $\int_{\frac{1}{2}}^2 \frac{1}{x} \sin \left(x - \frac{1}{x} \right) dx = \dots\dots$

- (A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) π

Sol. : $I = \int_{\frac{1}{2}}^2 \frac{1}{x} \sin \left(x - \frac{1}{x} \right) dx$

$$\text{Let, } \frac{1}{x} = t \quad \text{So, } -\frac{1}{x^2} dx = dt$$

$$\therefore dx = -x^2 dt \quad \text{So, } dx = -\frac{1}{t^2} dt$$

$$x = \frac{1}{2} \Rightarrow t = 2 \text{ and } x = 2 \Rightarrow t = \frac{1}{2}$$

$$\therefore I = \int_{\frac{1}{2}}^2 t \cdot \sin\left(\frac{1}{t} - t\right) \cdot \frac{1}{t^2} dt$$

$$= \int_{\frac{1}{2}}^2 \left\{ -\frac{1}{t} \sin\left(t - \frac{1}{t}\right) \right\} dt$$

$$= - \int_{\frac{1}{2}}^2 \frac{1}{x} \sin\left(x - \frac{1}{x}\right) dx$$

$$\therefore I = -I$$

$$\therefore 2I = 0. \quad \text{So, } I = 0$$

Ans. : (A)

$$(25) \int_0^{\infty} \frac{dx}{(x + \sqrt{1+x^2})^3} = \dots \quad (\text{Improper integral})$$

$$(A) \quad -\frac{1}{4}$$

$$(B) \quad -\frac{1}{8}$$

$$(C) \quad \frac{1}{2}$$

$$(D) \quad \frac{3}{8}$$

$$\text{Sol. : } I = \int_0^{\infty} \frac{dx}{(x + \sqrt{1+x^2})^3}$$

$$\text{Let, } x = \tan\theta. \quad \text{Hence } dx = \sec^2\theta d\theta$$

$$\therefore x = 0 \Rightarrow \theta = 0 \text{ and } x = \infty \Rightarrow \theta = \frac{\pi}{2}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta d\theta}{(\tan\theta + \sec\theta)^3}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1 + \sin\theta)^3} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 + \sin \theta)^{-3} (\cos \theta) d\theta$$

$$= \left[\frac{(1 + \sin \theta)^{-2}}{-2} \right]_0^{\frac{\pi}{2}} = -\frac{1}{2} \left[\frac{1}{4} - 1 \right] = -\frac{1}{2} \left(\frac{-3}{4} \right) = \frac{3}{8}$$

Ans. : (D)

(26) If $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \in [a, b]$, then $[a, b] = \dots$ where $[a, b]$ is the smallest possible interval.

- (A) $\left[\frac{1}{2}, 1\right]$ (B) $\left[\frac{1}{\sqrt{2}}, 1\right]$ (C) $[0, 1]$ (D) $\left[\frac{3}{4}, 1\right]$

Sol. : $0 \leq x \leq 1 \Rightarrow 0 \leq x^4 \leq 1$

$$\Rightarrow 1 \leq 1 + x^4 \leq 2$$

$$\Rightarrow 1 \leq \sqrt{1 + x^4} \leq \sqrt{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1 + x^4}} \leq 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \int_0^1 \frac{1}{\sqrt{1 + x^4}} dx \leq 1$$

$$\therefore [a, b] = \left[\frac{1}{\sqrt{2}}, 1\right]$$

Ans. : (B)

(27) $\int_{-2}^0 (x^3 + 3x^2 + 3x + 3 + (x+1) \cos(x+1)) dx = \dots$

- (A) -4 (B) 0 (C) 4 (D) 6

(IIT : 2005)

Sol. : $I = \int_{-2}^0 (x^3 + 3x^2 + 3x + 3 + (x+1) \cos(x+1)) dx$

$$= \int_{-2}^0 ((x+1)^3 + 2 + (x+1) \cos(x+1)) dx$$

Let, $x+1 = t$. So, $dx = dt$. Hence $x = -2 \Rightarrow t = -1$ and $x = 0 \Rightarrow t = 1$

$$\therefore I = \int_{-1}^1 (t^3 + 2 + t \cos t) dt$$

$$= \int_{-1}^1 (t^3 + t \cos t) dt + 2 \int_{-1}^1 dt \quad (t^3 + t \cos t \text{ is an odd function})$$

$$= 0 + 2[t]_{-1}^1 = 2(1 + 1) = 4$$

Ans. : (C)

(28) $\int_0^{\frac{3}{2}} [x^2] dx = \dots$ where $[\cdot]$ denotes the greatest integer function.

- (A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$ (C) $-2 - \sqrt{2}$ (D) $-2 + \sqrt{2}$

Sol. : $I = \int_0^{\frac{3}{2}} [x^2] dx$

$$0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0$$

$$1 < x < \sqrt{2} \Rightarrow 1 < x^2 < 2 \Rightarrow [x^2] = 1$$

$$\sqrt{2} < x < \frac{3}{2} \Rightarrow 2 < x^2 < 2.25 \Rightarrow [x^2] = 2$$

$$\therefore I = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\frac{3}{2}} [x^2] dx$$

$$= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\frac{3}{2}} 2 dx$$

$$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\frac{3}{2}}$$

$$= (\sqrt{2} - 1) + (3 - 2\sqrt{2})$$

$$= 2 - \sqrt{2}$$

Ans. : (B)

(29) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2 \sin x] dx = \dots$, where $[\cdot]$ denotes the greatest integer function.

- (A) $-\pi$ (B) 0 (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$ (IIT : 1999)

Sol. : $I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2 \sin x] dx$

Let $x = \frac{\pi}{2} + t$. So, $dx = dt$. $x = \frac{\pi}{2} \Rightarrow t = 0$ and $x = \frac{3\pi}{2} \Rightarrow t = \pi$

$$\therefore I = \int_0^{\pi} [2 \cos t] dt$$

$$0 < t < \frac{\pi}{3} \Rightarrow 1 < 2 \cos t < 2 \Rightarrow [2 \cos t] = 1$$

$$\frac{\pi}{3} < t < \frac{\pi}{2} \Rightarrow 0 < 2 \cos t < 1 \Rightarrow [2 \cos t] = 0$$

$$\frac{\pi}{2} < t < \frac{2\pi}{3} \Rightarrow -1 < 2 \cos t < 0 \Rightarrow [2 \cos t] = -1$$

$$\frac{2\pi}{3} < t < \pi \Rightarrow -2 < 2 \cos t < -1 \Rightarrow [2 \cos t] = -2$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{3}} 1 \, dt + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 0 \, dt + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} -1 \, dt + \int_{\frac{2\pi}{3}}^{\pi} (-2) \, dt \\
&= \frac{\pi}{3} - \frac{\pi}{6} - \frac{2\pi}{3} = -\frac{\pi}{2}
\end{aligned}$$

Ans. : (C)

$$(30) \int_1^5 (|x-3| + |x-1|) \, dx = \dots$$

- (A) $\frac{5}{6}$ (B) 10 (C) 12 (D) 21

$$\begin{aligned}
\text{Sol. : I} &= \int_1^5 (|x-3| + |x-1|) \, dx \\
&= \int_1^3 |x-3| \, dx + \int_3^5 |x-3| \, dx + \int_1^5 |x-1| \, dx \\
&= \int_1^3 (-x+3) \, dx + \int_3^5 (x-3) \, dx + \int_1^5 (x-1) \, dx \\
&= \left[\frac{-x^2}{2} + 3x \right]_1^3 + \left[\frac{x^2}{2} - 3x \right]_3^5 + \left[\frac{x^2}{2} - x \right]_1^5 \\
&= \left[\left(\frac{-9}{2} + 9 \right) - \left(\frac{-1}{2} + 3 \right) \right] + \left[\left(\frac{25}{2} - 15 \right) - \left(\frac{9}{2} - 9 \right) \right] + \left[\left(-5 + \frac{25}{2} \right) - \left(-1 + \frac{1}{2} \right) \right] \\
&= \frac{9}{2} - \frac{5}{2} - \frac{5}{2} + \frac{9}{2} + \frac{15}{2} + \frac{1}{2} = 12
\end{aligned}$$

Ans. : (C)

$$(31) \int_{-3}^3 \log |x + \sqrt{x^2+1}| \, dx = \dots$$

- (A) 0 (B) $\log 3$ (C) $\log (19 + 6\sqrt{10})$ (D) $\log \frac{1}{3}$

$$\text{Sol. : Let } f(x) = \log |x + \sqrt{x^2+1}|$$

$$\therefore f(-x) = \log |-x + \sqrt{x^2+1}|$$

$$= \log \left(\left(\sqrt{x^2+1} - x \right) \times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} \right)$$

$$= \log \left(\frac{x^2+1-x^2}{\sqrt{x^2+1} + x} \right)$$

$$= \log \frac{1}{\sqrt{x^2+1} + x} = -\log (\sqrt{x^2+1} + x) = -f(x)$$

$$\therefore \text{I} = 0$$

($f(x)$ is an odd function.) **Ans. : (A)**

$$(32) \int_0^1 \log (\sqrt{1-x} + \sqrt{1+x}) dx = \dots\dots$$

$$(A) \frac{1}{2} \left[\log 2 - \frac{\pi}{4} + 1 \right]$$

$$(B) \frac{1}{2} \left[\log 2 + \frac{\pi}{2} - 1 \right]$$

$$(C) \frac{1}{2} \left[\log 2 - \frac{3\pi}{2} + 1 \right]$$

$$(D) \frac{1}{2} \left[\log 2 + \frac{\pi}{3} - 1 \right]$$

$$\text{Sol. : I} = \int_0^1 \log (\sqrt{1-x} + \sqrt{1+x}) dx$$

$$= \left[x \log (\sqrt{1-x} + \sqrt{1+x}) \right]_0^1 - \int_0^1 \left(\frac{1}{\sqrt{1-x} + \sqrt{1+x}} \right) \left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}} \right) \cdot x dx$$

$$= \log \sqrt{2} + \frac{1}{2} \int_0^1 \frac{(\sqrt{1+x} - \sqrt{1-x}) \cdot x}{(\sqrt{1+x} + \sqrt{1-x}) \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{(\sqrt{1+x} - \sqrt{1-x})^2 \cdot x}{(\sqrt{1+x} + \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x}) \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{(2 - 2\sqrt{1-x^2}) \cdot x}{2x \sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int_0^1 dx$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} [\sin^{-1} x]_0^1 - \frac{1}{2} [x]_0^1$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2}$$

$$= \frac{1}{2} \left[\log 2 + \frac{\pi}{2} - 1 \right]$$

Ans. : (B)

$$(33) \int_1^{e^{37}} \frac{\pi \sin (\pi \log x)}{x} dx = \dots\dots$$

$$(A) 0$$

$$(B) 2$$

$$(C) 37$$

$$(D) 74$$

$$\text{Sol. : I} = \int_1^{e^{37}} \frac{\pi \sin(\pi \log x)}{x} dx$$

$$\text{Let } \pi \log x = t. \quad \text{So, } \frac{\pi}{x} dx = dt.$$

$$x = 1 \Rightarrow t = 0 \text{ and } x = e^{37} \Rightarrow t = 37\pi$$

$$= \int_0^{37\pi} \sin t \, dt$$

$$= [-\cos t]_0^{37\pi} = -[\cos 37\pi - \cos 0] = -(-1 - 1) = 2$$

Ans. : (B)

$$(34) \int_0^1 \cos \left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx = \dots$$

(A) $-\frac{1}{2}$

(B) -1

(C) $-\frac{3}{4}$

(D) $-\frac{3}{8}$

$$\text{Sol. : I} = \int_0^1 \cos \left(2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx$$

$$\text{Let, } x = \cos \theta. \quad \text{So, } dx = -\sin \theta \, d\theta$$

$$x = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ and } x = 1 \Rightarrow \theta = 0$$

$$\therefore \text{ I} = - \int_{\frac{\pi}{2}}^0 \cos \left(2 \cot^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \right) \sin \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos \left(2 \cot^{-1} \left(\tan \frac{\theta}{2} \right) \right) \sin \theta \, d\theta \quad \left(0 < \frac{\theta}{2} < \frac{\pi}{4} \right)$$

$$= \int_0^{\frac{\pi}{2}} \cos \left\{ 2 \cot^{-1} \left(\cot \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right) \right\} \sin \theta \, d\theta \quad \left(0 < \frac{\pi}{4} < \frac{\pi}{2} - \frac{\theta}{2} < \frac{\pi}{2} < \pi \right)$$

$$= \int_0^{\frac{\pi}{2}} \cos (\pi - \theta) \sin \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\cos\theta \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} -\sin 2\theta \, d\theta$$

$$= \frac{1}{2} \left[\frac{\cos 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} [\cos\pi - \cos 0] = \frac{1}{4} [-1 - 1] = -\frac{1}{2}$$

Ans. : (A)

$$(35) \quad \int_{\frac{1}{e}}^{\tan x} \frac{t \, dt}{1+t^2} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)} = \dots\dots$$

- (A) 1 (B) 0 (C) $\frac{1}{e}$ (D) e

$$\text{Sol. : I} = \int_{\frac{1}{e}}^{\tan x} \frac{t \, dt}{1+t^2} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{Let } I = I_1 + I_2$$

$$\text{where, } I_1 = \int_{\frac{1}{e}}^{\tan x} \frac{t \, dt}{1+t^2} \quad \text{and} \quad I_2 = \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{Now } I_2 = \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{Let, } t = \frac{1}{u} \quad \text{So, } dt = -\frac{1}{u^2} du$$

$$t = \frac{1}{e} \Rightarrow u = e \quad \text{and} \quad t = \cot x \Rightarrow u = \tan x$$

$$\therefore I_2 = \int_e^{\tan x} \frac{1}{\frac{1}{u}\left(1+\frac{1}{u^2}\right)} \left(-\frac{1}{u^2}\right) du$$

$$= - \int_e^{\tan x} \frac{u \, du}{u^2 + 1}$$

$$= - \left\{ \int_e^{\tan x} \frac{u \, du}{1+u^2} + \int_{\frac{1}{e}}^{\tan x} \frac{u \, du}{1+u^2} \right\}$$

$$I_2 = -\int_e^{\frac{1}{e}} \frac{t}{1+t^2} dt - \int_e^{\tan x} \frac{t}{1+t^2} dt = -\int_e^{\frac{1}{e}} \frac{t}{1+t^2} dt - I_1$$

$$\therefore I_1 + I_2 = \frac{-1}{2} \int_e^{\frac{1}{e}} \frac{2t}{1+t^2} dt$$

$$\begin{aligned} \therefore I &= \frac{-1}{2} \left[\log(t^2 + 1) \right]_e^{\frac{1}{e}} \\ &= \frac{-1}{2} \left[\log \left| \frac{1}{e^2} + 1 \right| - \log |e^2 + 1| \right] \\ &= \frac{-1}{2} \log \left(\frac{1}{e^2} \right) = \frac{-1}{2} \times -2 \times \log_e e = 1 \end{aligned}$$

Ans. : (A)

(36) If $\alpha f(x) + \beta f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, $x \neq 0$ and $\alpha \neq \beta$, then $\int_1^2 f(x) dx = \dots$

(A) $\frac{2\alpha \log 2 - 5\alpha + 3\beta}{2(\alpha^2 - \beta^2)}$ (B) $\frac{3\alpha \log 2 - 7\alpha + 5\beta}{2(\alpha^2 - \beta^2)}$ (C) $\frac{\alpha \log 2 - 5\alpha + 3\beta}{2(\alpha^2 - \beta^2)}$ (D) $\frac{2\alpha \log 2 - 10\alpha + 7\beta}{2(\alpha^2 - \beta^2)}$

(IIT : 1995)

Sol. : $\alpha f(x) + \beta f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ (1)

Replacing x by $\frac{1}{x}$ in (1), $\alpha f\left(\frac{1}{x}\right) + \beta f(x) = x - 5$ (2)

Multiply (1) by α and (2) by β , then subtract,

$$(\alpha^2 - \beta^2) f(x) = \frac{\alpha}{x} - 5\alpha - \beta x + 5\beta$$

$$f(x) = \frac{1}{(\alpha^2 - \beta^2)} \left\{ \frac{\alpha}{x} - \beta x - 5(\alpha - \beta) \right\}$$

$$\begin{aligned} \text{Now, } \int_1^2 f(x) dx &= \frac{1}{(\alpha^2 - \beta^2)} \int_1^2 \left[\frac{\alpha}{x} - \beta x - 5(\alpha - \beta) \right] dx \\ &= \frac{1}{(\alpha^2 - \beta^2)} \left\{ \alpha [\log x]_1^2 - \beta \left[\frac{x^2}{2} \right]_1^2 - 5(\alpha - \beta) [x]_1^2 \right\} \\ &= \frac{1}{(\alpha^2 - \beta^2)} \left\{ \alpha \log 2 - \frac{3}{2} \beta - 5(\alpha - \beta) \right\} \\ &= \frac{2\alpha \log 2 - 10\alpha + 7\beta}{2(\alpha^2 - \beta^2)} \end{aligned}$$

Ans. : (D)

$$(37) \int_0^1 \frac{dx}{x^2 + (2\cos\alpha)x + 1} = \dots\dots$$

$$(A) \frac{1}{2\alpha \sin \alpha}$$

$$(B) \frac{\alpha}{2 \sin \alpha}$$

$$(C) \alpha \sin \alpha$$

$$(D) \frac{\alpha}{2} \sin \alpha$$

$$\text{Sol. : I} = \int_0^1 \frac{dx}{x^2 + (2\cos\alpha)x + 1} \quad (0 < \alpha < \frac{\pi}{2})$$

$$= \int_0^1 \frac{dx}{(x^2 + (2\cos\alpha)x + \cos^2\alpha) + (1 - \cos^2\alpha)}$$

$$= \int_0^1 \frac{dx}{(x + \cos\alpha)^2 + (\sin\alpha)^2}$$

$$= \left[\frac{1}{\sin \alpha} \tan^{-1} \left(\frac{x + \cos \alpha}{\sin \alpha} \right) \right]_0^1$$

$$= \frac{1}{\sin \alpha} \left[\tan^{-1} \left(\frac{1 + \cos \alpha}{\sin \alpha} \right) - \tan^{-1} \left(\frac{\cos \alpha}{\sin \alpha} \right) \right]$$

$$= \frac{1}{\sin \alpha} \left[\tan^{-1} \frac{2 \cos^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} - \tan^{-1} (\cot \alpha) \right]$$

$$= \frac{1}{\sin \alpha} \left[\tan^{-1} \left(\cot \frac{\alpha}{2} \right) - \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \alpha \right) \right) \right]$$

$$= \frac{1}{\sin \alpha} \left[\tan^{-1} \left(\tan \left(\frac{\pi}{2} - \frac{\alpha}{2} \right) \right) - \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \alpha \right) \right) \right]$$

$$= \frac{1}{\sin \alpha} \left[\left(\frac{\pi}{2} - \frac{\alpha}{2} \right) - \left(\frac{\pi}{2} - \alpha \right) \right]$$

$$= \frac{\alpha}{2 \sin \alpha}$$

Ans. : (B)

$$(38) \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx = \dots\dots$$

$$(A) 2 \cos \frac{1}{2}$$

$$(B) \cos \frac{1}{3}$$

$$(C) 2 \log 2$$

$$(D) 0$$

$$\text{Sol. : I} = \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

$$= \int_{-\log 2}^{\log 2} \sin \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

$$\text{Let, } f(x) = \sin \left(\frac{e^x - 1}{e^x + 1} \right)$$

$$\therefore f(-x) = \sin \left(\frac{e^{-x} - 1}{e^{-x} + 1} \right)$$

$$\therefore f(-x) = \sin \left(\frac{1 - e^x}{1 + e^x} \right) = -\sin \left(\frac{e^x - 1}{e^x + 1} \right) = -f(x)$$

$\therefore f(x)$ is an odd function.

$$\therefore I = 0$$

Ans. : (D)

$$\text{Second method : } I = \int_{\log \frac{1}{2}}^{\log 2} \sin \left(\frac{\frac{x}{2} - e^{-\frac{x}{2}}}{\frac{x}{2} + e^{-\frac{x}{2}}} \right) dx = \int_{-\log 2}^{\log 2} \sin \left(\frac{\frac{x}{2} - e^{-\frac{x}{2}}}{\frac{x}{2} + e^{-\frac{x}{2}}} \right) dx$$

$$\begin{aligned} \therefore f(-x) &= \sin \left(\frac{e^{-\frac{x}{2}} - e^{\frac{x}{2}}}{e^{-\frac{x}{2}} + e^{\frac{x}{2}}} \right) \\ &= -\sin \left(\frac{\frac{x}{2} - e^{-\frac{x}{2}}}{\frac{x}{2} + e^{-\frac{x}{2}}} \right) = -f(x) \end{aligned}$$

$$\therefore I = 0$$

Ans. : (D)

$$(39) \int_1^{16} \tan^{-1} \sqrt{(\sqrt{x} - 1)} dx = \dots\dots$$

$$(A) \frac{3\pi}{10} - 3\sqrt{2}$$

$$(B) \frac{4\pi}{15} - 3\sqrt{3}$$

$$(C) \frac{16\pi}{3} - 2\sqrt{3}$$

$$(D) \frac{9\pi}{16} - 2\sqrt{2}$$

$$\text{Sol. : } I = \int_1^{16} \tan^{-1} \sqrt{(\sqrt{x} - 1)} dx$$

$$\text{Let, } x = \sec^4 \theta. \quad \text{So, } dx = 4 \sec^3 \theta (\sec \theta \tan \theta) d\theta$$

$$x = 1 \Rightarrow \theta = 0$$

$$x = 16 \Rightarrow \theta = \sec^{-1} \sqrt[4]{16} = \sec^{-1} 2 = \frac{\pi}{3}$$

$$\therefore I = 4 \int_0^{\frac{\pi}{3}} \theta \sec^3 \theta (\sec \theta \tan \theta) d\theta$$

$$\begin{aligned}
&= 4 \left\{ \left[\theta \cdot \frac{\sec^4 \theta}{4} \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \left(1 \cdot \frac{\sec^4 \theta}{4} \right) d\theta \right\} \quad \left(\int (\sec \theta)^3 (\sec \theta \tan \theta) d\theta = \frac{\sec^4 \theta}{4} \right) \\
&= 4 \left[\frac{4\pi}{3} - \frac{1}{4} \int_0^{\frac{\pi}{3}} \sec^2 \theta (1 + \tan^2 \theta) d\theta \right] \\
&= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^2 \theta d\theta - \int_0^{\frac{\pi}{3}} \sec^2 \theta \tan^2 \theta d\theta \\
&= \frac{16\pi}{3} - [\tan \theta]_0^{\frac{\pi}{3}} - \left[\frac{\tan^3 \theta}{3} \right]_0^{\frac{\pi}{3}} \\
&= \frac{16\pi}{3} - (\sqrt{3} - 0) - \left(\frac{3\sqrt{3}}{3} - 0 \right) = \frac{16\pi}{3} - (\sqrt{3} + \sqrt{3}) = \frac{16\pi}{3} - 2\sqrt{3} \quad \text{Ans. : (C)}
\end{aligned}$$

(40) If $I_{(m, n)} = \int_0^1 x^m (1-x)^n dx$, where $m, n \in \mathbb{N}$, then $\frac{I_{(m, n)}}{I_{(m+1, n-1)}} = \dots$

- (A) $\frac{n}{m+1}$ (B) $\frac{m}{n-1}$ (C) $\frac{m-1}{n+1}$ (D) $\frac{n-1}{m+1}$

$$\begin{aligned}
\text{Sol. : } I_{(m, n)} &= \int_0^1 x^m (1-x)^n dx \\
&= \left[(1-x)^n \cdot \frac{x^{m+1}}{m+1} \right]_0^1 - \int_0^1 n(1-x)^{n-1} \cdot (-1) \frac{x^{m+1}}{m+1} dx \\
&= 0 + \frac{n}{m+1} \int_0^1 x^{m+1} (1-x)^{n-1} dx \\
I_{(m, n)} &= \frac{n}{m+1} I_{(m+1, n-1)}
\end{aligned}$$

$$\therefore \frac{I_{(m, n)}}{I_{(m+1, n-1)}} = \frac{n}{m+1} \quad \text{Ans. : (A)}$$

(41) The value of $\int_0^1 (1 + e^{-x^2}) dx$ lies in interval.

- (A) $(0, 1)$ (B) $\left(\frac{1}{e}, 1\right)$ (C) $\left(\frac{1}{e^2}, 2\right)$ (D) $\left(1 + \frac{1}{e}, 2\right)$

Sol. : $0 < x < 1$

$$\therefore 0 < x^2 < 1$$

$$\therefore e^0 < e^{x^2} < e^1$$

(e^{x^2} is an increasing function)

$$\therefore 1 < e^{x^2} < e$$

$$\therefore \frac{1}{e} < \frac{1}{e^{x^2}} < 1$$

$$\therefore 1 + \frac{1}{e} < 1 + e^{-x^2} < 2$$

$$\therefore \left(1 + \frac{1}{e}\right) (1 - 0) < \int_0^1 (1 + e^{-x^2}) dx < 2(1 - 0)$$

$$\therefore \left(1 + \frac{1}{e}\right) < \int_0^1 (1 + e^{-x^2}) dx < 2$$

$$\therefore \text{The value of } \int_0^1 (1 + e^{-x^2}) dx \text{ lies in the interval } \left(1 + \frac{1}{e}, 2\right).$$

Ans. : (D)

(42) If the function f is defined for real positive numbers and

$$\text{if } I_1 = \int_{1-k}^k x \cdot f\{x(1-x)\} dx \text{ and } I_2 = \int_{1-k}^k f\{x(1-x)\} dx \text{ where } 2k-1 > 0, \text{ then } \frac{I_1}{I_2} = \dots$$

(A) 2

(B) 1

(C) $\frac{1}{2}$

(D) 3

(IIT : 1997)

$$\text{Sol. : } I_1 = \int_{1-k}^k x \cdot f\{x(1-x)\} dx$$

$$= \int_{1-k}^k (1-x) f\{(1-x)(1-(1-x))\} dx$$

$$(a + b - x = 1 - x)$$

$$= \int_{1-k}^k (1-x) f\{(1-x)x\} dx$$

$$I_1 = \int_{1-k}^k f\{x(1-x)\} dx - \int_{1-k}^k x f\{x(1-x)\} dx$$

$$\therefore I_1 = I_2 - I_1$$

$$\therefore 2I_1 = I_2$$

$$\therefore \frac{I_1}{I_2} = \frac{1}{2}$$

Ans. : (C)

$$(43) \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx = \dots$$

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{2} - 1$

(C) $\frac{\pi^4}{32}$

(D) $\frac{\pi^4}{32} + \frac{\pi}{2}$

$$\text{Sol. : I} = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx$$

$$= \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x + \pi)^3 + \cos^2(x + \pi)] dx$$

$$\text{Let, } x + \pi = t. \quad \text{So, } dx = dt$$

$$x = -\frac{3\pi}{2} \Rightarrow t = -\frac{\pi}{2} \quad \text{and} \quad x = -\frac{\pi}{2} \Rightarrow t = \frac{\pi}{2}$$

$$\therefore \quad \text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t^3 + \cos^2 t) dt$$

$$= 0 + 2 \int_0^{\frac{\pi}{2}} \cos^2 t dt \quad (t^3 \text{ is an odd function and } \cos^2 t \text{ is an even function})$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{(1 + \cos 2t)}{2} dt = \left[t + \frac{\sin 2t}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Ans. : (A)

$$(44) \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x + \sin x)} dx = \dots\dots$$

(A) $\frac{\pi}{4} - \frac{1}{2} \log 2$ (B) $\frac{\pi}{2} - 2 \log 2$ (C) $\pi - \frac{1}{2} \log 2$ (D) $\frac{\pi}{4} - 2 \log 2$

$$\text{Sol. : I} = \int_0^{\frac{\pi}{2}} \frac{\sin x}{(1 + \cos x + \sin x)} dx \quad (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin \left(\frac{\pi}{2} - x \right)}{1 + \cos \left(\frac{\pi}{2} - x \right) + \sin \left(\frac{\pi}{2} - x \right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x + \cos x} dx \quad (2)$$

Adding (1) and (2), we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1 + \sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1 + \sin x + \cos x) - (1)}{(1 + \sin x + \cos x)} dx$$

$$= \int_0^{\frac{\pi}{2}} 1 dx - \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx$$

$$2I = [x]_0^{\frac{\pi}{2}} - I_1$$

$$2I = \frac{\pi}{2} - I_1 \quad (3)$$

$$\text{Now, } I_1 = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx$$

$$\text{Let, } \tan \frac{x}{2} = t. \quad \text{So, } dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\therefore I_1 = \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{2(1+t)}$$

$$= [\log(1+t)]_0^1$$

$$= \log 2 - \log 1 = \log 2$$

$$\therefore 2I = \frac{\pi}{2} - \log 2 \quad (\text{substituting in (3)})$$

$$I = \frac{\pi}{4} - \frac{1}{2} \log 2$$

Ans. : (A)

$$(45) \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{(\sin^4 x + \cos^4 x)} dx = \dots$$

- (A) 0 (B) $\frac{\pi^2 - 1}{4}$ (C) $\frac{\pi^2}{16}$ (D) $\frac{\pi}{4} (\pi - 1)$

$$\text{Sol. : } I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{(\sin^4 x + \cos^4 x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\sin^4 x + \cos^4 x} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} \sin x \cos x}{\cos^4 x + \sin^4 x} dx - \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx - I$$

$$\therefore 2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x \sec^2 x dx}{\tan^4 x + 1} \quad \text{(dividing numerator and denominator by } \cos^4 x \text{)}$$

$$\text{Let, } \tan^2 x = t. \quad \text{So, } 2 \tan x \sec^2 x dx = dt$$

$$x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = \infty$$

$$\therefore 2I = \frac{\pi}{4} \int_0^{\infty} \frac{dt}{t^2 + 1} = \frac{\pi}{4} [\tan^{-1} t]_0^{\infty} = \frac{\pi}{4} \left[\frac{\pi}{2} - 0\right]$$

$$\therefore 2I = \frac{\pi^2}{8}. \quad \text{So, } I = \frac{\pi^2}{16} \quad \text{Ans. : (C)}$$

$$(46) \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx = \dots$$

(A) $-\pi$

(B) $\frac{\pi}{2}$

(C) π

(D) 2π

$$\text{Sol. : } I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \quad (i)$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx \quad (x \Rightarrow a+b-x)$$

$$= \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{a^x + 1} dx \quad (ii)$$

Adding (i) and (ii), we get,

$$\therefore 2I = \int_{-\pi}^{\pi} \left(\frac{\cos^2 x}{1+a^x} + \frac{a^x \cos^2 x}{1+a^x} \right) dx$$

$$2I = \int_{-\pi}^{\pi} \cos^2 x \, dx = 2 \int_0^{\pi} \left(\frac{1+\cos 2x}{2} \right) dx$$

$$\therefore 2I = \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= [\pi - (0)] = \pi$$

$$\therefore I = \frac{\pi}{2}$$

Ans. : (B)

$$(47) \text{ If } f(x) = \frac{e^x}{1+e^x}, I_1 = \int_{f(-a)}^{f(a)} x g\{x(1-x)\} dx \text{ and } I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\} dx, \text{ then } \frac{I_2}{I_1} = \dots$$

(A) $\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2 (AIEEE : 2004)

$$\text{Sol. : } f(x) = \frac{e^x}{1+e^x} \Rightarrow f(a) = \frac{e^a}{1+e^a} \text{ and } f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^a}$$

$$\therefore f(a) + f(-a) = 1$$

Let, $f(a) = k$. so $f(-a) = 1 - k$

$$\therefore I_1 = \int_{1-k}^k x g \{x(1-x)\} dx$$

$$= \int_{1-k}^k (1-x) g \{(1-x)(1-(1-x))\} dx \quad (x \Rightarrow a + b - x)$$

$$I_1 = \int_{1-k}^k (1-x) g \{x(1-x)\} dx$$

$$\therefore I_1 = \int_{1-k}^k g \{x(1-x)\} dx - \int_{1-k}^k x g \{x(1-x)\} dx$$

$$\therefore I_1 = I_2 - I_1$$

$$\therefore 2I_1 = I_2$$

$$\therefore \frac{I_2}{I_1} = 2$$

Ans. : (D)

$$(48) \text{ If } f(x) = \begin{cases} e^{\cos x} \cdot \sin x, & \text{where } |x| \leq 2 \\ 2, & \text{otherwise} \end{cases}, \text{ then } \int_{-2}^3 f(x) dx = \dots$$

(A) 0

(B) 1

(C) 2

(D) 3

(IIT : 2000)

$$\text{Sol. : } I = \int_{-2}^3 f(x) dx$$

$$= \int_{-2}^2 f(x) dx + \int_2^3 f(x) dx$$

$$= \int_{-2}^2 e^{\cos x} \cdot \sin x dx + \int_2^3 2 dx$$

$$= 0 + 2 [x]_2^3 = 2[3 - 2] = 2$$

($e^{\cos x} \cdot \sin x$ is an odd function) **Ans. : (C)**

$$(49) \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{(1 + \cos^2 x)} dx = \dots$$

(A) 0

(B) 2π

(C) $\frac{\pi}{4}$

(D) π^2

$$\text{Sol. : } I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{(1 + \cos^2 x)} dx$$

$$= 2 \int_{-\pi}^{\pi} \frac{x}{1 + \cos^2 x} dx + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = 0 + 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \left(\frac{x}{1 + \cos^2 x} \text{ is odd and } \frac{x \sin x}{1 + \cos^2 x} \text{ is even} \right) \quad (1)$$

$$= 4 \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \cos^2 (\pi - x)} dx$$

$$= 4 \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$$

$$I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - I \quad (\text{Result (1)})$$

$$2I = 4\pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x}$$

$$\text{Let, } \cos x = t. \quad \text{So, } -\sin x dx = dt$$

$$\text{and } x = 0 \Rightarrow t = 1 \text{ and } x = \pi \Rightarrow t = -1$$

$$2I = 4\pi \int_1^{-1} \frac{-dt}{1 + t^2}$$

$$2I = 4\pi \int_{-1}^1 \frac{dt}{1 + t^2} = 4\pi [\tan^{-1} t]_{-1}^1$$

$$\therefore I = 2\pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 2\pi \left(\frac{\pi}{2} \right) = \pi^2 \quad \text{Ans. : (D)}$$

$$(50) \quad \int_0^{10\pi} |\sin x| dx = \dots\dots$$

(A) 5

(B) 10

(C) 20

(D) 25

$$\begin{aligned}
\text{Sol. : } I &= \int_0^{10\pi} |\sin x| \, dx \\
&= 10 \int_0^{\pi} |\sin x| \, dx \quad (\pi \text{ is fundamental period of } |\sin x|) \\
&= 10 \int_0^{\pi} \sin x \, dx \\
&= 10 [-\cos x]_0^{\pi} = -10 [\cos \pi - \cos 0] = -10 (-1 - 1) = 20 \quad \text{Ans. : (C)}
\end{aligned}$$

(51) If $I_1 = \int_0^1 2x^2 \, dx$, $I_2 = \int_0^1 2x^3 \, dx$, $I_3 = \int_1^2 2x^2 \, dx$ and $I_4 = \int_1^2 2x^3 \, dx$, then

(A) $I_1 > I_2$ (B) $I_3 > I_4$ (C) $I_1 < I_2$ (D) $I_3 = I_4$

(AIEEE : 2005)

Sol. : Here, $x^2 > x^3$, $\forall x \in [0, 1]$, $x^2 < x^3$, $\forall x \in [1, 2]$

$\therefore 2x^2 > 2x^3$, $\forall x \in [0, 1]$, $2x^2 < 2x^3$, $\forall x \in [1, 2]$

$\therefore \int_0^1 2x^2 \, dx > \int_0^1 2x^3 \, dx$ and $\int_1^2 2x^2 \, dx < \int_1^2 2x^3 \, dx$

$\therefore I_1 > I_2$ and $I_3 < I_4$ Ans. : (A)

(52) If $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ and $F(x) = \int_0^x f(t) \, dt$ and $F(x^2) = x^2(1+x)$, then $f(4) = \dots$

(A) $\frac{5}{4}$ (B) 2 (C) 4 (D) 7

(IIT : 2004)

Sol. : Here, $F(x) = \int_0^x f(t) \, dt$

$\therefore F(x^2) = \int_0^{x^2} f(t) \, dt$

$\therefore x^2(1+x) = \int_0^{x^2} f(t) \, dt$ [F(x^2) = x^2(1+x)]

Differentiating on both sides,

$x^2(1) + 2x(1+x) = f(x^2) \cdot (2x)$

$\therefore 3x^2 + 2x = f(x^2) \cdot (2x)$

$\therefore f(x^2) = \frac{3}{2}x + 1$

Putting, $x = 2$, we get, $f(4) = 4$

Ans. : (C)

(53) If $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$, $x \in \left(0, \frac{\pi}{2}\right)$, then $f\left(\frac{1}{\sqrt{3}}\right) = \dots$

- (A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{3}$ (C) $\sqrt{3}$ (D) 3

(IIT : 2005)

Sol. : $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$

Differentiating on both sides w.r.t. x ,

$$0 - \sin^2 x f(\sin x) \cos x = -\cos x$$

$$\therefore f(\sin x) = \frac{1}{\sin^2 x}$$

$$\therefore f(x) = \frac{1}{x^2}$$

$$\therefore f\left(\frac{1}{\sqrt{3}}\right) = (\sqrt{3})^2 = 3$$

Ans. : (D)

(54) If $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, then $\frac{dy}{dx}$ at $x = \pi$ is ($x > 0$)

- (A) 0 (B) π (C) π^2 (D) 2π

(IIT : 2004)

Sol. : $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$

$$= \cos x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$$

Differentiating both sides w.r.t. x ,

$$\frac{dy}{dx} = -\sin x \cdot \left(\int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta \right) + \left(\frac{\cos x}{1 + \sin^2 x} \cdot 2x \right) \cos x$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=\pi} = [0] + \frac{2\pi(-1)^2}{1+0} = 2\pi$$

$$\therefore \left(\frac{dy}{dx} \right)_{x=\pi} = 2\pi$$

Ans. : (D)

(55) If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ is an increasing function in

- (A) the interval $[-2, 2]$ (B) the interval $[0, \infty]$
 (C) the interval $(-\infty, 0)$ (D) for no value of x .

(IIT : 2003)

Sol. : $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$

$$\therefore f'(x) = e^{-(x^2+1)^2} (2x) - e^{-(x^2)^2} \cdot 2x$$

$$\left(F(x) = \int_{h(x)}^{g(x)} f(t) dt \Rightarrow F'(x) = f[g(x)] \cdot g'(x) - f[h(x)] \cdot h'(x) \right)$$

$$\begin{aligned} \therefore f'(x) &= (2x) [e^{-(x^4+2x^2+1)} - e^{-x^4}] \\ &= (2x) (e^{-x^4}) [e^{-(2x^2+1)} - 1] \\ &= (2x) (e^{-x^4}) (1 - e^{2x^2+1}) e^{-(2x^2+1)} \end{aligned}$$

Now, $e^{(2x^2+1)} > e > 1 \Rightarrow 1 - e^{(2x^2+1)} < 0$

and $e^{-x^4} > 0, e^{-(2x^2+1)} > 0$

Now, if f is to be an increasing function $f'(x) > 0$ is necessary.

If $f'(x) > 0$, then $2x < 0$ i.e. $x < 0$

So, if $x < 0$, then f is an increasing function. So, f is an increasing in $(-\infty, 0)$.

Ans. : (C)

(56) Function $f(x) = \int_{-1}^x t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$, has a local minimum at $x = \dots\dots$

- (A) $x = 0$ and $x = 1$ (B) $x = 0$ and $x = 2$
 (C) $x = 1$ and $x = 2$ (D) $x = 1$ and $x = 3$

(IIT : 1999)

Sol. : $f(x) = \int_{-1}^x t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$

$$\therefore f'(x) = x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5$$

Now, for local maximum or minimum values, we must have $f'(x) = 0$

$$\therefore x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5 = 0$$

$$\therefore x = 0 \text{ or } x = 1 \text{ or } x = 2 \text{ or } x = 3$$

$\therefore f$ may be a local minimum at $x = 0, 1, 2$ or 3 .

Now, for using small positive number h ,

(i) $f'(0 - h) < 0$ and $f'(0 + h) < 0$

Hence, at $x = 0$, f has neither a maximum nor a minimum value.

(ii) $f'(1 - h) < 0$ and $f'(1 + h) > 0$

Hence at $x = 1$ there is a local minimum.

(iii) $f'(2 - h) > 0$ and $f'(2 + h) < 0$

Hence at $x = 2$ there is no local minimum.

(iv) $f'(3 - h) < 0$ and $f'(3 + h) > 0$

Hence at $x = 3$ there is a local minimum.

$\therefore$ At $x = 1$ and $x = 3$ there is a local minimum.

Ans. : (D)

(57) $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)^2} = \dots\dots$

(A) $\frac{3}{4}$

(B) $\frac{2}{3}$

(C) $\frac{2}{9}$

(D) $\frac{1}{2}$

Sol. : $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)^2}$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 \left(1 + \frac{r}{n}\right)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{\left(1 + \frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{(1+x)^2} dx$$

[Replace $\frac{r}{n}$ by x , $\frac{1}{n}$ by dx and Σ by $\int$]

$$= \left[\frac{-1}{1+x} \right]_0^1 = \left(-\frac{1}{2} - (-1) \right) = -\frac{1}{2} + 1 = \frac{1}{2}$$

Ans. : (D)

(58) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, then $\lim_{n \rightarrow \infty} n [I_n + I_{n+2}] = \dots\dots$

(A) 0

(B) $\frac{1}{2}$

(C) 1

(D) ∞

Sol. : $I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x \, dx + \int_0^{\frac{\pi}{4}} \tan^{n+2} x \, dx$

$$= \int_0^{\frac{\pi}{4}} \tan^n x (\tan^2 x + 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \cdot \sec^2 x \, dx$$

$$= \left[\frac{\tan^{n+1} x}{n+1} \right]_0^{\frac{\pi}{4}}$$

$$\begin{aligned}\therefore \lim_{n \rightarrow \infty} n [I_n + I_{n+2}] &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right) \\ &= 1\end{aligned}$$

Ans. : (C)

$$(59) \quad \lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cot t^2 dt}{x \sin x} = \dots\dots$$

(A) $-\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2

(IIT : 1997)

Sol. : $\lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cot t^2 dt}{x \sin x}$ is required.

Let $f(x) = \int_0^{x^2} \cos t^2 dt$ and $g(x) = x \sin x$

Now, $f(0) = \int_0^0 \cos t^2 dt = 0$ and $g(0) = 0 \sin 0 = 0$

$$\therefore \lim_{x \rightarrow 0} \int_0^{x^2} \frac{\cos t^2 dt}{x \sin x} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \quad (\text{L-Hospital's rule})$$

$$= \lim_{x \rightarrow 0} \frac{(\cos x^4) (2x)}{x \cos x + \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2[-x(\sin x^4)4x^3 + \cos x^4(1)]}{-x \sin x + \cos x + \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{2[-4x^4 \sin x^4 + \cos x^4]}{2\cos x - x \sin x} = \frac{2(0+1)}{2(1)-0} = \frac{2}{2} = 1 \quad \text{Ans. : (B)}$$

$$(60) \quad \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt - \int_{x+y}^c e^{\sin^2 t} dt \right\} = \dots\dots$$

(A) $\sin y \cdot e^y$

(B) 0

(C) $e^{\sin^2 y}$

(D) $\sin 2y \cdot e^{\sin^2 y}$

Sol. : Let $X = \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt - \int_{x+y}^c e^{\sin^2 t} dt \right\}$

$$= \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \int_y^c e^{\sin^2 t} dt + \int_c^{x+y} e^{\sin^2 t} dt \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{\int_y^{x+y} e^{\sin^2 t} dt}{x} \right\}$$

Now, let $f(x) = \int_y^{x+y} e^{\sin^2 t} dt$ and $g(x) = x$

$$f(0) = 0 \text{ and } g(0) = 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \quad (\text{L-Hospital's rule})$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{e^{\sin^2(x+y)} \frac{d}{dx}(x+y)}{1} \right\}$$

$$= \lim_{x \rightarrow 0} e^{\sin^2(x+y)} = e^{\sin^2 y}$$

Ans. : (C)

(61) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function, $f(2) = 6$ and $f'(2) = \frac{1}{48}$, then $\lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{x-2} = \dots\dots$

(A) 32

(B) 26

(C) 18

(D) 12 (AIEEE : 2005)

Sol. : Let, $g(x) = \int_6^{f(x)} 4t^3 dt$ and $h(x) = x - 2$

$$\text{then } g(2) = \int_6^{f(2)} 4t^3 dt = \int_6^6 4t^3 dt = 0 \text{ and } h(2) = 2 - 2 = 0$$

$$\therefore \lim_{x \rightarrow 2} \frac{g(x)}{h(x)} = \lim_{x \rightarrow 2} \frac{g'(x)}{h'(x)} \quad (\text{L-Hospital rule})$$

$$= \lim_{x \rightarrow 2} \frac{4\{f(x)\}^3 \cdot f'(x)}{1}$$

$$= 4 \{f(2)\}^3 \cdot f'(2) = 4 \times 6^3 \times \frac{1}{48} = 18$$

Ans. : (C)

(62) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right) = \dots\dots$

(A) $\log \left(\frac{27}{4e} \right)$

(B) $\log \left(\frac{4}{e} \right)$

(C) $\log \left(\frac{25}{3e^2} \right)$

(D) $\log \left(\frac{27}{e^2} \right)$

Sol. : $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right)$

Replacing $\frac{r}{n}$ by x , $\frac{1}{n}$ by dx and Σ by $\int$,

$$\text{and lower limit of integration} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1 \quad (r = n+1)$$

$$\text{Upper limit of integration} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{2n}{n} = 2 \quad (r = 2n)$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log \left(1 + \frac{r}{n} \right) &= \int_1^2 \log (1+x) dx \\ &= [x \log (1+x)]_1^2 - \int_1^2 \frac{x dx}{1+x} \\ &= 2 \log 3 - \log 2 - \int_1^2 \frac{x+1-1}{1+x} dx \\ &= \log \frac{9}{2} - \int_1^2 dx + \int_1^2 \frac{dx}{1+x} \\ &= \log \frac{9}{2} - [x]_1^2 + [\log (1+x)]_1^2 \\ &= \log \frac{9}{2} + \log 3 - \log 2 - 1 \\ &= \log \frac{27}{4e} \quad (1 = \log e) \end{aligned} \quad \text{Ans. : (A)}$$

$$(63) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} = \dots$$

(A) $\sqrt{2} - 1$ (B) $\sqrt{5} - 1$ (C) $2\sqrt{3} - 5$ (D) $5\sqrt{2} - 3$

$$\begin{aligned} \text{Sol. : } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{\frac{r}{n}}{\sqrt{1 + \left(\frac{r}{n}\right)^2}} \\ &= \int_0^2 \frac{x}{\sqrt{1+x^2}} dx \\ &= \frac{1}{2} \int_0^2 \frac{2x}{\sqrt{1+x^2}} dx = \frac{1}{2} \left[2\sqrt{1+x^2} \right]_0^2 = (\sqrt{5} - 1) \end{aligned} \quad \text{Ans. : (B)}$$

$$(64) \quad \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{4n} \right\} = \dots$$

(A) $\log 2$ (B) $2 \log 3$ (C) $\log 4$ (D) $\frac{1}{2} \log 5$

$$\text{Sol. : } \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{4n} \right\}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=0}^{3n} \frac{1}{n+r}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=0}^{3n} \frac{1}{n} \left(\frac{1}{1 + \frac{r}{n}} \right)$$

$$= \int_0^3 \frac{1}{1+x} dx = [\log(1+x)]_0^3 = \log 4$$

Ans. : (C)

$$(65) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right] = \dots$$

(A) $\frac{2}{\pi}$

(B) $\frac{\pi}{2}$

(C) $\frac{4}{\pi}$

(D) 1

$$\text{Sol. : } \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \sec^2 \frac{r\pi}{4n}$$

$$= \int_0^1 \sec^2 \left(\frac{\pi}{4} x \right) dx$$

$$= \left[\frac{\tan \left(\frac{\pi}{4} x \right)}{\left(\frac{\pi}{4} \right)} \right]_0^1 = \frac{4}{\pi} \left[\tan \frac{\pi}{4} - \tan 0 \right] = \frac{4}{\pi}$$

Ans. : (C)

$$(66) \quad \lim_{n \rightarrow \infty} \left(\frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} \right) = \dots$$

(A) $\frac{101}{100}$

(B) $\frac{1}{101}$

(C) $\frac{99}{100}$

(D) $\frac{1}{100}$

$$\text{Sol. : } \lim_{n \rightarrow \infty} \left(\frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^{99}$$

$$= \int_0^1 x^{99} dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}$$

Ans. : (D)

$$(67) \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right] = \dots$$

(A) $2 \tan 1$

(B) $\frac{1}{2} \tan \frac{1}{2}$

(C) $2 \tan \frac{1}{2}$

(D) $\frac{1}{2} \tan 1$

(AIEEE : 2005)

$$\begin{aligned}
\text{Sol. : } & \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1^2}{n^2} + \frac{2}{n^2} \sec^2 \frac{2^2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3^2}{n^2} + \dots + \frac{n}{n^2} \sec^2 \frac{n^2}{n^2} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{r}{n} \sec^2 \left(\frac{r^2}{n^2} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right) \sec^2 \left(\frac{r}{n} \right)^2 \\
&= \int_0^1 x \sec^2(x)^2 dx \\
&= \frac{1}{2} \int_0^1 \sec^2 t dt \\
&= \frac{1}{2} [\tan t]_0^1 = \frac{1}{2} (\tan 1 - \tan 0) = \frac{1}{2} \tan(1)
\end{aligned}$$

Ans. : (D)

(68) If $f(x) = A \sin\left(\frac{\pi}{2}x\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x) dx = \frac{2A}{\pi}$, then constant A and B are respectively.

- (A) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (B) $\frac{2}{\pi}$ and $\frac{3}{\pi}$ (C) 0 and $\frac{-4}{\pi}$ (D) $\frac{4}{\pi}$ and 0

(IIT : 1993)

Sol. : Here, $f(x) = A \sin\left(\frac{\pi}{2}x\right) + B$

Let, $x^2 = t$.

So, $2x dx = dt$.

So, $x dx = \frac{1}{2} dt$

$$\therefore f'(x) = \frac{A\pi}{2} \cos\left(\frac{\pi}{2}x\right)$$

$$x = 0 \Rightarrow t = 0, \quad x = 1 \Rightarrow t = 1$$

$$f'\left(\frac{1}{2}\right) = \frac{A\pi}{2} \cos\left(\frac{\pi}{4}\right) = \frac{A\pi}{2\sqrt{2}} = \sqrt{2}.$$

$$\text{So, } A = \frac{4}{\pi}$$

Note : Here we can decide that answer is D.

$$\text{Now, } \int_0^1 f(x) dx = \frac{2A}{\pi}$$

$$\therefore \int_0^1 \left(A \sin\left(\frac{\pi x}{2}\right) + B \right) dx = \frac{2A}{\pi}$$

$$\therefore \left[\frac{-2A}{\pi} \cos \frac{\pi x}{2} + Bx \right]_0^1 = \frac{2A}{\pi}$$

$$\therefore B + \frac{2A}{\pi} = \frac{2A}{\pi}.$$

$$\text{So, } B = 0$$

$$\therefore A = \frac{4}{\pi} \text{ and } B = 0$$

Ans. : (D)

(69) If $g(x) = \int_0^x \cos^4 t \, dt$, then $g(x + \pi) = \dots$

- (A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x) g(\pi)$ (D) $\frac{g(x)}{g(\pi)}$ (IIT : 1997)

Sol. : $g(x) = \int_0^x \cos^4 t \, dt$

$$\begin{aligned} \therefore g(x + \pi) &= \int_0^{\pi+x} \cos^4 t \, dt \\ &= \int_0^{\pi} \cos^4 t \, dt + \int_{\pi}^{\pi+x} \cos^4 t \, dt \end{aligned}$$

$\therefore g(x + \pi) = I_1 + I_2$

$$I_1 = \int_0^{\pi} \cos^4 t \, dt = g(\pi)$$

$$I_2 = \int_{\pi}^{\pi+x} \cos^4 t \, dt$$

Let, $t = \pi + \theta$. So, $dt = d\theta$, $t = \pi \Rightarrow \theta = 0$; $t = \pi + x \Rightarrow \theta = x$

$$\therefore I_2 = \int_0^x \cos^4 (\pi + \theta) \, d\theta = \int_0^x \cos^4 \theta \, d\theta = \int_0^x \cos^4 t \, dt = g(x)$$

$\therefore g(x + \pi) = I_1 + I_2 = g(\pi) + g(x)$

Ans. : (A)

(70) $\frac{\sqrt{\log 3}}{\sqrt{\log 2}} \int \frac{x \sin x^2}{\sin x^2 + \sin (\log 6 - x^2)} \, dx = \dots$

- (A) $\frac{1}{4} \log \frac{3}{2}$ (B) $\frac{1}{2} \log \frac{3}{2}$ (C) $\log \frac{3}{2}$ (D) $\frac{1}{6} \log \frac{3}{2}$ (IIT : 2011)

Sol. : $\frac{\sqrt{\log 3}}{\sqrt{\log 2}} \int \frac{x \sin x^2}{\sin x^2 + \sin (\log 6 - x^2)} \, dx$

Let, $x^2 = t$. So, $x \, dx = \frac{1}{2} \, dt$

$x = \sqrt{\log 2} \Rightarrow t = \log 2$ and $x = \sqrt{\log 3} \Rightarrow t = \log 3$

$$\therefore I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin t \, dt}{\sin t + \sin (\log 6 - t)} \quad \dots(i)$$

$$\therefore I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin (\log 3 + \log 2 - t) \, dt}{\sin (\log 3 + \log 2 - t) + \sin (\log 6 - (\log 3 + \log 2 - t))}$$

$$I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin (\log 6 - t) dt}{\sin (\log 6 - t) + \sin t} \quad \dots(ii)$$

Adding (i) and (ii), we get,

$$2I = \frac{1}{2} \int_{\log 2}^{\log 3} 1 dt = \frac{1}{2} [t]_{\log 2}^{\log 3} = \frac{1}{2} (\log 3 - \log 2) = \frac{1}{2} \log \left(\frac{3}{2} \right)$$

$$I = \frac{1}{4} \log \left(\frac{3}{2} \right) \quad \text{Ans. : (A)}$$

$$(71) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log \frac{\pi - x}{\pi + x} \right) \cos x dx = \dots$$

(A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$ (IIT : 2013)

$$\begin{aligned} \text{Sol. : } I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log \frac{\pi - x}{\pi + x} \right) \cos x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left(\frac{\pi - x}{\pi + x} \right) \cdot \cos x dx \end{aligned}$$

$$\therefore I = 2 \int_0^{\frac{\pi}{2}} x^2 \cos x dx + 0$$

($\because x^2 \cos x$ is an even function and $\cos x \cdot \log \left(\frac{\pi - x}{\pi + x} \right)$ is an odd function.)

$$\begin{aligned} \therefore I &= 2 \left\{ [x^2 \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2x \sin x) dx \right\} \\ &= 2 \left\{ \left[\frac{\pi^2}{4} \right] - 2(-x \cos x)_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} 1 \cdot (-\cos x) dx \right\} \\ &= 2 \left\{ \frac{\pi^2}{4} - 2[\sin x]_0^{\frac{\pi}{2}} \right\} = 2 \left[\frac{\pi^2}{4} - 2 \right] = \left(\frac{\pi^2}{2} - 4 \right) \quad \text{Ans. : (B)} \end{aligned}$$

- (72) The intercepts made on X-axis by tangents to the curve, $y = \int_0^x |t| dt$ and which are parallel to the line $y = 2x$ are equal to

(A) ± 1 (B) ± 2 (C) ± 3 (D) ± 4 (JEE : 2013)

Sol. : Here, $y = \int_0^x |t| dt$

$$\therefore \frac{dy}{dx} = |x| \cdot 1 = |x|$$

Now, if the tangent is parallel to the line $y = 2x$, the slope of the tangent = the slope of the line

$$\therefore |x| = 2$$

$$\therefore x = \pm 2$$

$$\text{Now, } y = \int_0^{\pm 2} |t| dt = \pm 2 \quad \left(\int_0^2 t dt = \left[\frac{t^2}{2} \right]_0^2 = 2, \int_0^{-2} -t dt = \int_{-2}^0 t dt = \left[\frac{t^2}{2} \right]_{-2}^0 = -2 \right)$$

$\therefore$ Required points are (2, 2) and (-2, -2) and slope = 2

$\therefore$ The equations of the tangents are $(y - 2) = 2(x - 2)$ and $y + 2 = 2(x + 2)$

Now, for x-intercept, let $y = 0$,

$$\therefore 0 - 2 = 2(x - 2) \text{ and } 0 + 2 = 2(x + 2)$$

$$\therefore x = \pm 1$$

Ans. : (A)

(73) $\int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2} - 4 \sin \frac{x}{2}} dx = \dots\dots$

(A) $\pi - 4$ (B) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$ (C) $4\sqrt{3} - 4$ (D) $4\sqrt{3} - 4 - \frac{\pi}{3}$

(JEE : 2014)

Sol. : I = $\int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2} - 4 \sin \frac{x}{2}} dx$

$$= \int_0^{\pi} \sqrt{\left(1 - 2 \sin \frac{x}{2}\right)^2} dx$$

$$= \int_0^{\pi} \left| 1 - 2 \sin \frac{x}{2} \right| dx$$

$$= \int_0^{\frac{\pi}{3}} \left(1 - 2 \sin \frac{x}{2} \right) dx - \int_{\frac{\pi}{3}}^{\pi} \left(1 - 2 \sin \frac{x}{2} \right) dx$$

$$= \left[x + 4 \cos \frac{x}{2} \right]_0^{\frac{\pi}{3}} - \left[x + 4 \cos \frac{x}{2} \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \left[\left(\frac{\pi}{3} + \frac{4\sqrt{3}}{2} \right) - (0+4) \right] - \left[\left(\pi - \frac{\pi}{3} - \frac{4\sqrt{3}}{2} \right) \right]$$

$$= \left(4\sqrt{3} - 4 - \frac{\pi}{3} \right)$$

Ans. : (D)

$$(74) \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \operatorname{cosec} x)^{17} dx = \dots$$

$$(A) \int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$$

$$(B) \int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$$

$$(C) \int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$$

$$(D) \int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du \quad (\text{IIT : 2014})$$

$$\text{Sol. : I} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \operatorname{cosec} x)^{17} dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2^{17} (\operatorname{cosec} x)^{16} \operatorname{cosec} x dx$$

$$\text{Let } \operatorname{cosec} x + \cot x = t$$

$$\therefore -\operatorname{cosec} x dx = \frac{dt}{\operatorname{cosec} x + \cot x}$$

$$\therefore \operatorname{cosec} x dx = \frac{-dt}{t}$$

$$\therefore t + \frac{1}{t} = 2 \operatorname{cosec} x$$

$$x = \frac{\pi}{4} \Rightarrow t = \sqrt{2} + 1 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\therefore \text{I} = \int_{\sqrt{2}+1}^1 -2^{17} \left(\frac{t+\frac{1}{t}}{2} \right)^{16} \frac{dt}{t}$$

$$\text{Let } t = e^u \Rightarrow dt = e^u du$$

$$t = 1 \Rightarrow u = 0 \text{ and } t = \sqrt{2} + 1 \Rightarrow u = \log(\sqrt{2} + 1)$$

$$\therefore \text{I} = - \int_{\log(\sqrt{2}+1)}^0 2(e^u + e^{-u})^{16} \frac{e^u du}{e^u}$$

$$= \int_0^{\log(\sqrt{2}+1)} 2(e^u + e^{-u})^{16} du$$

Ans. : (A)

$$(75) \int_2^4 \frac{\log x^2}{\log x^2 + \log (36 - 12x + x^2)} dx = \dots$$

- (A) 2 (B) 4 (C) 1 (D) 6 (JEE : 2015)

$$\text{Sol. : } I = \int_2^4 \frac{\log x^2}{\log x^2 + \log (36 - 12x + x^2)} dx$$

$$= \int_2^4 \frac{\log x^2}{\log x^2 + \log (6 - x)^2} dx$$

$$= \int_2^4 \frac{2 \log x}{2 \log x + 2 \log (6 - x)} dx$$

$$I = \int_2^4 \frac{\log x}{\log x + \log (6 - x)} dx \quad (i)$$

$$I = \int_2^4 \frac{\log (6 - x)}{\log (6 - x) + \log x} dx \quad (ii)$$

Adding (i) and (ii), we get,

$$\therefore 2I = \int_2^4 1 dx = [x]_2^4 = 2$$

$$\therefore I = 1$$

Ans. : (C)

: Questions With More Than One Correct Answer :

$$(76) \text{ If } S_n = \sum_{k=1}^n \frac{n}{n^2 + nk + k^2} \text{ and } T_n = \sum_{k=1}^{n-1} \frac{n}{n^2 + nk + k^2}, \text{ where } n = 1, 2, 3, \dots \text{ then,}$$

- (A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$ (IIT : 2008)

$$\text{Sol. : } S_n < \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2 + nk + k^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \left(\frac{1}{1 + \left(\frac{k}{n}\right) + \left(\frac{k}{n}\right)^2} \right)$$

$$= \int_0^1 \frac{dx}{1 + x + x^2}$$

$$= \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^1 \\
&= \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}} \right] = \frac{\pi}{3\sqrt{3}}
\end{aligned}$$

$$\therefore S_n < \frac{\pi}{3\sqrt{3}}$$

$$\text{Now, } h \sum_{k=1}^{n-1} f(kh) > \int_0^1 f(x) dx > h \sum_{k=1}^n f(kh)$$

$$\therefore T_n > \int_0^1 \frac{dx}{x^2 + x + 1}$$

$$\therefore T_n > \frac{\pi}{3\sqrt{3}}$$

$$\therefore S_n < \frac{\pi}{3\sqrt{3}} \text{ and } T_n > \frac{\pi}{3\sqrt{3}}$$

Ans. : (A), (D)

$$(77) \text{ If } I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx, \quad n = 0, 1, 2, 3, \dots \text{ then,}$$

$$(A) \quad I_n = I_{n+2} \quad (B) \quad \sum_{m=1}^{10} I_{2n+1} = 10\pi \quad (C) \quad \sum_{m=1}^{10} I_{2m} = 0 \quad (D) \quad I_n = I_{n+1}$$

(IIT : 2009)

$$\text{Sol. : } I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx \quad (i)$$

$$\therefore I_n = \int_{-\pi}^{\pi} \frac{\sin(-nx)}{(1 + \pi^{-x})(-\sin x)} dx \quad (\text{Replacing } x \text{ by } a + b - x) \quad (ii)$$

$$\therefore 2I_n = \int_{-\pi}^{\pi} \left(\frac{1}{1 + \pi^x} + \frac{1}{1 + \pi^{-x}} \right) \frac{\sin nx}{\sin x} dx \quad (\text{Adding (i) and (ii)})$$

$$\therefore 2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$$

$$\therefore I_n = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$$

$$\begin{aligned}
\text{Now, } I_{n+2} - I_n &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \frac{2 \sin x \cdot \cos(n+1)x}{\sin x} dx \\
&= \int_{-\pi}^{\pi} \cos(n+1)x dx \\
&= \left[\frac{\sin(n+1)x}{(n+1)} \right]_{-\pi}^{\pi} = 0
\end{aligned}$$

$$\therefore I_{n+2} - I_n = 0$$

$$\therefore I_{n+2} = I_n$$

$$\therefore I_0 = I_2 = I_4 = \dots = I_{20}$$

$$I_1 = I_3 = I_5 = \dots = I_{21}$$

$$\therefore I_0 = \frac{1}{2} \int_{-\pi}^{\pi} 0 dx = 0$$

$$I_1 = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin x}{\sin x} dx = \frac{1}{2} (2\pi) = \pi$$

$$\therefore \sum_{m=1}^{10} I_{2m} = 0 \text{ and } \sum_{n=1}^{10} I_{2m+1} = 10\pi$$

Ans. : (A), (B), (C)

(78) If f is continuous and non-decreasing on $[a, b]$ and $f(a) > 0$, then $\frac{1}{b-a} \int_a^b f(x) dx$ has

(A) maximum value $f(b)$.

(B) minimum value $f(a)$.

(C) maximum value $bf(b)$

(D) maximum value $\frac{f(a)}{b-a}$

Sol. : $f(x)$ is continuous and non-decreasing function.

$$\therefore f(a) \leq f(x) \leq f(b)$$

$$\therefore \int_a^b f(a) dx \leq \int_a^b f(x) dx \leq \int_a^b f(b) dx$$

$$\therefore f(a) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f(b)$$

Ans. : (A), (B)

(79) The values of α which satisfy $\int_{\frac{\pi}{2}}^{\alpha} \sin x \, dx = \sin 2\alpha$, $\alpha \in [0, 2\pi]$ are

(A) $\frac{\pi}{2}$

(B) $\frac{3\pi}{2}$

(C) $\frac{7\pi}{6}$

(D) $\frac{11\pi}{6}$

Sol. : $\int_{\frac{\pi}{2}}^{\alpha} \sin x \, dx = \sin 2\alpha$

$$\therefore -[\cos x]_{\frac{\pi}{2}}^{\alpha} = \sin 2\alpha$$

$$\therefore -[\cos \alpha - 0] = \sin 2\alpha$$

$$\therefore 2\sin \alpha \cos \alpha + \cos \alpha = 0$$

$$\therefore \cos \alpha (2\sin \alpha + 1) = 0$$

$$\therefore \cos \alpha = 0 \text{ or } \sin \alpha = -\frac{1}{2}$$

$$\therefore \alpha = \frac{\pi}{2}, \frac{3\pi}{2} \text{ and } \alpha = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$\therefore \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Ans. : (A), (B), (C), (D)

(80) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, then

(A) $I_n + I_{n-2} = \frac{1}{n-1}$

(B) $I_{n+1} + I_{n-1} = \frac{1}{n}$

(C) $\frac{1}{n+1} < 2 I_n < \frac{1}{n-1}$, where $n \in \mathbb{N}$, $n > 1$

(D) $(I_{n+2} + I_n)(n+1) = 1$

Sol. : $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$I_n = \left[\frac{(\tan x)^{n-1}}{n-1} \right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

...(i)

(A) is true.

Now, replacing n by $(n + 1)$ in equation (i),

$$I_{n+1} + I_{n-1} = \frac{1}{n} \quad \dots(ii) \quad \text{(B) is true.}$$

Now, replacing n by $(n + 1)$ in equation (ii),

$$I_{n+2} + I_n = \frac{1}{n+1} \quad \dots(iii)$$

$$\therefore (n+1)(I_{n+2} + I_n) = 1 \quad \text{(D) is true.}$$

$$\text{Now, } 0 < x < \frac{\pi}{4} \Rightarrow 0 < \tan x < 1$$

$$\therefore \tan^{n+2}x < \tan^n x < \tan^{n-2}x$$

$$\therefore \tan^n x + \tan^{n+2}x < 2 \tan^n x < \tan^n x + \tan^{n-2}x$$

$$\therefore \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n+2}x) dx < 2 \int_0^{\frac{\pi}{4}} \tan^n x dx < \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2}x) dx$$

$$\therefore I_n + I_{n+2} < 2 I_n < I_n + I_{n-2}$$

$$\therefore \frac{1}{n+1} < 2 I_n < \frac{1}{n-1} \quad \text{(From (i) and (iii))}$$

(C) is also true.

Ans. : (A), (B), (C), (D)

$$(81) \int_0^{\pi} x f(\sin x) dx = \dots\dots$$

$$(A) \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \quad (B) \frac{\pi}{4} \int_0^{\pi} f(\sin x) dx \quad (C) \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx \quad (D) \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\text{Sol. : } I = \int_0^{\pi} x f(\sin x) dx$$

$$I = \int_0^{\pi} (\pi - x) f(\sin(\pi - x)) dx$$

$$= \int_0^{\pi} (\pi - x) f(\sin x) dx$$

$$= \int_0^{\pi} \pi f(\sin x) dx - \int_0^{\pi} x f(\sin x) dx$$

$$I = \pi \int_0^{\pi} f(\sin x) dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} f(\sin x) dx$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \quad (A)$$

$$= \frac{\pi}{2} \cdot 2 \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx \quad (C)$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx \quad (D) \quad \text{Ans. : (A), (C), (D)}$$

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, option (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

(A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.

(B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is false.

(D) Statement 1 is false, statement 2 is true.

$$(82) \text{ Statement 1 : } \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(3 + 4x^3) dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} = 4\sqrt{3} \tan^{-1} \frac{1}{2}$$

$$\text{Statement 2 : } \int_{-a}^a f(x) dx = \begin{cases} 0, & \text{If } f \text{ is an odd function,} \\ 2 \int_0^a f(x) dx, & \text{If } f \text{ is an even function,} \end{cases}$$

$$\text{Sol. : } I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{(3 + 4x^3) dx}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$$

$$I = 2 \int_0^{\frac{\pi}{3}} \frac{3}{2 - \cos\left(x + \frac{\pi}{3}\right)} dx + 0$$

$$\therefore I = 2 \int_0^{\frac{\pi}{3}} \frac{3 dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

$$x + \frac{\pi}{3} = t \Rightarrow dx = dt$$

$$x = 0 \Rightarrow t = \frac{\pi}{3} \quad \text{and} \quad x = \frac{\pi}{3} \Rightarrow t = \frac{2\pi}{3}$$

$$\therefore I = 6 \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dt}{2 - \cos t}$$

$$\text{Let, } \tan \frac{t}{2} = u \quad \text{So, } dt = \frac{2du}{1+u^2}, \cos t = \frac{1-u^2}{1+u^2}$$

$$t = \frac{\pi}{3} \Rightarrow u = \frac{1}{\sqrt{3}} \text{ and } t = \frac{2\pi}{3} \Rightarrow u = \sqrt{3}$$

$$\begin{aligned} \therefore I &= 12 \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{du}{1+3u^2} \\ &= 4\sqrt{3} [\tan^{-1} \sqrt{3}u]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= 4\sqrt{3} [\tan^{-1} 3 - \tan^{-1} 1] \\ &= 4\sqrt{3} \tan^{-1} \left(\frac{3-1}{1+3} \right) \\ &= 4\sqrt{3} \tan^{-1} \left(\frac{1}{2} \right) \end{aligned}$$

$\therefore$ Statements 1 and 2 are both true and statement 2 is the correct explanation of statement 1.

Ans. : (A)

(83) **Statement 1 :** If $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, then $\frac{dy}{dx} = \pi$ at $x = \pi$. ($x > 0$)

Statement 2 : $\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x)) \frac{d(b(x))}{dx} - f(a(x)) \frac{d(a(x))}{dx}$

Sol. : $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cdot \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + \cos x \frac{d}{dx} \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta \\ &= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + \frac{(\cos x)}{1 + \sin^2 x} \cdot 2x \cdot \cos \theta \end{aligned}$$

Now, if $x = \pi$, then $\frac{dy}{dx} = 0 + 2\pi$

$$\therefore \frac{dy}{dx} = 2\pi$$

$\therefore$ Statement 1 is false and statement 2 is true.

Ans. : (D)

(84) **Statement 1** : If $I_n = \int_0^1 (1 - x^5)^n dx$, then $\frac{I_{10}}{I_{11}} = \frac{55}{54}$

Statement 2 : If $u(x)$ and $v(x)$ are differentiable functions, then $\int u dv = uv - \int v du$

$$\begin{aligned} \text{Sol. : } I_{11} &= \int_0^1 (1 - x^5)^{11} dx \\ &= \int_0^1 (1 - x^5)(1 - x^5)^{10} dx \\ &= \int_0^1 (1 - x^5)^{10} dx - \int_0^1 x^5 (1 - x^5)^{10} dx \\ I_{11} &= I_{10} + \int_0^1 \frac{x}{5} (1 - x^5)^{10} (-5x^4) dx \\ I_{11} &= I_{10} + \left[\frac{x}{5} \frac{(1 - x^5)^{11}}{11} \right]_0^1 - \frac{1}{55} \int_0^1 (1 - x^5)^{11} dx \\ &= I_{10} - \frac{1}{55} I_{11} \\ \therefore \left(1 + \frac{1}{55}\right) I_{11} &= I_{10} \\ \therefore \frac{I_{10}}{I_{11}} &= \frac{56}{55} \end{aligned}$$

$\therefore$ **Statement 1 is false and statement 2 is true.**

Ans. : (D)

(85) **Statement 1** : $\int_1^3 \left\{ \tan^{-1} \frac{x}{1+x^2} + \tan^{-1} \frac{x^2+1}{x} \right\} dx = \pi$

Statement 2 : $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ and $\tan^{-1}x = \cot^{-1} \frac{1}{x}$, $x > 0$

$$\begin{aligned} \text{Sol. : } &\int_1^3 \left\{ \tan^{-1} \left(\frac{x}{1+x^2} \right) + \tan^{-1} \left(\frac{x^2+1}{x} \right) \right\} dx \\ &= \int_1^3 \left\{ \tan^{-1} \frac{x}{1+x^2} + \cot^{-1} \frac{x}{1+x^2} \right\} dx \\ &= \int_1^3 \frac{\pi}{2} dx = \frac{\pi}{2} [x]_1^3 = \frac{\pi}{2} (3 - 1) = \pi \end{aligned}$$

$\therefore$ **Statement 1 and statement 2 both are true and statement 2 is the correct explanation of statement 1.**

Ans. : (A)

(86) **Statement 1** : If $n > 1$, then $\int_0^{\infty} \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$ (Improper integral)

Statement 2 : $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

Sol. : Let $I_1 = \int_0^{\infty} \frac{dx}{1+x^n}$

Let, $x^n = \tan^2 \theta$. So, $n \cdot x^{n-1} dx = 2 \tan \theta \sec^2 \theta d\theta$

$$dx = \frac{2 \tan \theta \sec^2 \theta d\theta}{n x^{n-1}}$$

$$= x \frac{2 \tan \theta \sec^2 \theta d\theta}{n x^n}$$

$$= \tan^{\frac{2}{n}} \theta \frac{2 \tan \theta \sec^2 \theta d\theta}{n \tan^2 \theta}$$

$$= \frac{2}{n} \tan^{\left(\frac{2}{n}-1\right)} \theta \sec^2 \theta d\theta$$

$x = 0 \Rightarrow \theta = 0$ and $x = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{2}{n} \cdot \tan^{\left(\frac{2}{n}-1\right)} \theta \cdot d\theta$$

$$\therefore I_1 = \frac{2}{n} \int_0^{\frac{\pi}{2}} (\tan \theta)^{\left(\frac{2}{n}-1\right)} d\theta \quad (i)$$

Now let $I_2 = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$

Let $x^n = \sin^2 \phi$ So, $1 - x^n = 1 - \sin^2 \phi = \cos^2 \phi$

$$\therefore (1-x^n)^{\frac{1}{n}} = (\cos \phi)^{\frac{2}{n}}$$

When $x = 0 \Rightarrow \phi = 0$ and $x = 1 \Rightarrow \phi = \frac{\pi}{2}$

$$\therefore dx = \frac{2}{n} \sin^{\left(\frac{2}{n}-1\right)} \phi \cos \phi d\phi$$

$$\therefore I_2 = \int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$$

$$= \frac{2}{n} \int_0^{\frac{\pi}{2}} \left(\frac{\sin \phi}{\cos \phi} \right)^{\frac{2}{n}-1} d\phi$$

$$= \frac{2}{n} \int_0^{\frac{\pi}{2}} (\tan \phi)^{\frac{2}{n}-1} d\phi = I_1$$

$$\therefore I_2 = I_1$$

Statement 1 and statement 2 both are true but statement 2 is not the correct explanation of statement 1. **Ans. : (B)**

(87) **Statement 1 :** $\int_0^{10\pi} |\cos x| dx = 20$

Statement 2 : If $\int_a^b f(x) dx \geq 0$, then $f(x) \geq 0, \forall x \in [a, b]$.

Sol. : $\int_0^{10\pi} |\cos x| dx = 10 \int_0^{\pi} |\cos x| dx$
 $= 20 \int_0^{\frac{\pi}{2}} \cos x dx = 20$

$\therefore$ Statement 1 is true. Statement 2 is false. Because correct statement is 'if $f(x) \geq 0$ in

$[a, b]$, then $\int_a^b f(x) dx \geq 0$ '.

Ans. : (C)

Note : If $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

$f(x) < 0$ if $x \in (1, 2)$

$$\int_0^3 (x^2 - 3x + 2) dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^3$$

$$= \left(9 - \frac{27}{2} + 6 \right) = \frac{3}{2} > 0$$

(88) **Statement 1 :** If $g(x) = \int_0^{\frac{\pi}{2}} (\sin^6 x + \cos^6 x) dx$, then $\frac{\pi}{8} < g(x) < \frac{\pi}{2}$

Statement 2 : If m and M are the smallest and greatest values of a function $f(x)$ such that

$m < f(x) < M$, then $m(b-a) < \int_a^b f(x) dx < M(b-a)$.

Sol. : $f(x) = \sin^6 x + \cos^6 x$
 $= (\sin^2 x)^3 + (\cos^2 x)^3$
 $= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x$
 $= 1 - \frac{3}{4} \sin^2 2x$

Now, $0 \leq \sin^2 2x \leq 1$

$$\therefore 0 \geq -\frac{3}{4} \sin^2 2x \geq -\frac{3}{4}$$

$$\therefore -\frac{3}{4} \leq -\frac{3}{4} \sin^2 2x \leq 0$$

$$\therefore \frac{1}{4} \leq 1 - \frac{3}{4} \sin^2 2x \leq 1$$

$$\therefore \frac{1}{4} \leq f(x) \leq 1$$

$$\therefore m = \frac{1}{4} \text{ and } M = 1$$

$$\therefore \left(\frac{\pi}{2} - 0\right) \frac{1}{4} < \int_0^{\frac{\pi}{2}} (\sin^6 x + \cos^6 x) dx < \left(\frac{\pi}{2} - 0\right) \cdot 1$$

$$\therefore \frac{\pi}{8} < g(x) < \frac{\pi}{2} \text{ by statement 2.}$$

∴ Statement 1 and statement 2 are both true and statement 2 is the correct explanation of statement 1. **Ans. : (A)**

$$(89) \text{ Statement 1 : } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1 + \sqrt{\cot x}} dx = \frac{\pi}{12}$$

$$\text{Statement 2 : } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(x) dx}{f(x) + f\left(\frac{\pi}{2} - x\right)} = \frac{\pi}{12}$$

$$\text{Sol. : Let, } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f(x) dx}{f(x) + f\left(\frac{\pi}{2} - x\right)} \quad \dots(i)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)}{f\left(\frac{\pi}{6} + \frac{\pi}{3} - x\right) + f\left(\frac{\pi}{2} - \left(\frac{\pi}{6} + \frac{\pi}{3} - x\right)\right)} dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{f\left(\frac{\pi}{2} - x\right)}{f\left(\frac{\pi}{2} - x\right) + f(x)} dx \quad \dots(ii)$$

Adding (i) and (ii), we get,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6}$$

$$\therefore 2I = \frac{\pi}{6}. \quad \text{So, } I = \frac{\pi}{12}$$

$\therefore$ Statement 2 is true.

$$\text{Now let } f(x) = \frac{1}{1 + \sqrt{\cot x}} = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$\therefore f(x) = \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\cos(\frac{\pi}{2} - x)}}$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) dx = \frac{\pi}{12}$$

$\therefore$ Statement 1 and 2 are true and Statement 2 is the correct explanation of statement 1.

Ans. : (A)

(90) **Statement 1 :** $\int_0^{100} (x - [x]) dx = 50$, where $[x]$ denotes greatest integer function.

Statement 2 : If $f(x)$ is a periodic function with period T, then

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx.$$

Sol. : If the function $f(x)$ is a periodic function with period T, then

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx.$$

$\therefore$ Statement 2 is true.

(Property of periodic function)

$$\text{Now, } \int_0^{100} (x - [x]) dx = \int_0^{100(1)} (x - [x]) dx$$

$$= 100 \int_0^1 (x - [x]) dx \quad (\text{Period of } x - [x] \text{ is } 1)$$

$$= 100 \int_0^1 x dx \quad ([x] = 0 \text{ in } [0, 1])$$

$$= 100 \left[\frac{x^2}{2} \right]_0^1$$

$$= 100 \times \frac{1}{2} = 50$$

$\therefore$ Statement 1 and 2 are true and Statement 2 is the correct explanation of statement 1.

Ans. : (A)

