

Principle of Mathematical Induction :

If a statement $p(n)$ with variable natural number n ,

- (1) is true for $n = 1$ i.e. $p(1)$ is true.
- (2) for $k \in \mathbb{N}$, $p(k)$ is true $\Rightarrow p(k+1)$ is true.

Then $p(n)$ is true $\forall n \in \mathbb{N}$.

Use of Principle of Mathematical Induction in Some Special Types of Variable :**Variable type 1 :**

The statement $p(n)$, $n \in \mathbb{N}$ is given. If for some positive integer k_0 , $p(k_0)$ is true and for $k \geq k_0$, $k \in \mathbb{N}$, $p(k)$ is true $\Rightarrow p(k+1)$ is true, then $p(n)$ is true for all $n \geq k_0$, $n \in \mathbb{N}$.

Variable type 2 :

The statement $p(n)$, $n \in \mathbb{N}$ is given. If (i) $p(1)$ and $p(2)$ are true and

- (ii) for positive integer k , $p(k)$ and $p(k+1)$ are true $\Rightarrow p(k+2)$ is true, then $\forall n \in \mathbb{N}$, $p(n)$ is true.

Limitations of Principle of Mathematical Induction :

- (1) A result involving natural number variable cannot be derived, but only its validity can be verified.
- (2) The principle of mathematical induction verifies the validity of statements involving natural number variable only.

: Multiple Choice Questions :

- (1) Statement $p(n) : 3^{2n+1} + 2^{n-1}$, $\forall n \in \mathbb{N}$ is divisible by
 (A) 2 (B) 5 (C) 7 (D) 4

Sol. : For $n = 1$, $3^3 + 2^0 = 28$ is divisible by 7 and 2.

For $n = 2$, $3^5 + 2^1 = 245 = 7 \times 35$ is divisible by 7 but is not divisible by 2.

$\therefore$ We may infer that $p(n) : 3^{2n+1} + 2^{n-1}$ is divisible by 7. So, $p(1)$ is true.

Suppose $p(k) : 3^{2k+1} + 2^{k-1}$, $k \in \mathbb{N}$ is divisible by 7, $\forall n \in \mathbb{N}$.

$$\therefore 3^{2k+1} + 2^{k-1} = 7m \quad m \in \mathbb{N} \quad (1)$$

$$\begin{aligned} \text{For } n = k + 1, 3^{2k+3} + 2^k &= 9 \cdot 3^{2k+1} + 2^k \\ &= 9(7m - 2^{k-1}) + 2^k \\ &= 63m - 9 \cdot 2^{k-1} + 2^k \\ &= 63m - 2^{k-1}(9 - 2) \\ &= 7(9m - 2^{k-1}) \text{ is divisible by 7.} \end{aligned} \quad (\because 1)$$

$\therefore p(k)$ is true. $\Rightarrow p(k+1)$ is true.

Because of $p(2)$, options (A), (D) are not true. By $p(1)$, option (B) is not true.

$\therefore$ option (C) is true.

$\therefore$ By the principle of mathematical induction, $p(n)$ is true $\forall n \in \mathbb{N}$.

Ans. : (C)

(2) Statement $p(n) : n^2 \geq 3n$ is true.

(A) $\forall n \in \mathbb{N}$

(B) $\forall n \in \mathbb{N}, n > 1$

(C) $\forall n \in \mathbb{N}, n \geq 3$

(D) for all odd natural numbers n

Sol. : $p(1) : 1 \geq 3$ is not true.

$p(2) : 4 \geq 6$ is not true.

$p(3) : 9 \geq 9$ is true.

$p(4) : 16 \geq 12$ is true.

$p(5) : 25 \geq 15$ is true.

$\therefore$ By the principle of mathematical induction, it can be proved as shown below that $p(n)$ is true $\forall n \geq 3$:

Suppose $p(k) : k^2 \geq 3k, k \geq 3$ is true. (1)

We have to prove that $p(k+1) : (k+1)^2 \geq 3(k+1)$.

Here, $(k+1)^2 = k^2 + 2k + 1$

$$\geq 3k + 2k + 1 \quad (\text{By (1)})$$

$$\geq 3k + 3 = 3(k+1) \quad (k \geq 3 \Rightarrow 2k + 1 \geq 3)$$

$$\therefore (k+1)^2 \geq 3(k+1)$$

$$\therefore p(k+1) \text{ is true.}$$

$\therefore$ By the principle of mathematical induction $p(n) : n^2 \geq 3n; \forall n \in \mathbb{N}, n \geq 3$ is true.

Ans. : (C)

(3) $p(n) : 2^{2^n}$ has at unit's place for $n > 1$.

(A) 2

(B) 6

(C) 0

(D) 4

Sol. : For $n = 2, 2^{2^2} = 16$ has 6 at unit's place.

For $n = 3, 2^{2^3} = 2^8 = 256$ has 6 at unit's place.

$\therefore$ We may infer that $\forall n > 1, 2^{2^n}$ has 6 at unit's place.

Suppose $p(k) : 2^{2^k}$ has 6 at unit's place for $k > 1$.

Taking $n = k + 1, 2^{2^{k+1}} = 2^{2^k \cdot 2}$

$$= \left(2^{2^k}\right)^2$$

$$= (10m + 6)^2 \quad \left(2^{2^k} = 10m + 6, m \in \mathbb{N}\right)$$

$$= 100m^2 + 120m + 36$$

$$= 100m^2 + 120m + 30 + 6$$

$$= 10 (10 m^2 + 12m + 3) + 6 \text{ has 6 at unit's place.}$$

$\therefore p(k+1)$ is true.

$\therefore p(k)$ is true. $\Rightarrow p(k+1)$ is true, $\forall k > 1$

$\therefore p(n)$ is true, $\forall n > 1$.

Ans. : (B)

(4) If $1^3 + 2^3 + 3^3 + \dots + 50^3 = m^2$, then $m = \dots$

(A) 1275

(B) 1225

(C) 2450

(D) 1375

Sol. : $m^2 = 1^3 + 2^3 + 3^3 + \dots + 50^3$

$$= \left[\sum n^3 \right]_{n=50} \quad (n = 50)$$

$$= \left[\frac{n^2 (n+1)^2}{4} \right]_{n=50} \quad (\text{By principle of mathematical induction})$$

$$= \frac{50^2 \times 51^2}{4}$$

$$m = 25 \times 51 = 1275$$

Ans. : (A)

(5) The least positive integer which makes $n! < \left[\frac{n+1}{2} \right]^n$ is, where $[]$ is the greatest integer function.

(A) 2

(B) 4

(C) 3

(D) 6

Sol. : For $n = 1$, $1! < \left[\frac{1+1}{2} \right]^1$

i.e. $1! < 1$ is not true.

$$\text{For } n = 2, 2! < \left[\frac{2+1}{2} \right]^2$$

i.e. $2 < 1$ is not true.

$$\text{For } n = 3, 3! < \left[\frac{3+1}{2} \right]^3$$

$\therefore 6 < 8$ is true.

Ans. : (C)

(6) If $\sum_{r=1}^k r = \frac{1}{2} (n^2 + 11n + 30)$, then $k = \dots$

(A) $n+4$

(B) $n+7$

(C) $n+6$

(D) $n+5$

Sol. : $n^2 + 11n + 30 = (n + 5)(n + 6),$

Here, $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$

By replacing n by $(n + 5),$

$$\therefore 1 + 2 + 3 + \dots + (n + 4) + (n + 5) = \frac{(n+5)(n+6)}{2}$$

$$\therefore k = (n + 5)$$

Ans. : (D)

(7) If $\sum_{r=6}^{n+5} (r-5)^3 = \left\{ \sum_{r=2}^p (r-1) \right\}^2$, then $p = \dots\dots$

(A) $n + 2$

(B) $n - 1$

(C) $n + 1$

(D) $n + 3$

Sol. : Using the principle of mathematical induction, it can be proved that,

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

$$\therefore \sum_{r=6}^{n+5} (r-5)^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2} \right)^2 = \left\{ \sum_{r=2}^{n+1} (r-1) \right\}^2$$

$$\therefore P = n + 1$$

Ans. : (C)

(8) Statement $p(n) : 8n \leq 2^n - 16$ is true for $n > \dots\dots\dots, n \in \mathbb{N}.$

(A) 1

(B) 2

(C) 5

(D) 4

Sol. : $p(1) : 8 \leq 2^1 - 16$ is not true.

$p(2) : 16 \leq 2^2 - 16$ is not true.

$p(3) : 24 \leq 2^3 - 16$ is not true.

$p(4) : 32 \leq 2^4 - 16$ is not true.

$p(5) : 40 \leq 2^5 - 16$ is not true.

$p(6) : 48 \leq 2^6 - 16$ is true.

So, $p(6)$ is true.

$p(7) : 56 \leq 2^7 - 16$ is true.

$\therefore$ Suppose $p(k)$ is true, $k \geq 6, k \in \mathbb{N}$

$\therefore 8k \leq 2^k - 16$ is true.

$$\therefore 8k + 8 \leq 2^k - 16 + 8$$

$$(k \geq 6 \Rightarrow 8 \leq 2^k)$$

$$\therefore 8(k+1) \leq 2^k + 2^k - 16$$

$$\therefore 8(k+1) \leq 2 \cdot 2^k - 16$$

$$\therefore 8(k+1) \leq 2^{k+1} - 16$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true. $\Rightarrow p(k+1)$ is true. $k > 5, k \in \mathbb{N}$

$\therefore p(n), \forall n > 5$ is true, $n \in \mathbb{N}$

Ans. : (C)

- (9) The greatest positive integer which divides $(n + 1) (n + 2) (n + 3) \dots (n + r)$, $\forall n \in \mathbb{N}$, is
 (A) $(n + r) !$ (B) $n + r + 1$ (C) $n!$ (D) $r !$

Sol. : Here, $\forall n \in \mathbb{N}$, $(n + 1) (n + 2) (n + 3) \dots (n + r) = {}_{(n+r)}P_r$

$${}_{(n+r)}P_r = \frac{(n+r)!}{n!} = \frac{(n+r)!}{n! r!} \times r! = \binom{n+r}{n} r! \text{ is divisible by } r!.$$

Note : Thus, product of n consecutive integers is divisible by $n!$, $\forall n \in \mathbb{N}$.

Ans. : (D)

- (10) Inequality $3^n < (n + 1) !$, $n \in \mathbb{N}$

- (A) is false for $n = 4$. (B) is false for $n = 13$.
 (C) is true for $n \geq 4$. (D) is true for $n > 2$.

Sol. : For $n = 1$, $3 < 2$ is not true.

For $n = 2$, $9 < 6$ is not true.

For $n = 3$, $27 < 24$ is not true.

For $n = 4$, $81 < 120$ is true.

$p(n) : 3^n < (n + 1)!$ is true, $\forall n \geq 4$.

Suppose, for $n = k$, $3^k < (k + 1)!$ is true. $k \geq 4$

$$\therefore 3^k < (k+1)!$$

$$\therefore 3^k \cdot 3 < (k+1)! \cdot 3$$

$$\therefore 3^{k+1} < 3 (k+1)! < (k+2) (k+1)! \quad (k \geq 4 \Rightarrow k+2 > 3)$$

$$\therefore 3^{k+1} < (k+2)!$$

$\therefore$ The statement is true for $n = k + 1$.

$\therefore p(k)$ is true. $\Rightarrow p(k + 1)$ is true, $\forall k \geq 4$

$\therefore p(n) : 3^n < (n + 1)!$ $\forall n \geq 4$, $n \in \mathbb{N}$ is true.

Ans. : (C)

- (11) If $p(n) : \frac{n^n}{3^n} < n! < \frac{n^n}{2^n}$, $\forall n \geq k$, $n \in \mathbb{N}$ is true, then $k =$

- (A) 3 (B) 4 (C) 5 (D) 6

Sol. : $p(1) : \frac{1}{3} < 1 < \frac{1}{2}$ is not true.

$p(2) : \frac{4}{9} < 2 < 1$ is not true.

$p(3) : 1 < 6 < \frac{27}{8}$ is not true.

$$p(4) : \frac{256}{81} < 24 < \frac{256}{16} \text{ is not true.}$$

$$p(5) : \frac{5^5}{3^5} < 120 < \frac{5^5}{2^5} \text{ i.e. } \frac{3125}{243} < 120 < \frac{3125}{32} \text{ is not true.}$$

$$p(6) : \frac{6^6}{3^6} < 6! < \frac{6^6}{2^6} \text{ i.e. } \frac{6^5}{3^6} < 5! < \frac{6^5}{2^6} \text{ i.e. } \frac{2^5}{3} < 120 < \frac{3^5}{2}$$

$$\text{i.e. } \frac{32}{3} < 120 < \frac{243}{2} \text{ is true.}$$

Similarly, it can be proved that $p(n)$ is true, $\forall n \geq 6$.

Here, by the principle of mathematical induction, it can be proved that

$$n! < \frac{n^n}{2^n} \text{ and } \frac{n^n}{3^n} < n!, n \geq 6.$$

Ans. : (D)

(12) If $S(k) : 1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$, then which of following is true ? (AIEEE : 2004)

(A) $S(1)$ is true.

(B) $S(k) \Rightarrow S(k + 1)$ is true.

(C) $S(k) \Rightarrow S(k + 1)$ is not true.

(D) The principle of mathematical induction can be used to prove the formula.

Sol. : $S(1) : 1 = 3 + (1)^2$ is not true.

Suppose, $S(k) : 1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$ is true.

$$\begin{aligned} 1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) &= 3 + k^2 + (2k + 1) \\ &= 3 + (k + 1)^2 \end{aligned}$$

Thus, $S(k + 1)$ is true.

$\therefore S(k)$ is true. $\Rightarrow S(k + 1)$ is true. Thus, (B) is true.

Since $S(k) \Rightarrow S(k + 1)$ is true, (C) is not true.

$p(1)$ is not true. So, (D) is not true.

Ans. : (B)

(13) $p(n) : 49^n + 24^{n-1}$ is divisible by for $n \in \mathbb{N}$.

(A) 4

(B) 49

(C) 7

(D) 25

Sol. : $p(1) : 49^1 + 24^0 = 50$ is divisible by 25. So, $p(1)$ is true.

Suppose, $p(n) : 49^n + 24^{n-1}$ is divisible by 25, $\forall n \in \mathbb{N}$.

Assume $p(k)$ is true.

$$\text{i.e. } 49^k + 24^{k-1} = 25m, \text{ for some } m \in \mathbb{N}$$

Now, $p(k + 1) : 49^{k+1} + 24^k$ is divisible by 25 has to be proved.

$$\begin{aligned}
\text{Now, } 49^{k+1} + 24^k &= 49 (25m - 24^{k-1}) + 24^k \\
&= 49 \cdot 25m - 49 \cdot 24^{k-1} + 24^k \\
&= 49 \cdot 25m - 24^{k-1} (49 - 24) \\
&= 25 (49m - 24^{k-1}) \text{ is divisible by 25.}
\end{aligned}$$

$\therefore p(k+1)$ is true.

$\therefore p(k)$ is true. $\Rightarrow p(k+1)$ is true, $k \in \mathbb{N}$

$\therefore$ So, $49^n + 24^{n-1}$ is divisible by 25, $\forall n \in \mathbb{N}$.

Ans. : (D)

$$(14) \quad \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = \dots \quad \forall n \in \mathbb{N}$$

$$(A) \quad 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \quad (B) \quad 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \quad (C) \quad 1 \quad (D) \quad 0$$

$$\text{Sol. : For } n = 1, \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\begin{aligned}
\text{For } n = 2, \int_0^{\frac{\pi}{2}} \frac{\sin^2 2x}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{4\sin^2 x \cos^2 x}{\sin x} dx \\
&= -4 \int_0^{\frac{\pi}{2}} \cos^2 x (-\sin x) dx = -4 \left[\frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}} = -4 \left[0 - \frac{1}{3} \right] = \frac{4}{3} = 1 + \frac{1}{3}
\end{aligned}$$

For $n = 1$ and 2 , (B) is true.

$$\therefore \text{ Let } \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \text{ for } n \in \mathbb{N}$$

Let $p(k)$ be true for $k \in \mathbb{N}$.

$$\therefore p(k) : \int_0^{\frac{\pi}{2}} \frac{\sin^2 kx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1}$$

$$\begin{aligned}
\text{Now, } \int_0^{\frac{\pi}{2}} \frac{\sin^2 (k+1)x}{\sin x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin^2 kx}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 (k+1)x - \sin^2 kx}{\sin x} dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin (2k+1)x \sin x}{\sin x} dx
\end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \sin(2k+1)x \, dx = \left[\frac{-\cos(2k+1)x}{(2k+1)} \right]_0^{\frac{\pi}{2}} = \frac{-1}{2k+1} \left[\cos(2k+1)\frac{\pi}{2} - \cos 0 \right] = \frac{1}{2k+1}$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \frac{\sin^2(k+1)x}{\sin x} \, dx &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 kx}{\sin x} \, dx + \frac{1}{2k+1} \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} + \frac{1}{2k+1} \end{aligned}$$

$\therefore p(k+1)$ is true, $\forall n \in \mathbb{N}$.

$\therefore p(n)$ is true, $\forall n \in \mathbb{N}$.

So, (B) is true.

Ans. : (B)

(15) Using the principle of mathematical induction, prove that $p(n) : \cos(x - \pi n) = (-1)^n \cos x$, $\forall n \in \mathbb{N}$

Sol. : For $n = 1$, $\cos(x - \pi) = \cos(\pi - x) = -\cos x = (-1)^1 \cos x$

$\therefore p(1)$ is true.

Suppose, $p(k)$ is true. $k \in \mathbb{N}$

$p(k) : \cos(x - k\pi) = (-1)^k \cos x$, $k \in \mathbb{N}$

Here, $\cos(x - \pi(k+1)) = \cos(x - k\pi - \pi)$

$$= -\cos(x - k\pi)$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$= -(-1)^k \cos x$$

$$= (-1)^{k+1} \cos x$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true. $\Rightarrow p(k+1)$ is true, $\forall k \in \mathbb{N}$

The statement $p(n)$ is true, $\forall n \in \mathbb{N}$

(16) Using the principle of mathematical induction, prove that : $p(n) : |z^n| = |z|^n$, $\forall n \in \mathbb{N}$ and $z \in \mathbb{C}$

Sol. : For $n = 1$, $p(1) = |z^1| = |z|^1$ is true.

Suppose, $p(k)$ is true for $k \in \mathbb{N}$.

$\therefore p(k) : |z^k| = |z|^k$

Now, $p(k+1) : |z^{k+1}| = |z|^{k+1}$ is to be proved.

Here, $|z^{k+1}| = |z^k \cdot z|$

$$= |z^k| \cdot |z|$$

$$(|z_1 z_2| = |z_1| \cdot |z_2|)$$

$$= |z|^k \cdot |z|$$

$$= |z|^{k+1}$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true. $\Rightarrow p(k+1)$ is true, $\forall k \in \mathbb{N}$

$\therefore$ Statement $p(n)$ is true, $\forall n \in \mathbb{N}$

(17) For $n > 1$, prove that : $n! < \left(\frac{n+1}{2}\right)^n$

Sol. : $p(n) : n! < \left(\frac{n+1}{2}\right)^n, n > 1$

For $n = 2$, $2! < \left(\frac{3}{2}\right)^2$ i.e. $2 < \frac{9}{4}$ is true.

Suppose $p(k)$ is true, ($k \geq 2$). Thus, $p(k) : k! < \left(\frac{k+1}{2}\right)^k$ is true. (1)

Multiplying by $(k + 1)$ on both the sides of result (1),

$$\therefore k! (k + 1) < \frac{(k+1)^k}{2^k} (k + 1)$$

$$\therefore (k + 1)! < \frac{(k+1)^{k+1}}{2^k \times 2} \times 2$$

$$\therefore (k + 1)! < \frac{(k+1)^{k+1}}{2^{k+1}} \times \left(1 + \frac{1}{k+1}\right)^{k+1}, \left[\left(1 + \frac{1}{k+1}\right)^{k+1} = 1 + (k+1) \frac{1}{k+1} + \text{positive number} > 2\right]$$

$$\therefore (k + 1)! < \frac{(k+1)^{k+1}}{2^{k+1}} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}}$$

$$\therefore (k + 1)! < \left(\frac{k+2}{2}\right)^{k+1}$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true $\Rightarrow p(k + 1)$ is true, $k \geq 2, k \in \mathbb{N}$.

$\therefore$ By the principle of mathematical induction, $p(n)$ is true, $\forall n > 1, n \in \mathbb{N}$.

(18) Using the principle of mathematical induction, prove that : $n! < \left(\frac{n}{2}\right)^n, \forall n > 5, n \in \mathbb{N}$

Sol. : $p(n) : n! < \left(\frac{n}{2}\right)^n, n > 5 \quad n \in \mathbb{N}$

For $n = 6$, $6! < \left(\frac{6}{2}\right)^6 \Leftrightarrow 6! < (3)^6 \Leftrightarrow 720 < 729$ is true.

$\therefore p(6)$ is true.

Suppose, $p(k)$ is true, $k > 5$

$$\therefore k! < \left(\frac{k}{2}\right)^k, k > 5 \quad (1)$$

Multiplying by $(k + 1)$ on both the sides of result (1),

$$k! (k+1) < \frac{k^k}{2^k} (k+1)$$

$$\therefore (k+1)! < \frac{k^k (k+1)}{2^k \cdot 2} \times 2$$

$$\therefore (k+1)! < \frac{k^k (k+1)}{2^{k+1}} \times \left(1 + \frac{1}{k}\right)^k$$

$$\left[\left(1 + \frac{1}{k}\right)^k = 1 + k \frac{1}{k} + \text{positive number} \Rightarrow \left(1 + \frac{1}{k}\right)^k > 2 \right]$$

$$\therefore (k+1)! < \frac{k^k (k+1)}{2^{k+1}} \times \frac{(k+1)^k}{k^k}$$

$$\therefore (k+1)! < \frac{(k+1)^{k+1}}{2^{k+1}}$$

$$\therefore (k+1)! < \left(\frac{k+1}{2}\right)^{k+1}$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true. $\Rightarrow p(k+1)$ is true, $k > 5, k \in \mathbb{N}$.

$\therefore p(n)$ is true for $\forall n > 5, n \in \mathbb{N}$.

(19) Prove that $n^n < 3^n \cdot n!$, $n \geq 6$.

Sol. : $p(n) : n^n < 3^n \cdot n!$ $n \geq 6$

For $n = 6$, $6^6 < 3^6 \cdot 6! \Leftrightarrow \frac{6^6}{3^6} < 720 \Leftrightarrow 2^6 < 720$. i.e. $64 < 720$ is true.

Thus $p(6)$ is true.

Suppose $p(k)$ is true, $k \geq 6$

$$\therefore k! > \frac{k^k}{3^k}$$

$$\therefore (k+1)k! > \frac{k^k}{3^k} \cdot \frac{(k+1)}{3} \cdot 3$$

$$\therefore (k+1)! > \frac{k^k}{3^{k+1}} (k+1) \cdot 3 > \frac{k^k}{3^{k+1}} (k+1) \left(1 + \frac{1}{k}\right)^k \quad \left[\left(1 + \frac{1}{k}\right)^k < 3 \right]$$

$$\therefore (k+1)! > \frac{k^k}{3^{k+1}} (k+1) \cdot \frac{(k+1)^k}{k^k}$$

$$\therefore (k+1)! > \frac{(k+1)^{k+1}}{3^{k+1}}$$

$$\therefore (k+1)^{k+1} < 3^{k+1} (k+1)!$$

$\therefore p(k+1)$ is true.

$\therefore p(k)$ is true $\Rightarrow p(k+1)$ is true. $k \geq 6, k \in \mathbb{N}$.

$\therefore p(n)$ is true, $n \in \mathbb{N}, \forall n \geq 6$.

Note : Here, $\left(1 + \frac{1}{k}\right)^k = 1 + \binom{k}{1} \frac{1}{k} + \binom{k}{2} \left(\frac{1}{k}\right)^2 + \binom{k}{3} \left(\frac{1}{k}\right)^3 + \dots (k+1) \text{ terms.}$

$$= 1 + 1 + \frac{k(k-1)}{2k^2} + \frac{k(k-1)(k-2)}{3!k^3} + \dots (k+1) \text{ terms.}$$

$$= 1 + 1 + \frac{1}{2} \left(1 - \frac{1}{k}\right) + \frac{\left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right)}{3!} + \dots (k+1) \text{ terms.}$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \text{upto } \infty$$

$$< 1 + \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \text{upto } \infty\right) = 1 + \frac{1}{1 - \frac{1}{2}} = 3$$

$$\therefore \left(1 + \frac{1}{k}\right)^k < 3$$

(20) Using the principle of mathematical induction, prove that,

$$p(n) : 1^2 - 2^2 + 3^2 - 4^2 + \dots + (2n-1)^2 - (2n)^2 = -n(2n+1), n \in \mathbb{N}$$

Sol. : For $n = 1$, L.H.S. $= 1^2 - 2^2 = -3$. R.H.S. $= -1(2+1) = -3$

$\therefore p(1)$ is true. Suppose, $p(k)$ is true for $k \in \mathbb{N}$

$\therefore p(k) : 1^2 - 2^2 + 3^2 - 4^2 + \dots - (2k)^2 = -k(2k+1)$ is true. (1)

Adding $(2k+1)^2 - (2k+2)^2$ on both the sides of result (1),

$$1^2 - 2^2 + 3^2 - 4^2 + \dots - (2k)^2 + (2k+1)^2 - (2k+2)^2$$

$$= -k(2k+1) + (2k+1)^2 - (2k+2)^2$$

$$= -2k^2 - k + (4k+3)(-1)$$

$$= -(2k^2 + k + 4k + 3) = -(2k^2 + 5k + 3) = -(k+1)(2k+3)$$

$\therefore p(k+1)$ is true.

$\therefore p(k)$ is true $\Rightarrow p(k+1)$ is true. $k \in \mathbb{N}$

$\therefore p(n)$ is true, $\forall n \in \mathbb{N}$.

(21) Using the principle of mathematical induction prove that : $1^2 - 2^2 + 3^2 - 4^2 + \dots - n^2 = -\frac{n(n+1)}{2}$, n is even, $n \in \mathbb{N}$

Sol. : n is even. So, by taking $n = 2m$, the result can be proved as shown in the previous example.

- (22) Using the principle of mathematical induction prove that, if $(x+3)y = 1$, then $\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n!}{(x+3)^{n+1}}$, $\forall n \in \mathbb{N}$

Sol. : $p(n) : \frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n!}{(x+3)^{n+1}}$, where $y = \frac{1}{x+3}$

For $n = 1$, $\frac{dy}{dx} = \frac{(-1)^1 \cdot 1!}{(x+3)^{1+1}} = \frac{-1}{(x+3)^2}$ is true because $y = \frac{1}{x+3} \Rightarrow \frac{dy}{dx} = \frac{-1}{(x+3)^2}$

Suppose, $p(k)$ is true for $k \in \mathbb{N}$. $p(k) : \frac{d^k y}{dx^k} = \frac{(-1)^k \cdot k!}{(x+3)^{k+1}}$ is true. $k \in \mathbb{N}$ (1)

Here, $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx} \left(\frac{d^k y}{dx^k} \right)$

$$= \frac{d}{dx} \frac{(-1)^k \cdot k!}{(x+3)^{k+1}} \quad \text{(From (1))}$$

$$= (-1)^k k! \frac{d}{dx} (x+3)^{-(k+1)}$$

$$= (-1)^k k! (-1) (k+1) (x+3)^{-(k+1)-1} = (-1)^{k+1} (k+1)! (x+3)^{-(k+2)} = \frac{(-1)^{k+1} (k+1)!}{(x+3)^{k+2}}$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true $\Rightarrow p(k+1)$ is true. $\forall k \in \mathbb{N}$

$\therefore$ Statement $p(n)$ is true, $\forall n \in \mathbb{N}$.

- (23) If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}}$ n times then using the principle of mathematical induction which of the following is true? (AIEEE : 2002)

(A) $a_n > 7, \forall n \geq 1$ (B) $a_n > 3, \forall n \geq 1$ (C) $a_n < 4, \forall n \geq 1$ (D) $a_n < 3, \forall n \geq 1$

Sol. : $p(n) : a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}}$ n times

For $n = 1$, $a_1 = \sqrt{7}$. So, $a_1 < 7$. $a_n > 7$ is false.

$a_1 > 3$ is false. $a_1 = \sqrt{7} < 4$ and $a_1 = \sqrt{7} < 3$ is true.

$$a_2 = \sqrt{7 + \sqrt{7}}$$

Here, $2 < \sqrt{7} < 3 \Rightarrow 9 < 7 + \sqrt{7} < 10 < 16 \Rightarrow 3 < \sqrt{7 + \sqrt{7}} < 4 \Rightarrow 3 < a_2 < 4$

$\therefore a_2 < 3$ is false. But, $a_2 < 4$ is true.

Thus, $a_1 < 4$, $a_2 < 4$ is true.

Suppose $a_k < 4$

$$\begin{aligned}\text{Now, } a_{k+1} &= \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}} \quad (k+1) \text{ times} \\ &= \sqrt{7 + a_k}\end{aligned}$$

$$\therefore a_{k+1}^2 = 7 + a_k < 7 + 4.$$

$$\text{So, } a_{k+1} < \sqrt{11} < 4.$$

$$\text{So, } a_{k+1} < 4.$$

Thus, $a_n < 4$ is true for $n = k + 1$.

$$\therefore a_n < 4, \quad \forall n \in \mathbb{N}$$

Ans. : (C)

(24) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which of the following is true for $\forall n \geq 1, n \in \mathbb{N}$?

(AIEEE : 2005)

$$(A) A^n = 2^{n-1} A + (n-1) I$$

$$(B) A^n = nA + (n-1) I$$

$$(C) A^n = 2^{n-1} A - (n-1) I$$

$$(D) A^n = nA - (n-1) I$$

$$\text{Sol. : } A^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 2^{1-1} A + (1-1) I = A \quad (A)$$

$$= (1) A + (1-1) I = A \quad (B)$$

$$= 2^{1-1} A - (1-1) I = A \quad (C)$$

$$= (1) A - (1-1) I = A \quad (D)$$

For $n = 1$, (A), (B), (C), (D) are true.

$$\text{For } n = 2, A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2A - (2-1) I$$

$\therefore$ For $n = 2$, (A) is false.

$\therefore$ For $n = 2$, (B) is false. (C) and (D) are true.

$$\text{For } n = 3, A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \neq 4A + 2I$$

(A) is false. (B) is also not true.

$$\text{Let } n = 3 \text{ in R.H.S. of (C), } 2^2 A - 2I = 4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$$

$\therefore$ For $n = 3$, (C) is not true.

$$\text{Let } n = 3 \text{ in R.H.S. of (D), } 3A - (3-1)I = 3 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = A^3$$

∴ For $n = 3$, (D) is true.

For $n = 1, 2, 3$, (D) is true.

∴ Suppose $p(n) : A^n = nA - (n-1)I$ is true for $n = k$.

$$\begin{aligned}
 \text{Now, } A^{k+1} &= A^k \cdot A = (kA - (k-1)I)A \\
 &= kA^2 - (k-1)A \\
 &= k \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - (k-1) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} k-(k-1) & 0 \\ 2k-(k-1) & k-(k-1) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ k+1 & 1 \end{bmatrix} = \begin{bmatrix} k+1 & 0 \\ k+1 & k+1 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \\
 &= (k+1) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= (k+1)A - kI
 \end{aligned}$$

∴ $p(k+1)$ is true.

∴ $p(k)$ is true $\Rightarrow p(k+1)$ is true, $\forall k \in \mathbb{N}$

∴ $p(n)$ is true $\forall n \in \mathbb{N}$.

Ans. : (D)

(25) The remainder when $8^{2n} - 62^{2n+1}$ is divided by 9 is

(AIEEE : 2009)

(A) 2

(B) 7

(C) 8

(D) 0

Sol. : $8 \equiv -1 \pmod{9}$ and $62 \equiv -1 \pmod{9}$

$$\therefore 8^{2n} - (62)^{2n+1} \equiv [(-1)^{2n} - (-1)^{2n+1}] \pmod{9}$$

$$\equiv (1 + 1) \pmod{9}$$

$$\equiv 2 \pmod{9}$$

∴ $8^{2n} - (62)^{2n+1}$ leaves remainder 2 when divided by 9.

Ans. : (A)

(26) Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer $\forall n \in \mathbb{N}$.

(IIT-1990)

Sol. : $p(n) : \frac{1}{7} n^7 + \frac{1}{5} n^5 + \frac{2}{3} n^3 - \frac{1}{105} n$ is an integer. $n \in \mathbb{N}$

For $n = 1$, $\frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = \frac{1}{105} [15 + 21 + 70 - 1] = 1$ is an integer.

∴ $p(1)$ is true.

Let $p(k)$ be true for $k \in \mathbb{N}$.

$$p(k) : \frac{1}{7} k^7 + \frac{1}{5} k^5 + \frac{2}{3} k^3 - \frac{1}{105} k \text{ is an integer for } k \in \mathbb{N}. \quad (1)$$

For $n = k + 1$,

$$\begin{aligned}
& \frac{1}{7} (k+1)^7 + \frac{1}{5} (k+1)^5 + \frac{2}{3} (k+1)^3 - \frac{1}{105} (k+1) \\
&= \frac{1}{7} (k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1) + \frac{1}{5} (k^5 + 5k^4 + 10k^3 + 10k^2 \\
&\quad + 5k + 1) + \frac{2}{3} (k^3 + 3k^2 + 3k + 1) - \frac{1}{105} (k+1) \\
&= \left(\frac{1}{7} k^7 + \frac{1}{5} k^5 + \frac{2}{3} k^3 - \frac{1}{105} k \right) + (k^6 + 3k^5 + 6k^4 + 7k^3 + 7k^2 + 4k) + 1 \\
&= \text{integer} + \text{integer} + 1 = \text{integer} \quad \text{By (1)}
\end{aligned}$$

$\therefore p(k+1)$ is true.

Thus, $p(k)$ is true $\Rightarrow p(k+1)$ is true. $k \in \mathbb{N}$

$\therefore \forall n \in \mathbb{N}$, $p(n)$ is true. (by the principle of mathematical induction)

$$\therefore \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105} \text{ is an integer } \forall n \in \mathbb{N}.$$

(27) Using the principle of mathematical induction, prove that :

$$\sum_{m=0}^k (n-m) \frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!} \left[\frac{n}{r+1} - \frac{k}{r+2} \right] \text{ where } m, n, r, k \text{ are non-negative integers.}$$

(IIT : 1991)

Sol. : Here, for non-negative integer k , we shall use the principle of mathematical induction.

$$\text{For } k = 0, \text{ L.H.S.} = \sum_{m=0}^0 (n-m) \frac{(r+m)!}{m!} = \frac{(n-0)(r+0)!}{0!} = n(r!)$$

$$\text{R.H.S.} = \frac{(r+0+1)!}{0!} \left[\frac{n}{r+1} - \frac{0}{r+2} \right] = n \cdot r!$$

$\therefore$ The statement is true for $k = 0$.

Suppose, the statement is true for $k = p$ where p is a non-negative integer.

$$\therefore \sum_{m=0}^p (n-m) \frac{(r+m)!}{m!} = \frac{(r+p+1)!}{p!} \left[\frac{n}{r+1} - \frac{p}{r+2} \right] \quad (1)$$

$$\text{For } k = p + 1, \text{ L.H.S.} = \sum_{m=0}^{p+1} (n-m) \frac{(r+m)!}{m!}$$

$$= \sum_{m=0}^p (n-m) \frac{(r+m)!}{m!} + (n-p-1) \frac{(r+p+1)!}{(p+1)!}$$

$$= \frac{(r+p+1)!}{p!} \left(\frac{n}{r+1} - \frac{p}{r+2} \right) + (n-p-1) \frac{(r+p+1)!}{(p+1)!} \quad (\text{By (1)})$$

$$= \frac{(r+p+1)!}{p!} \left(\frac{n}{r+1} - \frac{p}{r+2} + \frac{n}{p+1} - 1 \right)$$

$$\begin{aligned}
&= \frac{(r+p+1)!}{p!} \left[\frac{np+n+nr+n}{(r+1)(p+1)} - \frac{p+r+2}{r+2} \right] \\
&= \frac{(r+p+1)!}{p!} \left[\frac{n(p+r+2)}{(r+1)(p+1)} - \frac{p+r+2}{r+2} \right] \\
&= \frac{(r+p+2)!}{(p+1)!} \left[\frac{n}{r+1} - \frac{p+1}{r+2} \right] = \text{R.H.S.}
\end{aligned}$$

∴ The statement is true for $n = k + 1$.

∴ Given statement is true for all non-negative integers, k .

∴ The statement is true for non-negative integers k, n, r, m , by the principle of mathematical induction.

- (28) Using mathematical induction, prove that for every integer $n \geq 1$, $(3^{2^n} - 1)$ is divisible by 2^{n+2} but not by 2^{n+3} , $\forall n \in \mathbb{N}$. (IIT : 1996)

Sol. : $p(n) : 3^{2^n} - 1$ is divisible by 2^{n+2} , but not by 2^{n+3} . $n \in \mathbb{N}$

∴ $p(1) : 3^{2^1} - 1 = 8$ is divisible by 2^3 , but not by 2^4 .

So, $p(1)$ is true.

Suppose $p(k)$ is true for $k \in \mathbb{N}$.

∴ $3^{2^k} - 1$ is divisible by 2^{k+2} , but not by 2^{k+3} .

Suppose, $3^{2^k} - 1 = m \cdot 2^{k+2}$ where m must be an odd positive integer, otherwise $3^{2^k} - 1$ would be divisible by 2^{k+3} .

Now, $3^{2^{k+1}} = (3^{2^k})^2 = (1 + m \cdot 2^{k+2})^2$ (By (1))

$$= 1 + 2m \cdot 2^{k+2} + m^2 \cdot 2^{2k+4}$$

$$= m \cdot 2^{k+3} + m^2 \cdot 2^{k+1} \cdot 2^{k+3} + 1$$

$$= m \cdot 2^{k+3} (1 + m \cdot 2^{k+1}) + 1$$

Here, as m and $1 + m \cdot 2^{k+1}$ both are odd, $3^{2^{k+1}} - 1$ is divisible by 2^{k+3} but not by 2^{k+4} .

∴ $p(k+1)$ is true.

Thus, $p(k)$ is true $\Rightarrow p(k+1)$ is true.

∴ $p(n)$ is true, $\forall n \in \mathbb{N}$.

: Assertion and Reason Type Questions :

Explanation : Each of these questions contains statement 1 (Assertion) and statement 2 (Reason). Each question has four choices, options (A), (B), (C) and (D), which are as follows. You have to select the correct choice.

- (A) Statement 1 is true, statement 2 is true. Statement 2 is the correct explanation for statement 1.
- (B) Statement 1 is true, statement 2 is true. Statement 2 is not a correct explanation for statement 1.
- (C) Statement 1 is true, statement 2 is false.
- (D) Statement 1 is false, statement 2 is true.

(29) **Statement 1** : $n^2 + n + 41$ is not a prime for some $n \in \mathbb{N}$.

Statement 2 : Every odd natural number is a prime.

Sol. : For, $n = 41$, $n^2 + n + 41 = 41^2 + 41 + 41 = 41 \times 43$ is not a prime.

$\therefore$ Statement 1 is true.

1 is an odd natural number which is neither composite nor a prime but it is a unit integer.

$\therefore$ Statement 2 is false.

$\therefore$ Statement 1 is true and statement 2 is false.

Ans. : (C)

(30) **Statement 1** : $(n+1)^7 - n^7 - 1$ is divisible by 7, $\forall n \in \mathbb{N}$.

Statement 2 : $n^7 - n$ is divisible by 7, $\forall n \in \mathbb{N}$.

(AIEEE : 2011)

Sol. : Consider statement 2.

$p(n) : n^7 - n$ is divisible by 7. $\forall n \in \mathbb{N}$

$\therefore p(1) : 1^7 - 1 = 0$ is divisible by 7.

$\therefore p(1)$ is true.

Suppose $p(k) : k^7 - k$ is divisible by 7 is true. Let $k^7 - k = 7m$, $m \in \mathbb{N}$ (1)

$$\begin{aligned}(k+1)^7 - (k+1) &= k^7 + \binom{7}{1}k^6 + \binom{7}{2}k^5 + \dots + 7k + 1 - k - 1 \\ &= (k^7 - k) + \binom{7}{1}k^6 + \binom{7}{2}k^5 + \dots + \binom{7}{6}k\end{aligned}$$

$k^7 - k$ is divisible by 7 and $\binom{7}{r}$ is divisible by 7. For $1 \leq r \leq 6$. (By (1))

$\therefore p(k+1)$ is true.

$\therefore p(n)$ is true, $\forall n \in \mathbb{N}$.

$\therefore$ Statement 2 is true.

Now, consider statement 1.

From statement 2, $n^7 - n$ is divisible by 7. $\forall n \in \mathbb{N}$

$\therefore (n+1)^7 - (n+1)$ is divisible by 7.

$\therefore (n+1)^7 - (n+1) = 7m$, $m \in \mathbb{N}$ and $n^7 - n = 7p$, $p \in \mathbb{N}$

$\therefore (n+1)^7 - n^7 + n^7 - n - 1 = 7m$

$\therefore [(n+1)^7 - n^7 - 1] + (n^7 - n) = 7m$

$\therefore (n+1)^7 - n^7 - 1 = 7(m - p)$

$\therefore (n+1)^7 - n^7 - 1$ is divisible by 7. $\forall n \in \mathbb{N}$

∴ Statement 1 is also true.

∴ Statement 1 and statement 2 both are true and statement 2 is the correct explanation for statement 1.

Ans. : (A)

(31) **Statement 1** : $p(n)$: 111...1 (n times) is not a prime for some $n \in \mathbb{N} - \{1\}$.

Statement 2 : $p(n)$: 111...1 (n times) is composite for $n = 91$.

Sol. : Take $n = 91$ in statement 2,

111...1 (91) times

$$= 1 + 10 + 10^2 + 10^3 + \dots + 10^{90}$$

$$= 1 \frac{(10^{91} - 1)}{10 - 1}$$

$$= \frac{10^{91} - 1}{9}$$

$$= \frac{(10^7)^{13} - 1}{9}$$

$$= \frac{(10^7 - 1)}{9} (10^{84} + 10^{77} + \dots + 10^7 + 1)$$

$$= \frac{(10 - 1)(10^6 + 10^5 + \dots + 10 + 1)(10^{84} + 10^{77} + \dots + 10^7 + 1)}{9}$$

$$= (10^6 + 10^5 + \dots + 10 + 1)(10^{84} + 10^{77} + 10^{70} + \dots + 10^7 + 1) \text{ is a composite integer.}$$

Statement 2 is true.

∴ 111...1 (91 times) is not a prime.

∴ Statement 1 is also true.

Thus, both the statements are true and statement 2 is the correct explanation for statement 1.

Ans. : (A)

